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PDE and Modelling

Exercise sheet 5

Problem 1 (3 points)

Let $\Omega \subset \mathbb{R}^n$ be bounded, and let $W : \Omega \times \mathrm{GL}_+(n) \to \mathbb{R}$ be a free energy function for a body with reference density $\rho_0 : \Omega \to (0, +\infty)$. Denote by $\hat{S}(X, F) := \frac{\partial W}{\partial F}$ the constitutive function for the Piola-Kirchhoff stress, and write

$$\left(\operatorname{div} \hat{S}(X, Dx(t, X))\right)_i = \sum_{j=1}^{n} \frac{\partial}{\partial X_j} \left(\hat{S}_{i,j}(X, Dx(t, X))\right).$$

Let x satisfy the equation of motion

$$\rho_0 \ddot{x} = \operatorname{div} \hat{S}(X, Dx(t, X))$$

(formulated in the reference configuration) with boundary condition

$$\hat{S}(X, Dx(t, X))\mathbf{n} = 0$$

for $X \in \partial\Omega$ (force balance in the normal direction on the boundary). Show that the total energy

$$\int_\Omega \rho_0(X) \frac{|\dot{x}(t, X)|^2}{2} + W(X, Dx(t, X)) \,\mathrm{d}X$$

is conserved.

Problem 2 (7 points)

Let $\Omega \subset \mathbb{R}^n$ be open, connected and bounded, and let $\{\rho_k\}_{k\in\mathbb{N}}$ be a family of mollification kernels, that is, $\rho_k \in C_c^\infty(\mathbb{R}^n)$, $\rho_k \geq 0$, $\rho_k(x) = 0$ for $|x| \geq \frac{1}{k}$ and $\rho_k * \varphi \to \varphi$ uniformly as $k \to \infty$ for any $\varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$. For a C^1-curve $\gamma : [0, 1] \to \Omega$ with $\gamma' \neq 0$, define the linear map $T : C_c^\infty(\Omega; \mathbb{R}^n) \to \mathbb{R}$ by

$$T(\phi) := \int_0^1 \phi(\gamma(s)) \cdot \gamma'(s) \,\mathrm{d}s.$$

(a) Define T_k by $T_k(\phi) = T(\rho_k * \phi)$. Show that

$$T_k(\phi) = \int_\Omega w_k(x) \cdot \phi(x) \,\mathrm{d}x$$

for some $w_k \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$.

(b) Show that

$$T(\nabla\psi) = \psi(\gamma(1)) - \psi(\gamma(0)), \qquad T_k(\nabla\psi) = (\rho_k * \psi)(\gamma(1)) - (\rho_k * \psi)(\gamma(0)).$$

(c) Assume that $v \in C^\infty(\Omega; \mathbb{R}^n)$ satisfies

$$\int_\Omega v \cdot w \, \mathrm{d}x = 0$$

for all $w \in C_c^\infty(\Omega; \mathbb{R}^n)$ with $\operatorname{div} w = 0$. Show that

$$T(v) = \int_0^1 v(\gamma(s)) \cdot \gamma'(s) \, \mathrm{d}s = 0$$

for all closed C^1 curves $\gamma : [0,1] \to \Omega$. Conclude that there exists $h \in C^\infty(\Omega)$ such that $v = \nabla h$.

Hint: First argue that w_k from part (a) satisfies $\operatorname{div} w_k = 0$ if γ is a closed curve.

(d) Suppose that $v \in L^2(\Omega; \mathbb{R}^n)$ satisfies

$$\int_\Omega v \cdot w \, \mathrm{d}x = 0$$

for all $w \in C_c^\infty(\Omega; \mathbb{R}^n)$ with $\operatorname{div} w = 0$. Show that there exists $h \in W^{1,2}(\Omega)$ such that $v = \nabla h$.

Problem 3 (6 points)

Suppose that $\Omega \subset \mathbb{R}^3$ is a bounded domain, and consider the Stokes' problem with no-slip boundary condition

$$\begin{cases} -\nu \Delta u + \nabla p = f, & \text{in } \Omega, \\ \operatorname{div} u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{*}$$

where $\nu \in (0, +\infty)$ and $f : \Omega \to \mathbb{R}^n$ are given.

(a) Let $f \in L^2(\Omega; \mathbb{R}^n)$, and suppose that $u \in C^2(\overline{\Omega}; \mathbb{R}^n)$ and $p \in C^1(\overline{\Omega})$ solve (*). Show that

$$\nu \int_\Omega \sum_{i,j=1}^n \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, \mathrm{d}x = \int_\Omega f \cdot v \, \mathrm{d}x \qquad \text{for all } v \in W_0^{1,2}(\Omega; \mathbb{R}^n) \text{ with } \operatorname{div} v = 0 \tag{†}$$

(b) Assume that the functional

$$I(w) := \int_\Omega \frac{\nu}{2} \sum_{i,j=1}^n \left| \frac{\partial w_i}{\partial x_j} \right|^2 - f \cdot w \, \mathrm{d}x$$

takes its minimum on the set of vector fields $w \in W_0^{1,2}(\Omega)$ that satisfy $\operatorname{div} w = 0$. Show that the minimizer u satisfies (†).

(c) Show that (†) has at most one solution $u \in W_0^{1,2}(\Omega)$ that satisfies $\operatorname{div} u = 0$.

(d) Suppose now that $u \in W^{2,2}(\Omega)$ satisfies $\operatorname{div} u = 0$ in Ω and $u = 0$ on $\partial\Omega$, and solves (†). Prove that there exists $p \in W^{1,2}(\Omega)$ such that $\int_\Omega p \, \mathrm{d}x = 0$ and (u, p) solve (*).

Hint: Apply the result of the previous problem.

Problem 4 (4 points)

Consider the compressible Euler equations

$$\begin{cases} \dfrac{\partial v}{\partial t} + (v \cdot \nabla) v = -\dfrac{\nabla p(\rho)}{\rho}, \\ \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0 \end{cases}$$

(a) Linearize the Euler equations around the hydrostatic equilibrium $\rho = \rho_0$ and $p = p_0$. More precisely, introduce a (small) velocity field v which results in small changes in ρ and p, i.e., $\rho = \rho_0 + \tilde{\rho}$ and $p = p_0 + \tilde{p}$. Show that to first order

$$\frac{\partial v}{\partial t} = -\frac{\nabla \tilde{p}}{\rho_0}, \qquad \frac{\partial \tilde{\rho}}{\partial t} = -\rho_0 \operatorname{div} v$$

(b) Show that $\tilde{p}$ satisfies a wave equation, i.e., there is $c \in \mathbb{R}$ such that

$$\frac{\partial^2 \tilde{p}}{\partial t^2} = c^2 \Delta \tilde{p}$$

Hint: Differentiate the second linearized equation with respect to t, insert the first one and use $p = p(\rho)$ to derive a linearized relation between $\tilde{p}$ and $\tilde{\rho}$.

Due: Wednesday, June 3 at the end of the lecture