LECTURE NOTES ON DIRECT METHODS AND REGULARITY IN THE CALCULUS OF VARIATIONS

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Contents

1. The direct methods in the calculus of variations

Let Ω be a bounded domain of $\mathbb{R}^n$ and $g : \partial\Omega \to \mathbb{R}$ be a continous function. We start by considering the classical example of a *multi-dimensional variational problem*:

$$\text{Find a function } u : \bar{\Omega} \to \mathbb{R} \text{ such that}$$

$$\begin{aligned} u &\in C(\bar{\Omega}) \cap C^2(\Omega)\,, \\ u &= g \text{ on } \partial\Omega\,, \\ \int_\Omega |\nabla u|^2\,dx &\le \int_\Omega |\nabla w|^2\,dx \end{aligned} \tag{1.1}$$

for all $w \in C(\bar{\Omega}) \cap C^2(\Omega)$, $w = g$ on $\partial\Omega$.
In other words: Find the minimum of the "variational integral" $\int_\Omega |\nabla u|^2\,dx$ in the class $\mathbb{K}$ of "admissible" functions where

$$\mathbb{K} = \{w \in C(\bar{\Omega}) \cap C^2(\Omega) | w = g \text{ on } \partial\Omega\,; \quad \int_\Omega |\nabla w|^2\,dx \le \infty\}\,.$$

If g is smooth, say $g \in C^1(\mathbb{R}^n)$, the class $\mathbb{K}$ is not empty.

During the last century the famous mathematician Riemann made the famous error of assuming that the minimum of the variational integral "always" exists. This was later criticized by Weierstraß, but it took a long time until satisfactory results on this question were found.
We first want to show that with the above formulation of the problem it may happen, that (1.1) *has no solution.* A simple example is the following: Let

$$\Omega = \{x \in \mathbb{R}^n |\, |x| \le 1\,,\ x \ne 0\} \text{ where } |x| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}, \text{ and}$$

$$g(x) = 0 \text{ on } \{x \in \mathbb{R}^n |\, |x| = 1\} \text{ and } g(x) = 1 \text{ for } x = 0\,.$$

Obviously, $g : \partial\Omega \to \mathbb{R}$ is a restriction of a $C^1(\mathbb{R}^n)$-function and the admissible class is not empty. We intend to show that for this example

$$\inf_{w\in\mathbb{K}} \left\{ \int_\Omega |\nabla w|^2\,dx \right\} = 0\,. \tag{1.2}$$

Therefore if there existed a minimum $u \in \mathbb{K}$ then $\int_\Omega |\nabla u|^2\,dx = 0$ and thus $u = \text{const}$ since Ω is connected. This contradicts the fact that $u \in \mathbb{K}$ and thus $u = 0$ on $\{|x| = 1\}$, $u(x) = 1$ for $x = 0$, i.e. $u \ne \text{const}$. So there is no solution to problem (1.1).
In order to prove (1.2) we construct a sequence $w_j \in \mathbb{K}$ such that $\int_\Omega |\nabla w_j|^2\,dx \to 0$. We define

$$w_j(x) = \frac{2}{\pi} \operatorname{arctg}\left(\frac{1}{j} \ln\ln\left(\frac{e}{|x|}\right)\right)\,, \qquad e = \exp(1)\,,\ \pi = 3,14\ldots\,.$$

First note that $w_j \in \mathbb{K}$. In fact, $w_j(x) = 0$ for $|x| = 1$ and $w_j \in C^2(B_1 - \{0\})$. For $x \to 0$, $B_1 = \{|x| \le 1\}, \ln\ln(\frac{e}{|x|}) \to \infty$ and $\operatorname{arctg}(\dots) \to \frac{\pi}{2}$, hence $w_j(0) = 1$ by continuous extension and $w_j \in C(\bar{\Omega})$.
By a simple calculation

$$\partial_i w_j(x) = \frac{2}{\pi}\frac{1}{j}\frac{1}{1+[\frac{1}{j}\ln\ln(\frac{e}{|x|})]^2}\frac{1}{\ln(\frac{e}{|x|})}\frac{-x_i}{|x|^2}$$

and

$$|\nabla w_j(x)|^2 \le \frac{4}{\pi^2}\frac{1}{j^2}\frac{1}{|\ln(\frac{e}{|x|})|^2}\frac{1}{|x|^2}.$$

Using polar coordinates, it is simple to show that

$$C := \int\limits_\Omega \frac{dx}{|\ln(\frac{e}{|x|})|^2|x|^2}$$

is finite. Hence

$$\int\limits_\Omega |\nabla w_j(x)|^2\,dx \le \frac{1}{j^2}\frac{4}{\pi^2}C \to 0 \text{ as } j \to \infty.$$

Thus we obtain (1.2) and the knowledge that (1.1) is not solvable in general.

We want to analyze why this happens, and for this purpose we look at what happens if one minimizes a function $f : \mathbb{R}^n \to \mathbb{R}$: A possible way of trying to prove the existence of a minimum is the following:

(A) Consider a *minimizing sequence*, i.e. a sequence $u_m \in \mathbb{R}^n$ such that $f(u_m) \to \inf\limits_{v\in\mathbb{R}^n} f(v)$.

(B) Try to prove that the sequence $\{u_m\}$ is bounded. Usually, this is done via a *coerciveness-condition*:

$$f(u) \to \infty \text{ if } |u| \to \infty.$$

(C) Use the fact that *bounded sets in* $\mathbb{R}^n$ *are compact* and select a convergent subsequence $u_{m_i} \to u$ $(i \to \infty)$.

(D) Try to prove that f is continuous or at least *lower semi-continuous*, i.e.

$$\liminf_{i\to\infty} f(u_{m_i}) \ge f(u).$$

Since $f(u_{m_i}) \to \inf\limits_{v\in\mathbb{R}^n} f(v)$ we have

$$f(u) \le \inf_{v\in\mathbb{R}^n} f(v),$$

and thus $f(u) = \inf\limits_{v\in\mathbb{R}^n} f(v)$ and this means that f has a minimum.

