Prof. Dr. M. Disertori
Dr. M. Zaal
Institut für Angewandte Mathematik
Universität Bonn

PDE and Modelling

Exercise sheet 2

Problem 1

Let $\Omega = [0,1]^2$, and consider the domain $\mathbb{R} \times \Omega$ of the Lagrangian coordinates (t, X). Let $\mathcal{T}$ be the domain for the corresponding Eulerian coordinates (t, x), and suppose that the velocity field in Eulerian coordinates is $v(t, x) = (x_1, -x_2)$.

(a) Find the trajectories and streamlines of v.

(b) Fix some $t > 0$ and find $\Omega(t)$, that is, the domain of the spatial coordinate x at time t. Sketch the corresponding region on the plane (x_1, x_2).

(c) Sketch $\mathcal{T}$ as a subset of $\mathbb{R}^3$.

Problem 2

We want to model the movement of a planet and a sun by two point particles located at the points $x_P(t)$ and $x_S(t)$. The corresponding masses are denoted by m_p and m_S. The movement of the planet around the sun is described by the oriented distance:

$$x(t) = x_P(t) - x_S(t).$$

We assume the two particles interact only via gravitational forces

$$f_{PS}(x_P, x_S) = -G m_S m_P \frac{x_P - x_S}{\|x_P - x_S\|^3} = -f_{SP}(x_S, x_P)$$

and no external force is present

$$f_S = 0 = f_P.$$

(a) Write the equations of motion for $x_P(t)$ and $x_S(t)$. Write the expression for the total angular momentum L of the two-body system and verify by direct computation that this is a conserved quantity.

Write the expression for the kinetic energy of the two-body system planet-sun, then determine an expression for the potential energy E_{pot} such that the total energy $E = E_{\text{kin}} + E_{\text{pot}}$ is a conserved quantity i.e. $E'(t) = 0$.

Hint: remember that the kinetic energy for a system of n particles is given by

$$E_{\text{kin}} = \sum_{i=1}^{n} \frac{1}{2} m_i \|x_i\|^2.$$

(b) Show that the movement of the planet around the sun is described by the equation

$$m_P x''(t) = -G m_P m \frac{x(t)}{\|x(t)\|^3},$$

where $G > 0$ is the gravitational constant and $m = m_P + m_S$ the total mass. The corresponding angular momentum is $L = x \times m_P x'$. Show that this quantity is conserved (i.e. $L' = 0$).

(c) Prove *Kepler's second law of planetary motion*: The distance vector $x(t)$ between the planet and the sun sweeps out equal areas $A_{\Delta t}$ during equal intervals Δt of time.

Hint: Use and show, that for $\Delta t = t_2 - t_1$ it holds

$$A_{\Delta t} = \frac{1}{2} \int_{t_1}^{t_2} \|x(t) \times x'(t)\| \, \mathrm{d}t.$$

Problem 3

Suppose that $\Omega \subset \mathbb{R}^d$ is bounded, and let $x : \mathbb{R} \times \Omega \to \mathbb{R}^d$ be a smooth motion. Assume the reference mass distribution is constant $\rho_0(X) = 1 \; \forall X \in \Omega$. The *center of mass* of $\Omega(t)$ is defined by

$$x^*(t) = \frac{1}{|\Omega|} \int_{\Omega(t)} x \rho(t, x) \, \mathrm{d}x,$$

where $|\Omega|$ is the volume of Ω.

(a) Show that the mass density is given by

$$\rho(t, x) = \frac{1}{\det Dx(t, X)|_{X(t,x)}}.$$

(b) Show that

$$\int_{\Omega(t)} \rho(t, x) \, \mathrm{d}x = |\Omega|.$$

(c) Show that $\partial_t x^*(t)$ is constant whenever linear momentum is conserved.

(d) Assume that, in addition, angular momentum is conserved, and let $x_0 \in \mathbb{R}^n$. Show that

$$\int_{\Omega(t)} (x - x_0) \wedge (\rho v) \, \mathrm{d}x$$

is also conserved.

(e) Under the same assumptions, prove that

$$\int_{\Omega(t)} (x - x^*(t)) \wedge (\rho v) \, \mathrm{d}x$$

is conserved

Caution: as $x^*(t)$ is not constant in general, (c) cannot be applied directly.

Problem 4

Suppose that $\Omega \subset \mathbb{R}^d$ is bounded, $\rho : \Omega \to \mathbb{R}_+$ is a mass density on Ω, $v : \Omega \to \mathbb{R}^d$ is a vector field on Ω such that the linear and angular moment vanishes, i.e.

$$\int_\Omega \rho v \,\mathrm{d}x = 0, \qquad \int_\Omega x \wedge (\rho v) \,\mathrm{d}x = 0.$$

Show that

$$\int_\Omega \rho \|v\|^2 \,\mathrm{d}x \leq \int_\Omega \rho \|v(x) + Ax + b\|^2 \,\mathrm{d}x$$

for all skew $n \times n$ matrices A and all vectors $b \in \mathbb{R}^d$.

Hint: Consider

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \int_\Omega \rho(x) \|v(x) + \tau A x + \tau b\|^2 \,\mathrm{d}x \bigg|_{\tau=0},$$

and argue that the map $f : \mathbb{R}_+ \to \mathbb{R}$ defined by

$$f(\tau) = \int_\Omega \rho(x) \|v(x) + \tau A x + \tau b\|^2 \,\mathrm{d}x$$

is non decreasing for all $\tau \geq 0$.

Due: Friday, April 24 at the end of the lecture