Graduate seminar on analysis (S4B1), Spring term 2025: The mathematics of the immune system

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Preliminary meeting: Friday 31rst January at 10:15, Room SR 0.007, Endenicher Allee 60.

If you cannot come to preliminary meeting but are interested in the seminar, please feel free to contact the organizers by the e-mail given above.

The role of the immune system is to detect and to respond to various pathogens and to malignant cells. To understand how the immune system works many mathematical models have been formulated. The analysis of these models leads to challenging mathematical problems. For instance we can encounter combinatorial problems, problems of optimization and problems related with the study of certain dynamical systems. During this seminar we will see various models describing different processes taking place in the immune system. We will follow the review by Perelson [11] and we will study in detail some of the papers cited in there. In the last part of the seminar we will also study more recent papers as for instance [12] and [2]. The main properties of the immune system that we want to study during this seminar are the following.

1. The ability to detect essentially any antigen. In this direction we will study [9], [5] and [6]. To study this papers an elementary knowledge of probability theory is needed. The models studied in [9] use basic tools from Reliability theory that can be found in [5]. Instead, the model in [6] has analogies with the matching problem, which is a classical problem in combinatorics.

2. The ability of the immune system to discriminate between self antigens (against to which an immune response should not take place) and foreign antigens (against which we must have an immune response). The papers that we will study on this topic are [3] and [2] and are based on the analysis of the kinetic proofreading mechanism. In [3] the model is deterministic while in [2] the model is probabilistic. To study these papers a basic knowledge of complex analysis, probability theory and of dynamical systems is needed.

   A generalization of the kinetic proofreading model is studied in [12]. This paper uses some tools of dynamical systems and chemical network theory.

3. The various mechanisms that lead to an immune response after a foreign antigen is recognized. In this seminar we will study as a first step the mechanism of receptor clustering, see [7]. The models of receptor clustering are important in immunology because there exists experimental evidence that receptor clustering is one of the first mechanisms necessary to activate the immune response. These models consist of systems of differential equations. Both the equilibrium and the kinetics aspects of these equations are studied in these papers.

4. We will study optimal strategies in immunology. In particular, we will study the paper [8]. In this paper the optimal strategy of lymphocytes proliferation in order to minimize the time necessary to produce the minimal amount of antibody required for the immune response is studied. The model is simple, but illustrative.

5. Maturation of the affinity of antibodies with immunizing antigens (i.e. foreign antigens to which the immune system is exposed). On this topic we will discuss the paper [4]. In this paper a model of cell mutations is formulated and an optimal control problem is studied. Roughly speaking the goal is to choose a mutation function that maximizes the total affinity of the system.

6. Immunological memory. To understand this property we will study [14]. The main feature of the model studied in that paper is that the interactions between different antibodies are considered and are proven to lead, under certain conditions, to immunological memory.

A generalization of the model studied in [14] is the model in [1]. In this model the interaction between antibodies and antigens is considered

Knowledge required for the course: basic knowledge of probability theory, basic theory of dynamical systems.

References

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