This method, consisting of steps (A)–(D) applied to variational integrals $\int\limits_\Omega F(x,u,\nabla u)\,dx$, is called *the direct methods in the calculus of variations.*

This method does not work if one does not choose the set $\mathbb{K}$ of admissible functions (and it is topology) in an adequate way. For example if

$$\mathbb{K} = \{u \in C(\bar{\Omega}) \cap C^2(\Omega) | u = g \text{ on } \partial\Omega\,; \quad \int\limits_\Omega |\nabla u|^2\,dx < \infty\}$$

is equipped with the norms

$$\|u\|_\infty = \max_{x\in\bar{\Omega}} |u(x)| \qquad \text{or}$$

$$\|u\|_{1,2} = \Big(\int\limits_\Omega |u|^2\,dx\Big)^{1/2} + \Big(\int\limits_\Omega |\nabla u|^2\,dx\Big)^{1/2}$$

respectively then the steps (B)-(C) or (C) are not possible. To see this, we define $v_\varepsilon(x) = \ln(\ln(e\sqrt{1+\varepsilon}\,\sqrt{x^2+\varepsilon}))\,,\ \varepsilon > 0\,,\ x \in \Omega = \{x \in \mathbb{R}^2 \,|\, |x| \le 1\}$. Then

$$\nabla v_\varepsilon = -\frac{x_i}{|x|^2+\varepsilon}\,\frac{1}{\ln\Big(\frac{e}{\sqrt{|x|^2+\varepsilon)}}\Big)} \qquad \text{and}$$

$$\|v_\varepsilon\|_\infty \to \infty\,, \ \int\limits_\Omega |\nabla v_\varepsilon|^2\,dx \le K \text{ as } \varepsilon \to 0\,.$$

On the other hand, $\int\limits_\Omega |\nabla v_j|^2\,dx = 1$ and it follows that the *variational integral* $\int\limits_\Omega |\nabla u|^2\,dx$ *is not coercive with respect to the maximum norm.* Note, however, that $\int\limits_\Omega |\nabla u|^2\,dx$ is coercive with respect to the $\|.\|_{1,2}$-norm if we fix the boundary values. So, we have seen that step (B) fails if we use the wrong norm. Step (C) fails if we use as norms $\|.\|_\infty$ or $\|.\|_{1,2}$ since bounded sets in infinite dimensional spaces are not compact. This leads us to the idea of using the *weak topology.* This, however, does not work if we take the spaces $C(\bar{\Omega}) \cap C^2(\Omega)$ as the set of admissible functions since they are not complete (and reflexive). So, the appropriate method is to use the completion of $C(\bar{\Omega}) \cap C^2(\Omega)$ with respect to the norm $\|.\|_{1,2}$ as the set of admissible functions. The completion of $C(\bar{\Omega}) \cap C^2(\Omega)$ with respect to $\|.\|_{1,2}$ is denoted by $H^{1,2}(\Omega)$, i.e. the Sobolev space $H^{1,2}$, and its correct definition is:
$H^{1,2}(\Omega)$ *is the quotient space* $\tilde{H}^{1,2}/N$ where $\tilde{H}^{1,2}$ is the set of Cauchy-sequences $(u_j)_{j=1}^\infty$, $u_j \in C(\bar{\Omega}) \cap C^2(\Omega)\,,\ \int\limits_\Omega |\nabla u_j|^2\,dx + \int\limits_\Omega |u_j|^2\,dx < \infty\,,\ \|u_j - u_k\|_{1,2} \to 0$ if $i, k \to \infty$, and where N is the set of elements $(u_j)_{j=1}^\infty$ of $\tilde{H}^{1,2}$ such that $\|u_j\|_{1,2} \to 0$.

Since the elements of $H^{1,2}(\Omega)$ are classes of Cauchy-sequences we have to ask whether the variational integral $\int\limits_\Omega |\nabla u|^2\,dx$ can be defined in a reasonable way if $u \in H^{1,2}$. Indeed this is possible: If $u \in H^{1,2}$, then there is a Cauchy-sequence $(u_j) \in u\,,\ u_j \in C(\bar{\Omega}) \cap C^2(\bar{\Omega})\,,$ $\|u_j - u_k\|_{1,2} \to 0$.
Since ∇u_j is a Cauchy-sequence in L^2, the numbers $\int\limits_\Omega |\nabla u_j|^2\,dx$ converge to a limit, which we define as the value of $\int\limits_\Omega |\nabla u|^2\,dx$. It is easy to see that this definition of $\int\limits_\Omega |\nabla u|^2\,dx$ does not

depend on the above choice of $(u_j) \in u$. Since ∇u_j is a Cauchy-sequence in L^2 we can associate to each $u \in H^{1,2}$ a *generalized gradient* $\nabla u \in L^2$ which is defined up to a set of measure zero. Similarly, to every $u \in H^{1,2}$ one can associate a function $u \in L^2$ which is defined up to a set of measure zero. Interestingly one can say even more, namely one can prove that to each $u \in H^{1,2}$ one can associate a function which is defined up to a set of capacity zero. The capacity of a set E of points $\in \mathbb{R}^n$ is defined in the following way: Let Q and Q' be cubes $\subset \mathbb{R}^n$ such that Q is contained in the interior of Q'. Let E be a *closed* set $\subset Q$. Then the *capacity* of E is defined by

$$\operatorname{cap} E = \inf \Big\{ \int_{Q'} |\nabla \varphi|^2 \, dx | \varphi \in C_0^\infty(Q') \,, \ \varphi \geq 1 \text{ on } E \Big\} .$$

If E is an arbitrary set contained in Q i.e. not necessarily closed, then we define the *(inner) capacity* by

$$\operatorname{cap} E = \sup\{\operatorname{cap} K | K \subset E \,, \ K \text{ is closed } \} .$$

One can define the *p-capacity* of E by

$$p - \operatorname{cap} E = \inf \Big\{ \int_{Q'} |\nabla \varphi|^p \, dx| \ \varphi \in C_0^\infty(Q') \,, \ \varphi \geq 1 \text{ on } E \Big\}$$

if E is closed and extend the definition to arbitrary E in the above way.
Note that a point in $\mathbb{R}^n$, $n \geq 2$, has 2-capacity zero.
If $u \in H^{1,2}$, then every Cauchy-sequence $(u_j) \in u$ has the property, that it converges to a function $\tilde{u}$ up to a set of capacity zero. A different representative of u may converge to another function $\bar{u}$ up to a set of capacity zero, and $\tilde{u}$ and $\bar{u}$ will differ at most at a set of capacity zero. So, the space $H^{1,2}$ may be considered as equivalence classes of square-integrable functions which differ only on sets of capacity zero and to which one can assign generalized derivatives $\in L^2$ in the preceding sense.
Note that a "function" $u \in H^{1,2}$ "forgets" sets of capacity zero. This is another reason why we could not prove the existence of a minimum in our example at the beginning. We had prescribed boundary data at an isolated point - which has capacity zero - and since $H^{1,2}$ is the natural class of admissible functions - the boundary data may be violated in sets of capacity zero.

