Nonlinear PDEs II Prof. Dr. J. J. L. Velázquez
Problem Sheet 6 Dr. M. Helmers, Dr. M. Zaal
Due in class on 05.06.2013

Problem 1 (An unbounded entropy solution to Burgers' equation)
Show that

$$u(t,x) = \begin{cases} -\frac{2}{3}\left(t+\sqrt{3x+t^2}\right) & \text{if } 4x+t^2>0, \\ 0 & \text{if } 4x+t^2<0 \end{cases}$$

is an unbounded entropy solution to Burgers' equation $\partial_t u + \frac{1}{2}\partial_x(u^2) = 0$. Verify the Rankine-Hugoniot condition and sketch the projected characteristics in the (x,t)-plane.

Problem 2 (A few shock curves)
Compute the unique entropy solution to

$$\begin{aligned} \partial_t u + \frac{1}{2}\partial_x(u^2) &= 0 && \text{in } \mathbb{R}\times\{t>0\}, \\ u(0,x) &= g(x) && \text{on } \mathbb{R}\times\{t=0\}, \end{aligned}$$

where

$$g(x) = \begin{cases} 1 & \text{if } x<-1, \\ 0 & \text{if } -1<x<0, \\ 2 & \text{if } 0<x<1, \\ 0 & \text{if } 1<x. \end{cases}$$

Sketch the characteristic diagram, including all shock curves etc.

Problem 3 (Self similarity and Burgers' equation – 8 MARKS)
Let $g_0 : \mathbb{R} \to [0,1]$ be

$$g_0(x) = \begin{cases} 1, & \text{if } x \le 0, \\ 1-x, & \text{if } 0<x<1, \\ 0, & \text{if } x \ge 1, \end{cases}$$

and define g_n ($n \in \mathbb{N}$) by

$$g_n(x) = \begin{cases} \frac{1}{2} + \frac{1}{2}g_{n-1}(3x), & \text{if } x \le \frac{1}{2}, \\ \frac{1}{2}g_{n-1}(3x-2), & \text{if } x > \frac{1}{2}. \end{cases}$$

1. Sketch the graphs of g_0, g_1 and g_2. Note that $g_n \to g$, where $g = 1-V$ and V is the Cantor function or "Devil's staircase".

2. Show that $h_n(x) := \int_0^x g_n(x)\,\mathrm{d}x$ satisfies

$$h_n(x) = \begin{cases} \frac{x}{2} + \frac{1}{6}h_{n-1}(3x), & \text{if } x \le \frac{1}{2}, \\ \frac{5}{12} + \frac{1}{6}h_{n-1}(3x-2), & \text{if } x > \frac{1}{2}. \end{cases}$$

Consider the initial-value problem

$$\begin{cases} \partial_t u_n + \frac{1}{2}\partial_x(u_n^2) = 0, & \text{in } (0,\infty)\times\mathbb{R}, \\ u_n = g_n, & \text{on } \{0\}\times\mathbb{R}. \end{cases}$$

Denote

$$\Phi_n(t,x,y) := \frac{|x-y|^2}{2t} + h_n(y),$$

and if $\Phi_n(t,x,.)$ has a unique minimizer, denote it by $y_n(t,x)$.

3. Show that, for $n \geq 1$,

$$\Phi_n(t,x,y) = \begin{cases} \frac{1}{6}\Phi_{n-1}(\frac{3}{2}t, 3x-\frac{3}{2}t, 3y) + \frac{x}{2} - \frac{t}{8}, & \text{if } y \leq \frac{1}{2}, \\ \frac{1}{6}\Phi_{n-1}(\frac{3}{2}t, 3x-2, 3y-2) + \frac{5}{12}, & \text{if } y > \frac{1}{2}. \end{cases}$$

4. Let $t \geq \frac{4}{3}$. Prove that

$$y_n(t,x) = \begin{cases} x-t, & \text{if } x < \frac{1+t}{2}, \\ x, & \text{if } x > \frac{1+t}{2}, \end{cases}$$

   and

$$\Phi_n(t,x,y_n(t,x)) = \begin{cases} x-\frac{t}{2}, & \text{if } x < \frac{1+t}{2}, \\ \frac{1}{2}, & \text{if } x > \frac{1+t}{2}, \end{cases}$$

   for all $n \geq 0$. Use the Lax-Oleinik formula to conclude that

$$u_n(t,x) = \begin{cases} 1, & \text{if } x < \frac{1+t}{2}, \\ 0, & \text{if } x > \frac{1+t}{2}. \end{cases}$$

5. Let $t < \frac{4}{3}$. Show that

$$y_n\left(t, \frac{1+t}{2}\right) = \frac{1}{2}$$

   for all $n \geq 1$. Use the fact that $y_n(t,.)$ is nondecreasing to argue that

$$y_n(t,x) = \begin{cases} \frac{1}{3}y_{n-1}\left(\frac{3}{2}t, 3x-\frac{3}{2}t\right) & \text{if } x < \frac{1+t}{2}, \\ \frac{1}{3}y_{n-1}\left(\frac{3}{2}t, 3x-2\right) + \frac{2}{3}, & \text{if } x > \frac{1+t}{2}, \end{cases}$$

   for all whenever $\frac{4}{3}\left(\frac{2}{3}\right)^k \leq t < \frac{4}{3}\left(\frac{2}{3}\right)^{k-1}$ and $n \geq k$.

6. Under the same assumptions on t, show that $y_n(t,x) = x - \frac{j}{2^{-n}}t$ for some $j \in \{0,\ldots,2^n\}$, and sketch the characteristic diagram of u_n by using the Lax-Oleinik formula.
