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PDE and Modelling

Exercise sheet 3

Problem 1

Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$. Let V be a simplex in $\mathbb{R}^n$ with faces $S_j \subset \{x \in \mathbb{R}^n \,|\, x_j = 0\}$ for $j = 1, \ldots, n$ and face S having the normal $\vec{n} = \sum_{j=1}^n n_j e_j$. We assume $n_j > 0$ and $x_j \geq 0$ for $x \in V$ and $j = 1, \ldots, n$. Show

$$\frac{|S_j|}{|S|} = n_j, \qquad \text{where } |S|, |S_j| \text{ are the areas of the faces.}$$

Hint: Use the divergence theorem for appropriate vector fields.

Problem 2

Suppose that conservation of mass, momentum and energy (with $r \equiv 0$) hold, i.e.

$$\begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho v) &= 0 \\ \varrho(\partial_t v + (v \cdot \nabla) v) &= \operatorname{div} \sigma + f \\ \varrho \frac{D\varepsilon}{Dt} &= \sigma : Dv - \operatorname{div} q, \end{aligned}$$

where we defined $A : B = tr A^T B$. Furthermore assume the following thermodynamic relations: $\varepsilon = a(\theta, \varrho) + \theta\eta$, where $\theta(t,x)$ denotes the temperature at (t,x) and

$$\eta = -\frac{\partial a}{\partial \theta} \quad \text{and} \quad p = \varrho^2 \frac{\partial a}{\partial \varrho}.$$

Deduce that

$$\partial_t(\varrho\eta) + \operatorname{div}\left(\frac{q}{\theta} + \varrho\eta v\right) = \frac{1}{\theta}(\sigma + p\mathrm{Id}) : Dv + q \cdot \nabla \frac{1}{\theta}.$$

Problem 3

Consider an ideal incompressible fluid of constant density ρ. It was shown in the lecture that there exists a scalar function p such that the Cauchy stress tensor is of the form $\sigma = -p\mathrm{Id}$. Assume that the fluid is subject to the gravitational body force density $f(x) = -g\rho e_3$, where $g > 0$ is a constant.

(a) Show that a stationary (time independent) solution of the equation of motion $\operatorname{div} \sigma + f = 0$ is given by $p = -g\rho x_3$.

(b) Consider a rigid body of a different constant density $\rho_1 > \rho$ which occupies a domain ω and is surrounded by the fluid. Compute the total force on the body by adding the gravitational force on the body and the force

$$-\int_{\partial\omega} pn \, \mathrm{d}\mathcal{H}^{d-1}$$

on the body. Does the body sink or float?

Hint: Gauß.

Problem 4

Suppose that $\hat{\sigma} : \mathrm{GL}_+(n) \to \mathbb{R}^{n\times n}_{\mathrm{sym}}$ satisfies

(i) $\hat{\sigma}(QF) = Q\hat{\sigma}(F)Q^T$ for all $Q \in \mathrm{SO}(n)$ (frame indifference),

(ii) $\hat{\sigma}(FG) = \hat{\sigma}(F)$ for all $G \in \mathrm{SL}(n)$ (isotropy group of a fluid).

(a) Use *(ii)* to show that $\hat{\sigma}(F) = \hat{\sigma}(\sqrt[n]{\det F}\,\mathrm{Id})$.

(b) Show that $Q\hat{\sigma}(F)Q^T = \hat{\sigma}(F)$.

(c) Let M be a symmetric matrix such that $M = QMQ^T$ for all $Q \in \mathrm{SO}(n)$. Show that $M = \alpha\mathrm{Id}$ for some $\alpha \in \mathbb{R}$.

Hint: You may consider a diagonal matrix M and permutation matrices Q first. Then consider the general case. Alternatively you may prove that M cannot have two distinct eigenvalues.

(d) Prove that there exists a function $\hat{p} : (0, \infty) \to \mathbb{R}$ such that $\hat{\sigma}(F) = -\hat{p}(\frac{1}{\det F})\mathrm{Id}$.

Due: Wednesday, May 6 at the end of the lecture