In order to preserve the boundary data, we need also the space $H_0^{1,2}$ which we define as the closure of the space $C_0^\infty(\Omega)$ of testfunctions with respect to $\|.\|_{1,2}$. Clearly, $H_0^{1,2} \subset H^{1,2}$ and one can prove that $H_0^{1,2}$ *consists of all* $H^{1,2}$-functions *which vanish on* $\partial\Omega$ *up to a set of capacity zero and vice versa.* This seems to us the best way to understand why one uses the space $H_0^{1,2}$ to express generalized zero boundary conditions.
So, the linear manifold $g + H_0^{1,2}$ where $g \in H^{1,2}$ (or, say, $g \in C^1(\bar{\Omega})$) consists of all functions which are equal to g at the boundary up to a set of capacity zero and this is our set $\mathbb{K}$. $H^{1,2}$ can be equipped with the inner product

$$(u, v)_1 = \int_\Omega \bar{u}\bar{v} \, dx + \int_\Omega \overline{\nabla u} \; \overline{\nabla v} \, dx \text{ for } u, v \in H^{1,2}$$

where $\bar{u}, \bar{v}$ and $\overline{\nabla u}\ \overline{\nabla v}$ are the representatives of $u, v, \nabla u, \nabla v$ in the sense described.

$H^{1,2}$ is a Hilbert-space with respect to $(.,.)_1$. Since the variational integral is coercive with respect to the norm in $H^{1,2}$ and since bounded sets in $H^{1,2}$ are weakly compact we have found a space and a topology where the steps (B) and (C) of the direct methods of the calculus of variations work. It is a famous theorem that step (D) works as well, i.e. the *variational integral is lower semi-continuous in the weak topology of* $H^{1,2}$. The weak convergence $u_j \rightharpoonup u$ in $H^{1,2}$ is defined by $(u_j, w)_{1,2} \to (u, w)_{1,2}$ for all $w \in H^{1,2}(j \to \infty)$.

Theorem 1.1 (Lower-semi-continuity theorem)**.** *Let* $u_j \in H^{1,2}(\Omega)$ *and* $u_j \rightharpoonup u$ *weakly in* $H^{1,2}(\Omega)(j \to \infty)$. *Then*

$$\int_\Omega |\nabla u|^2\,dx \le \liminf_{j\to\infty} \int_\Omega |\nabla u_j|^2\,dx =: \mathrm{LINF}\ .$$

Proof. We may select a subsequence which we still denote by (u_j) such that

$$\int_\Omega |\nabla u_j|^2\,dx \to \mathrm{LINF} \qquad\qquad (j \to \infty)\,.$$

Then for every $\varepsilon > 0$ there is a $j(\varepsilon)$ such that

$$\int_\Omega |\nabla u_j|^2\,dx \le \mathrm{LINF} + \varepsilon\,, \qquad\qquad j \ge j(\varepsilon)\,.$$

By the theorem of *Banach-Saks* there is a subsequence (u_{j_i}), $j_i \ge j(\varepsilon)$ such that the arithmetic means

$$\frac{1}{K}\sum_{i=1}^{K} u_{j_i} \to u \text{ strongly in } H^{1,2}\,, \qquad (K \to \infty)\,.$$

Since quadratic functions are convex

$$\int_\Omega \Big|\frac{1}{K}\sum_{i=1}^{K} \nabla u_{j_i}\Big|^2\,dx \le \frac{1}{K}\sum_{i=1}^{K}\int_\Omega \Big|\nabla u_{j_i}\Big|^2\,dx \le \mathrm{LINF} + \varepsilon\,,\ j_i \ge j(\varepsilon)\,.$$

From Fatou's lemma we obtain

$$\int_\Omega |\nabla u|^2\,dx \le \liminf_{j_i\to\infty} \int_\Omega \Big|\frac{1}{K}\sum_{i=1}^{K} \nabla u_{j_i}\Big|^2\,dx\,,$$

since $\frac{1}{K}\sum\limits_{i=1}^{K} \nabla u_{j_i} \to \nabla u$ in measure and the integrals are non negative. So, we conclude

$$\int_\Omega |\nabla u|^2\,dx \le \mathrm{LINF} + \varepsilon$$

and passing to the limit $\varepsilon \to 0$, we obtain theorem 1.1. □

Since we have established coerciveness and weak compactness (by the choice of the basic space), the direct methods of the calculus of variations give us

Theorem 1.2. *There is a function $u \in H_0^{1,2} + g\,,\ g \in H^{1,2}$ such that*

$$\int_\Omega |\nabla u|^2\,dx = \min\left\{\int_\Omega |\nabla w|^2\,dx | w \in H_0^{1,2} + g\right\}.$$

bf Remark: Uniqueness follows by strict convexity.

We want to extend this method to the case of the variational integral $\int_\Omega F(\nabla u)\,dx$ where F is continuous and convex and non negative. We first remark that it is more general to formulate this problem in the Sobolev space $H^{1,1}(\Omega)$, which is the closure of $C(\bar{\Omega}) \cap C^1(\Omega)$ with respect to the norm $\|u\|_{1,1} = \int_\Omega |u|\,dx + \int_\Omega |\nabla u|\,dx$. The reason is that the variational integral may be coercive in the $\|.\|_{1,1}$-norm but not in the $\|.\|_{1,2}$-norm. An example is the minimal-surface integral

$$\int_\Omega \sqrt{1 + |\nabla u|^2}\,dx\,,$$

which is convex and coercive with respect to the $\|.\|_{1,1}$-norm in $H_0^{1,1}$.
Since we did not impose any growth condition for F we have to allow $+\infty$ as an admissible value for the variational integral (observe $F \geq 0$). However, since bounded sets in $H^{1,1}$ are not necessarily weakly compact, we have to impose an additional condition on F which guarantees the weak compactness of minimizing sequences. This condition is given by the following

Lemma 1.1. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be continuous, non negative, and assume the condition*

$$\frac{F(\eta)}{|\eta|} \to \infty \ \textit{if}\ |\eta| \to \infty\,.$$

Then every sequence $(u_j) \subset H^{1,1}(\Omega)$ with the property $\int_\Omega F(\nabla u_j)\,dx \leq K\,,\ j = 1, 2, \ldots\,,$ has a subsequence such that (∇u_j) converges weakly in $L^1(\Omega)$.

