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PDE and Modelling

Exercise sheet 4

Problem 1 (8 points)

Let $n = 3$, and assume that $\mathcal{L} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}_{\text{sym}}$ is linear, and

$$\mathcal{L}\left(QFQ^T\right) = Q\mathcal{L}(F)Q^T$$

for all $F \in \mathbb{R}^{n\times n}$ and all $Q \in \mathrm{SO}(n)$.

(a) Let $F = e_1 \otimes e_2 - e_2 \otimes e_1$, and show that $F\mathcal{L}(F) = \mathcal{L}(F)F$.

Hint: Consider $Q_\tau = e^{\tau F} \in \mathrm{SO}(n)$.

(b) Conclude from (a) that

$$\mathcal{L}(F) = \begin{pmatrix} a & c & 0 \\ c & d & 0 \\ 0 & 0 & b \end{pmatrix}$$

for some $a, b, c \in \mathbb{R}$.

(c) Use (a) again to show that the matrices

$$\begin{pmatrix} a & c \\ c & d \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

commute, and argue that $c = 0$ and $a = d$ using Problem 4 of sheet 3.

(d) Use the hypothesis for $\mathcal{L}$ for an appropriate Q to obtain $a = b = 0$.

(e) Argue that $\mathcal{L}(F) = 0$ for every skew matrix F, and that (for general F) $\mathcal{L}(F)$ only depends on $\frac{1}{2}(F + F^T)$, that is, $\mathcal{L}(F) = \mathcal{F}(F + F^T))$.

(f) Argue that the arguments from parts (a)-(c) can be repeated to show

$$\mathcal{L}(e_3 \otimes e_3) = \lambda \mathrm{Id} + \mu e_3 \otimes e_3,$$

for some $\lambda, \mu \in \mathbb{R}$, and conclude that

$$\mathcal{L}(e_j \otimes e_j) = \lambda \mathrm{Id} + \mu e_j \otimes e_j,$$

for $j = 1, 2, 3$.

(g) Show that

$$\mathcal{L}(F) = \mu \frac{F + F^T}{2} + \lambda (\operatorname{tr} F)\mathrm{Id}$$

for all F.

Hint: Assume that F is diagonal at first.

Problem 2 (5 points)

Let $W : \mathrm{GL}_+(n) \to \mathbb{R}$ be C^1, set $S = \nabla W$, that is

$$S_{i,j}(F) = \frac{\partial W}{\partial F_{i,j}},$$

and let $\sigma(F) \operatorname{cof} F = S(F)$.

For a subgroup g of $\mathrm{SL}(n)$, consider the following statements:

(i) $W(F) = W(FA)$ for all $A \in g$,

(ii) $S(FA)A^T = S(F)$ for all $A \in g$,

(iii) $\sigma(FA) = \sigma(F)$ for all $A \in g$.

(a) Show that (i) implies (ii).

(b) Show that (ii) and (iii) are equivalent.

(c) Consider

$$g = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, t \in \mathbb{R} \right\}, \qquad W(F) = \frac{Fe_1 \cdot Fe_2}{|Fe_1|^2}.$$

Show that (ii) holds, but (i) does not.

Problem 3 (3 points)

Suppose $W : \mathrm{GL}_+(n) \to \mathbb{R}$ satisfies

$$W(QF) = W(F) = W(FQ)$$

for all $Q \in \mathrm{SO}(n)$.

(a) Show that there exists a function $g : (0, \infty)^n \to \mathbb{R}$ such that

$$W(F) = g(\lambda_1(F), \ldots, \lambda_n(F)),$$

where $\lambda_i(F)$ are the singular values of F.

(b) Argue that g is invariant under permutation of the arguments, that is,

$$g(z_1, \ldots z_n) = g(z_{i(1)}, \ldots, z_{i(n)})$$

if i is a permutation.

Due: Wednesday, May 13 at the end of the lecture