Graduate Seminar in PDEs (S4B2)
Self similarity and singular behaviour in PDEs

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A usual example of explicit solutions of some PDEs are those of self-similar solutions. These correspond to solutions that are invariant under rescaling of its variables. They arise as a consequence of the fact that the corresponding equations are invariant under these transformations. A typical example is that of the heat equation

$$\partial_t u = \Delta u, \text{ for } (t,x) \in (0,\infty) \times \mathbb{R}^d. \tag{0.1}$$

It is straightforward to check that, if u solves (0.1), then $u_\lambda = \lambda^\alpha u(\lambda^2 t, \lambda x)$ solves (0.1) for any $\alpha \in \mathbb{R}$ as well. Then, if we looked for a solution of the form $u(t,x) = f(x/\sqrt{t})$, we would obtain that $u_\lambda = u$, i.e. we would obtain a solution that is invariant under the transformation $(t,x) \mapsto (\lambda t, \lambda^2 x)$. The resulting solution is the well-known Gaussian profile

$$u(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{4t}}.$$

These kind of solutions are much more than a mere mathematical curiosity, as they appear constantly when describing the asymptotic behaviour of solutions of PDEs. They are also very useful in order to study the form of solutions near singularities.

In this seminar we will study the use of self-similar solutions by means of different examples from different PDEs. We will study how they are useful to describe the asymptotic behaviour of solutions of the Heat Equation, the Vorticity Equations [5], the Porous Medium Equations [13] or even free boundary problems like the Stefan Problem [4, 11] when $t \to \infty$. On the other hand, we will also study how to use self-similar variables to describe the behaviour of solutions for non-linear equations when we approach a singular point, as in the semilinear Heat Equation [6, 7] or the Mean Curvature Flow Equation [5]. If time allows, we will also study one famous case of a situation where the self-similar blow up is excluded, as in the case of the 3D Navier-Stokes Equations [10].

These ideas can also be applied to other non-parabolic equations, as in the Schrödinger's Equation [2] and kinetic equations [1, 9, 12], or in order to analyze the behaviour of solutions for the Laplace's Equation near a corner [3, 8].

Prerequisites: Functional Analysis and basic notions of PDE.

Preliminary meeting: Wednesday 15th of July at 14:15 in room 2.040.

References

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