Proof. By the above condition we can find an $L \in \mathbb{R}$ such that $\frac{F(\eta)}{|\eta|} \geq N$ for any given N provided that $|\eta| \geq L = L(N)$. Since $F \geq 0$ we conclude

$$K \geq \int_\Omega F\left(\nabla u_j\right)\,dx \geq \int_{\{x\in\Omega|\,|\nabla u_j|\geq L\}} F\left(\nabla u_j\right)\,dx \geq N \int_{\{x|\,|\nabla u_j|\geq L\}} \left|\nabla u_j\right|\,dx\,.$$

If ε is given and we choose N such that $\frac{K}{N} < \frac{\varepsilon}{2}$ we obtain

$$\int_{\{x|\,|\nabla u_j|\geq L\}} \left|\nabla u_j\right|\,dx < \frac{\varepsilon}{2}\,.$$

Now, choose $\delta > 0$ such that $\delta < \frac{\varepsilon}{2L}$. Then for every measurable set $e \subset \Omega$ with the property $\mu(e)$ (= measure of e) $\leq \delta$ one has

$$\int_e \left|\nabla u_j\right| dx \leq \int_{e\cap\{|\nabla u_j|\geq L\}} |\nabla u_j|\, dx + \int_{e\cap\{|\nabla u_j|\leq L\}} |\nabla u_j|\, dx < \frac{\varepsilon}{2} + L\mu(e) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

uniformly for j.

This es exactly the well known criterium for weak compactness in L^1.
Cf. Dunford-Schwartz, Linear Operators I. □

With this tool we can prove

Theorem 1.3. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be continuous, convex and non negative. Then the integral $\int_\Omega F(\nabla u)\, dx$ is lower semi-continuous with respect to the weak topology of $H^{1,1}(\Omega)$, i.e. if $u_j \rightharpoonup u$ weakly in $H^{1,1}$, then $\int_\Omega F(\nabla u)\, dx \leq \liminf_{j\to\infty} \int_\Omega F(\nabla u_j)\, dx$.*

Proof. We select a subsequence still denoted by (u_j) such that

$$\int_\Omega F(\nabla u_j)\, dx \to \liminf_{k\to\infty} \int_\Omega F(\nabla u_k)\, dx =: \liminf \qquad (j \to \infty)\,.$$

Thus we have for given $\varepsilon > 0$

$$\int_\Omega F\left(\nabla u_j\right)\, dx \leq \lim\ \inf + \varepsilon\,, \qquad j \geq j(\varepsilon)\,.$$

By the theorem of Alaoglu-Bourbaki there is a subsequence of convex-linear combinations of ∇u_j which converge strongly, i.e., there exist numbers $C_i^K \geq 0$ such that $\sum_{i=1}^K C_i^K = 1$ and indices $j_{iK} \geq j(\varepsilon)$ such that

$$\sum_{i=1}^K C_i^K \nabla u_{j_{iK}} \to \nabla u \text{ strongly in } L^1(\Omega) \qquad (K \to \infty)\,.$$

By Fatou's lemma

$$\int_\Omega F(\nabla u)\, dx \leq \liminf_{K\to\infty} \int_\Omega F\left(\sum_{i=1}^K C_i^K \nabla u_{j_{iK}}\right)\, dx$$

and by the convexity of F

$$\int_\Omega F\left(\sum_{i=1}^K C_i^K \nabla u_{j_{iK}}\right)\, dx \leq \sum_{i=1}^K C_i^K \int_\Omega F\left(\nabla u_{iK}\right)\, dx \leq \sum_{i=1}^K C_i^K (\liminf + \varepsilon) =$$

$$= \liminf + \varepsilon\,.$$

Thus we obtain

$$\int_\Omega F(\nabla u)\,dx \le \liminf + \varepsilon$$

and the result follows as $\varepsilon \to 0$.

Combining Lemma 1.1 and Theorem 1.3 we obtain

Theorem 1.4. *Let $g \in H^{1,1}(\Omega)$, let $F : \mathbb{R}^n \to \mathbb{R}$, $F \ge 0$, be continuous and convex and assume that*

$$\frac{F(\eta)}{|\eta|} \to \infty \qquad (|\eta| \to \infty)\,.$$

Then there is a minimum of the variational integral

$$\int_\Omega F(\nabla u)\,dx$$

in the class $g + H_0^{1,1}(\Omega)$.

Proof. We use the direct methods of the calculus of variations.
Suppose that $\int_\Omega F(\nabla u)\,dx < \infty$ for at least one $u \in g + H_0^{1,1}(\Omega)$. Let $u_j \in g + H_0^{1,1}(\Omega)$ be a minimizing sequence.
By lemma 1.1, the sequence ∇u_j is weakly compact and there is a subsequence u_{j_i} such that ∇u_{j_i} converges weakly. By Rellich's theorem, which we discuss in detail in the next section, the functions u_{j_i} themselves converge strongly in L^1. (Note, that $u - g \in H_0^{1,1}(\Omega)$.) Thus $u_{j_i} \rightharpoonup u$ weakly in $H^{1,1}$ and by the lower semi continuity theorem we obtain that u is a minimum of the variational integral.

Note that this theorem is not applicable to the minimal-surface case since then the condition $\frac{F(\eta)}{|\eta|} \to \infty$ ($|\eta| \to \infty$) is violated. In fact, it is known that the minimal-surface integral may have *no* minimum if the domain is not convex.

We finally remark that the techniques discribed can be applied to the following problem:
Find $u \in g + H_0^{1,2}(\Omega)$ such that $u \ge \psi$ (ψ given) in Ω and

$$\int_\Omega |\nabla u|^2\,dx = \min!$$

However, one has to understand the inequality $u \ge \psi$ in the "capacity-sense".

2. The direct methods in the calculus of variations. Continuation.

In this section we want to prove the existence of minima of variational integrals $\int_\Omega F(x,u,\nabla u)\,dx$ where F is convex in ∇u but not necessarily convex in u. Note that one *cannot* expect lower semi-continuity for integrals of the type $\int_\Omega F(x,\nabla u)\,dx$ where f is *not* convex in ∇u. If u is a scalar function, then lower semi-continuity in the weak topology of $H^{1,1}$ *implies that* F is *convex in* ∇u. In fact, the lower semi-continuity theorem of section 1 is already remarkable in view of the theorem that the Nemytski-Operator

$$(Tw)(x) = F(x,w(x))\,, \qquad T: L^1 \to L^p\,,$$

is *necessarily linear* if T is continuous from the weak topology of L^1 into the weak topology of L^p. Thus, it is clear that we cannot obtain a lower semi-continuity theorem for $\int F(x,u,\nabla u)\,dx$ when f is not necessarily convex in u if we *only know that the sequence* $(u_j)_{j=1}^\infty$ for which we want to show the lower semi-continuity relation *converges weakly.* We need a theorem which says that a sequence of functions u_j converging weakly in $H^{1,1}(\Omega)$ converges strongly in $L^1(\Omega)$. Such a theorem is indeed true if one imposes some mild restrictions on the boundary $\partial\Omega$; this theorem is called *Rellich's theorem.* (In fact, for lower semi-continuity we need only convergence of u_j in measure and so we need no restriction on $\partial\Omega$.)
We state Rellich's theorem here for the special case of $H_0^{1,p}(\Omega)$-functions:

Theorem 2.1. *Let $(u_j)_{j=1}^\infty$ be a sequence in $H_0^{1,p}(\Omega)$, $1 \le p < \infty$, such that $u_j \rightharpoonup u$ weakly in $H^{1,p}(\Omega)$. Then*

$$u_j \to u \text{ strongly in } L^p(\Omega)\,.$$

We will not prove this here. One possible proof uses the compactness criterium in L^p, which says that a sequence $(u_j)_{j=1}^\infty$ in L^p is strongly sequentially compact if the translation operators E_h defined by $E_h w(x) = w(x+h)$ are uniformly continuous on the sequence (u_j) as $h \to \infty$. (This can be proved by representing u_j by its derivative.) Using the uniform continuity of E_h, one can prove that the sequence $(u_j)_{i=1}^\infty$ has a finite ε-net for any ε. The compactness then follows from a Lemma of Hausdorff.
Another way of proving Rellich's lemma is to prove it first in $H^{1,2}$ via Fourier-series and then to extend it to $H^{1,p}$, $p>2$, by deducing from the $H^{1,2}$-case that u_j converges in measure and from Sobolev's inequality that (u_j) is uniformly bounded in $L^{p+\varepsilon}$. The L^p-convergence of u_j then follows from Vitali's theorem.

We remark that the following interesting question is open.
Conjecture (Stampacchia). *Let $u_j \rightharpoonup u$ weakly in $H^{1,p}$. Then for every $\varepsilon > 0$ there is a set E_ε of p-capacity $p\text{-cap}E_\varepsilon < \varepsilon$ such that $u_j \to u$ uniformly $(j \to \infty)$* (possibly only for a subsequence) *in $\Omega - E_\varepsilon$.*

At the present moment we only know (by Egoroff's theorem) that this holds with an exception set E_ε having Lebesgue-measure $\mu(E_\varepsilon) < \varepsilon$.

For our lower semi-continuity "project", we start with the simple example

$$J(u) = \int_\Omega F(\nabla u)\,dx + \int_\Omega F_0(u)\,dx\,,$$

where F is continuous and convex, F_0 merely continuous, and both F and F_0 nonnegative. Then J is lower semi-continuous with respect to the weak topology of $H^{1,1}$.

In fact, the sum of lower semi-continuous functions is lower semi-continuous, and since we already know that $\int F(\nabla u)\,dx$ is lower semi-continuous, we have to show that $\int F_0(u)\,dx$ is lower semi-continuous in the weak topology of $H^{1,1}$. But this is true since by Rellich's theorem $u_j \to u$ in measure. Thus $F_0(u_j) \to F_0(u)$ in measure since F_0 is continuous, and by Fatou's theorem and the non-negativity of F_0

$$\int_\Omega F_0(u)\,dx \le \liminf_{j\to\infty} \int_\Omega F_0(u_j)\,dx\,.$$

We intend to extend this idea to the general variational integral $\int_\Omega F(x,u,\nabla u)\,dx$. This leads to the following

Theorem 2.2. *Let $F : \bar\Omega \times \mathbb{R}^{1+n} \to \mathbb{R}$ be continuous, non negative and let $F(x,u,\eta)$ be convex in $\eta \in \mathbb{R}^n$. Then the integral*

$$J(u) = \int_\Omega F(x,u,\nabla u)\,dx$$

is lower semi-continuous with respect to the weak topology of $H^{1,1}(\Omega)$.

Proof. We shall prove a slight generalization: If $u_j \to u$ in measure and $p_j \rightharpoonup p$ weakly in $L^1 (j \to \infty)$, where the p_j are L^1-functions having n components, then

$$\int_\Omega F(x,u,p)\,dx \le \liminf_{j\to\infty} \int_\Omega F(x,u_j,p_j)\,dx\,. \tag{2.1}$$

Note that the lim inf on the right hand side of the last inequality may be ∞.
Then there is nothing to prove.
The theorem will be proved in several steps:
(i) Let u be measurable and $p_m \rightharpoonup p$ weakly in $L^1 (j \to \infty)$. Then

$$\int_\Omega F(x,u,p)\,dx \le \liminf_{m\to\infty} \int_\Omega F(x,u,p_m)\,dx\,.$$

This is proved with the same method as in section 1 using the theorem of Alaoglu-Bourbaki and Fatou's lemma. One uses the fact that $F(x,u,p)$ is convex in p.

(ii) Let $u_m \to u$ uniformly[1] and $p_m \rightharpoonup p$ weakly in L^1 and $\|P_m\|_\infty \le C$ uniformly. Then (2.1) holds.

[1]such that $u \in L^\infty$

Let $\Lambda \subset \{1,2,\dots\}$ be a subsequence such that

$$\int\limits_\Omega F(x,u_j,p_j)\,dx \to \liminf_{m\to\infty} \int\limits_\Omega F(x,u_m,p_m)\,dx =: \liminf\,. \tag{2.2}$$

By (i)

$$\int\limits_\Omega F(x,u,p)\,dx \le \liminf_{\substack{m\to\infty\\ m\in\Lambda}} \int\limits_\Omega F(x,u,p_m)\,dx\,. \tag{2.3}$$

Since F is continuous, $\|p_m\|_\infty \le C$ uniformly and $u_m \to u$ uniformly we have

$$F(x,u_m,p_m) - F(x,u,p_m) \to 0 \qquad (m\to\infty)$$

pointwise. Since $F(x,u_m,p_m)$, $F(x,u,p_m)$ are uniformly bounded, we have by the Lebesgue theorem of dominated convergence

$$\Big|\int F(x,u_m,p_m)\,dx - \int F(x,u,p_m)\,dx\Big| < \varepsilon$$

for $m \ge m(\varepsilon)$.
Together with (2.3) we obtain

$$\int\limits_\Omega F(x,u,p)\,dx \le \liminf_{\substack{m\to\infty\\ m\in\Lambda}} \int\limits_\Omega F(x,u_m,p_m)\,dx + \varepsilon\,,$$

and by (2.2)

$$\int\limits_\Omega F(x,u,p)\,dx \le \liminf + \varepsilon\,.$$

The statement follows as $\varepsilon \to 0$.

(iii) Let $u_m \to u$ uniformly, $u \in L^\infty$, and $p_m \rightharpoonup p$ weakly in L^1. Then (2.1) holds.

First we select a subsequence Λ such that (2.2) holds. Then we define for $L \in \mathbb{R}$, $L > 0$

$$p_m^L(x) = \begin{cases} p_m\ (x) & \text{if } \ |p_m(x)| \le L\,,\\ 0 & \text{if } \ |p_m(x)| > L\,,\end{cases}$$

Since p_m^L is uniformly bounded in $L^2(\Omega)$ for fixed L (recall Ω is bounded), there is a subsequence (may depend on L) such that

$$p_m^L \rightharpoonup p^L \qquad (m\to\infty\,,\ m\in\Lambda_L \subset \Lambda)$$

weakly in $L^2(\Omega)$ and thus in $L^1(\Omega)$.

Since $\|p_m^L\|_\infty \le L$ we may apply (ii) and obtain

$$\int\limits_\Omega F(x,u,p^L)\,dx \le \liminf_{\substack{m\to\infty\\ m\in\Lambda_L}} \int\limits_\Omega F(x,u_m,p_m^L)\,dx\,. \tag{2.4}$$

Now we observe that

$$\int_\Omega F\left(x, u_m, p_m^L\right)\, dx = \int_{\{x||p_m(x)|\le L\}} F\left(x, u_m, p_m\right)\, dx +$$

$$+ \int_{\{x||p_m(x)|>L\}} F\left(x, u_m, 0\right)\, dx \underset{\text{since } F\ge 0}{\le}$$

$$\le \int_\Omega F\left(x, u_m, p_m\right)\, dx \quad + \int_{\{x||p_m(x)|\le L\}} F\left(x, u_m, p_m\right)\, dx\,.$$

Since $\|u_m\|_\infty \le K$ uniformly and F is continuous, we have that $F(x, u_m, 0)$ is uniformly bounded by a constant K'. Furthermore, for $L \ge L(\varepsilon)$ we have that $\mu\{x||p_m(x)| > L\} < \varepsilon$ ($\mu =$ Lebesgue measure). We prove this below. So, we arrive at the inequality

$$\int_\Omega F\left(x, u_m, p_m^L\right)\, dx \le \int_\Omega F\left(x, u_m, p_m\right)\, dx + \varepsilon\,,\ L \ge L(\varepsilon)\,,$$

and by (2.2) and (2.4) we obtain

$$\int_\Omega F\left(x, u, p^L\right)\, dx \le \liminf +\varepsilon\,,\ L \ge L(\varepsilon)\,. \tag{2.5}$$

We shall show that $p^L \to p$ strongly in L^1, $L \to \infty$. Then by Fatou's lemma we obtain

$$\liminf_{L\to\infty} \int_\Omega F\left(x, u, p^L\right)\, dx \ge \int_\Omega F(x, u, p)\, dx$$

and

$$\int_\Omega F(x, u, p)\, dx \le \liminf +\varepsilon\,.$$

The statement follows as $\varepsilon \to 0$.

For step (iii) it remains to show that $\mu\{x||p_m(x)| > L\} < \varepsilon/K'$ and $p^L \to p$ strongly in L^1. The first inequality follows from the inequalities

$$K'' \ge \int_\Omega |p_m(x)|\, dx \ge \int_{\{x|\,|p_m(x)|>L\}} p_m(x)\, dx \ge L\mu\{x|\,|p_m(x)| > L\}$$

where the first inequality holds because "weak convergence implies uniform boundedness" and thus

$$\mu\{x|\,|p_m(x)| > L\} \le \frac{K''}{L} < \varepsilon' \text{ for } L > L(\varepsilon')\,.$$

For the other statement we use the (equivalent) criterium for weak compactness in L^1, i.e.: For every $\varepsilon > 0$ there is a $\delta > 0$ such that $\int_e |p_m(x)|\, dx < \varepsilon$ for all $e \subset \Omega$ such that $\mu(e) < \delta$, uniformly for m.

Since $\mu\{x|\,|p_m(x)| > L\} \le \frac{K''}{L} < \varepsilon'$ for $L > L(\varepsilon')$ we obtain that

$$\int\limits_{\{x|\,|p_m(x)|>L\}} |p_m(x)|\,dx < \varepsilon \qquad \text{for} \qquad L > L(\varepsilon)\,. \tag{2.6}$$

Now, for all $\varphi \in L^\infty$ with $\|\varphi\|_\infty \le 1$ we have

$$\Big|\int\limits_\Omega \left(p_m - p_m^L\right)\varphi\,dx\Big| \le \int\limits_\Omega \left|p_m - p_m^L\right| dx =$$

$$= \int\limits_{\{x|\,|p_m(x)|>L\}} |p_m|\,dx < \varepsilon \qquad \text{for} \qquad L > L(\varepsilon)\,.$$

Passing to the limit as $m \to \infty$ we obtain

$$\int\limits_\Omega \left(p - p^L\right)\varphi\,dx < \varepsilon\,, \qquad L > L(\varepsilon)\,.$$

Passing to the $\sup\limits_{\|\varphi\|_\infty \le 1}$ in the left hand side of the last inequality, we obtain

$$\int\limits_\Omega |p - p^L|\,dx < \varepsilon\,, \qquad L > L(\varepsilon)\,,$$

i.e. $p^L \to p$ strongly in L^1.

(iv)(Proof of the theorem:) Choose a subsequence Λ such that (2.2) holds. Choose $\varepsilon > 0$. By Egoroff's theorem there is a subsequence $\Lambda' \subset \Lambda$ and a set $E_\varepsilon \subset \Omega$ with $\mu(E_\varepsilon) < \varepsilon$ such that $u_m \to u$ uniformly on $\Omega - E_\varepsilon$.

By (iii)

$$\int\limits_{\Omega - E_\varepsilon} F(x,u,p)\,dx \le \liminf_{\substack{m\to\infty\\ m\in\Lambda'}} \int\limits_{\Omega - E_\varepsilon} F(x,u_m,p_m)\,dx \text{ since (2.2) holds}$$

$$\underset{\text{since} F \ge}{\le} \liminf_{\substack{m\to\infty\\ m\in\Lambda'}} \int\limits_\Omega F(x,u_m,p_m)\,dx =$$

$$= \liminf\,.$$

The theorem follows as $\varepsilon \to 0$. □

Now, being the proud owner of a lower semi-continuity theorem, we can state the following existence theorem:

Theorem 2.3. *Let $F : \bar\Omega \times \mathbb{R}^{1+n} \to \mathbb{R}$ be continuous, non negative and let $F(x,u,\eta)$ be convex in $\eta \in \mathbb{R}^n$. Assume that*

$$\frac{F(x,u,\eta)}{|\eta|} \to \infty \text{ for } |\eta| \to \infty\,, \qquad \eta \in \mathbb{R}^n\,,$$

uniformly in $u \in \mathbb{R}$. Then there is a minimum of the variational integral

$$\int_\Omega F(x, u, \nabla u)\, dx$$

in the class $\mathbb{K} = \left\{ v \in g + H_0^{1,1}(\Omega \right\}$, with a given function $g \in H^{1,1}(\Omega)$.

Proof. Let $(u_m)_{m=1}^\infty$ be a minimizing sequence. As in the proof of Lemma 1.1 we obtain that $(u_m)_{m=1}^\infty$ is weakly sequentially compact in $H^{1,1}$ and there is a subsequence $(u_{m_i})_{i=1}^\infty$ converging weakly to u in $H^{1,1}$. By theorem 2.2, one has

$$\int_\Omega F(x, u, \nabla u)\, dx \le \liminf_{i \to \infty} \int_\Omega F\left(x, u_{m_i}, \nabla u_{m_i}\right)\, dx = \inf$$

and the theorem is proved. □

Several generalizations are possible. Using a sharper form of Sobolev's imbedding theorem[2] we obtain the theorem

Theorem 2.4. *Let $F : \bar{\Omega} \times \mathbb{R}^{1+n} \to \mathbb{R}$ be continuous and let $p > 1$ and*

$$\int_\Omega F(x, u, \nabla u)\, dx \to \infty \text{ as } \|u\|_{1,p} \to \infty\,, \qquad u \in g + H_0^{1,p}\,.$$

Furthermore, let K, s be constants such that

$$F(x, u, \eta) \ge -K - K|u|^s\,, \qquad s < \frac{np}{n-p} \text{ if } p < n$$

and s arbitrary if $p \ge n$.
Then there is a minimum of the variational integral $\int_\Omega F(x, u, \nabla u)\, dx$ in the class $g + H_0^{1,p}$.

If $p > n$ one can replace the condition $F(x, u, \eta) \ge -K - K|u|^s$ by

$$F(x, u, \eta) \ge -K - f(u)\,,$$

where f is any continuous function.
Theorem 2.4 relies on the fact that the coerciveness condition in L^p gives minimizing sequences which are weakly compact in $H^{1,p}$, $p > 1$.
By theorem 2.2 the integral

$$\int [F(x, u, \nabla u) + K + K|u|^s]\, dx$$

is lower semi-continuous since its integrand is non negative, and the lower semi-continuity of $\int_\Omega F(x, u, \nabla u)\, dx$ follows since the perturbation $\int [K + K|u|^s]\, dx$ is contiuous in the weak topology because of Sobolev's theorem.

[2]which states that bounded sets in $H^{1,p}$, $p < n$ are compact in L^s, $s < \frac{np}{n-p}$

Note that it is not hard to remove the condition that $F(x,u,\eta)$ is continuous with respect to x and to replace it by measurability (use Lusin and the non-negativity of F!).

Open problems:

One might expect lower semi-continuity if $F(x,u,\eta)$ is measurable in x, lower semi-continuous in (u,η) with respect to the $\mathbb{R}^n$-topology and convex in η. (If F does not depend on u, this can be proved easily with our methods.)

The main open problems arise in the case when u is an r-vector-function since then it is no longer natural to assume convexity of $F(x,u,\nabla u)$ in ∇u. The natural condition is the so called *Legendre-Hadamard-condition*, which we state in the case when F is twice differentiable. In the scalar case then the convexity of F is equivalent to the positive semi-definiteness of the matrix $(F_{ik}(x,u,\eta))$ where

$$F_{ik}(x,u,\eta) = \frac{\partial}{\partial \eta_i}\frac{\partial}{\partial \eta_k} F(x,u,\eta)\,.$$

In the non-scalar case the corresponding condition is the *Legendre-Hadamard-condition*:

$$\sum_{i,k=1}^{n}\sum_{\nu,\mu=1}^{r} F_{ik}^{\nu\mu}(x,u,\eta)\lambda_i\lambda_k\xi_\nu\xi_\mu \geq 0$$

for all λ_j , $j = 1,\ldots,n$, and all ξ_i , $i = 1,\ldots,r$.
Here $F_{ik}^{\nu\mu}$ is the second derivative with respect to the argument η_i^ν and η_k^μ which stand for the i-th resp. k-th derivative of the ν-th resp. μ-th component of u. One can prove that *the lower-semi-continuity of* $\int F(x,u,\nabla u)\,dx$ (with respect to weak $H^{1,1}$-convergence) implies the *Legendre-Hadamard-condition*!

The converse result - i.e. that the Legendre-Hadamard-condition implies lower semi-continuity in the weak topology of $H^{1,1}$ - has been proved for important cases but the questions have not been solved completely.
A complete discussion of the known results can be found in Morrey's book "Multiple Integrals in the Calculus of Variations", Springer 1966.

Euler's equation

We conclude this section with the proof that the solutions to variational problems are weak solutions to Euler's equation - so our existence theorems also give us theorems for showing that a certain class of partial differential equations, namely those arising from variational problems, have weak solutions.

Theorem 2.5. *Let* $F : \bar{\Omega}\times\mathbb{R}^{1+n} \to \mathbb{R}$ *be continuous and suppose that* $F(x,u,\eta)$ *is continuously differentiable with respect to* $u \in \mathbb{R}$ *and* $\eta \in \mathbb{R}^n$ *and that there exist constants* $K > 0$ *and* $p \in [1,\infty[$ *such that*

$$\begin{aligned} |F(x,u,\eta)| &\leq K\big(1+|u|^p + |\eta|^p\big)\,,\\ |F_u(x,u,\eta)| &\leq K\big(1+|u|^p + |\eta|^p\big)\,,\\ |F_\eta(x,u,\eta)| &\leq K\big(1+|u|^p + |\eta|^p\big)\,. \end{aligned} \tag{2.7}$$

Then every minimum $u \in g + H_0^{1,p}(\Omega)$ is a weak solution to Euler's equation, namely for all $\varphi \in C_0^\infty(\Omega)$ the equality

$$\int_\Omega \sum_{i=0}^n F_i(x,u,\nabla u)\partial_i\varphi\,dx = 0$$

holds, where $F_0 = F_u$, $(F_1,\dots,F_n) = F_\eta$, $\partial_0 =$ identity.

Proof. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(t) := \int_\Omega F(x, u+t\varphi\,,\ \nabla u + t\nabla\varphi)\,dx\,.$$

Since $u + t\varphi \in H^{1,p}$, and since $|F(x,u,\eta)| \le K(1+|u|^p+|\eta|^p)$, g is defined. We want to prove that g is differentiable. For this purpose, we consider the difference quotient

$$\frac{g(t+h)-g(t)}{h} = \int_\Omega \frac{1}{h} F(x, u+(t+h)\varphi\,,\ \nabla u + (t+h)\nabla\varphi) - \\ - F(x,u+t\varphi\,,\ \nabla u + t\nabla\varphi)\,dx\,.$$

Since $F(x,u,\eta)$ is continuously differentiable with respect to (u,η) we have

$$\frac{1}{h}[F(x,u+(t+h)\varphi\,,\ \nabla u+(t+h)\nabla\varphi) - F(x,u+t\varphi\,,\ \nabla u+t\nabla\varphi)] \\ \longrightarrow \sum_{i=0}^n F_i(x,u+t\varphi\,,\ \nabla u+t\nabla\varphi)\partial_i\varphi \qquad (h\to 0)$$

pointwise almost everywhere. We want to show that one may interchange the pointwise limit and the integration over Ω. For this, we write

$$\mathfrak{F}_h(x) := \frac{1}{h}[F(x,u+(t+h)\varphi\,,\ \nabla u+(t+h)\nabla\varphi) - F(x,u+t\varphi\,,\ \nabla u+t\nabla\varphi)] \\ = \int_0^h \sum_{i=0}^n F_i(x,u+(t+\tau)\varphi\,,\ \nabla u+(t+\tau)\nabla\varphi)\cdot\partial_i\varphi\,d\tau\,.$$

For any measurable subset we obtain via Fubini's theorem

$$\int_\Omega \mathfrak{F}_h\,dx = \frac{1}{h}\int_0^h\int_\Omega \sum_{i=0}^n F_i(x,u+(t+\tau)\varphi\,,\ \nabla u+(t+\tau)\nabla\varphi)\cdot\partial_i\varphi\,dx\,d\tau\,.$$

The integral exists since $u+(t+\tau)\varphi \in H^{1,p}$ and (2.7) holds. By (2.7)

$$|F_i(x,u+(t+\tau)\varphi\,,\ \nabla u+(t+\tau)\nabla\varphi)\partial_i\varphi| \le K_\varphi\,(1+|u|^p+|\nabla u|^p)$$

uniformly for $t,\tau\in[0,1]$.

Since $\mathfrak{F}_h \to \sum_{i=0}^{n} F_i(.,u+t\varphi\,,\ \nabla u + t\nabla\varphi)\partial_i\varphi$ pointwise a.e. we obtain by Lebesgue's theorem

$$\int_\Omega \mathfrak{F}_h\,dx \to \int_\Omega \sum_{i=0}^{n} F_i(.,u+t\varphi\,,\ \nabla u + t\nabla\varphi)\partial_i\varphi\,dx \qquad (h\to 0)$$

and we have proved that $g'(t)$ exists and that

$$g'(t) = \int_\Omega \sum_{i=0}^{n} F_i(.,u+t\varphi\,,\ \nabla u + t\nabla\varphi)\partial_i\varphi\,dx\,.$$

Since u is a minimum of the variational integral and since $u+\varphi \in g + H_0^{1,p}(\Omega)$ we conclude that g has a minimum for $t=0$. Thus $g'(0)=0$ and this means that Euler's equation

$$\int_\Omega \sum_{i=0}^{n} F_i(.,u,\nabla u)\partial_i\varphi\,dx = 0\,, \qquad \varphi\in C_0^\infty(\Omega)$$

is satisfied. □

If we knew that $F_i(.,u,\nabla u)\in C^1(\Omega)$, we could perform a partial integration and obtain

$$\int_\Omega \left[-\sum_{i=0}^{n}\partial_i F_i(.,u,\nabla u) + F_0(.,u,\nabla u)\right]\varphi\,dx = 0\,, \qquad \varphi\in C_0^\infty(\Omega)\,,$$

and since this holds for all φ, we would obtain

$$-\sum_{i=0}^{n}\partial_i F_i(.,u,\nabla u) + F_0(.,u,\nabla u) = 0 \text{ in } \Omega\,.$$

This means that u satisfies Euler's partial differential equation in the *classical sense*. If one can prove that $u\in C(\bar\Omega)$ then u would attain the boundary values in the classical sense. However, it is a long way of proving this (and not always true).

The next sections will be devoted to these questions (i.e. the question of the regularity of weak solutions).

3. The Hölder continuity for the minima of $\int_\Omega F(x,u,\nabla u)\,dx$ in the two dimensional case

In this section we go through the first step on the way to complete regularity of solutions to variational problems, namely the Hölder continuity of the minima of variational problems.
A function v is called *Hölder-continuous* if there exists an $\alpha \in]0,1[$ such that

$$\sup_{x\neq y} \frac{|v(x)-v(y)|}{|x-y|^\alpha} =: [v]_\alpha < \infty\,.$$

We need a classical tool due to Morrey with which one can show that a function v is Hölder continuous if a certain *growth condition for it's Dirichlet-integral* is satisfied:

Theorem 3.1 ("Morrey's Lemma"). *Let $u \in H^{1,p}(B_R(x_0))$, $1 \le p \le n$, and suppose that there are constants $\mu > 0$ and $L > 0$ such that*

$$\int_{B_r(x)} |\nabla u|^p\,dx \le L^p \left(\frac{r}{\delta}\right)^{n-p+\mu p}\,, \qquad 0 < r < \delta = R - |x-x_0| \tag{3.1}$$

for every ball $B_r(x)$, $x \in B_R(x_0)$. Then $u \in C[B_r(x_0)]$ for $r < R$ and

$$|u(\xi)-u(x)| \le \frac{4}{\mu} L \delta^{1-n/p-\mu} \Gamma_n^{-1/p} |\xi - x|^\mu\,, \qquad |\xi - x| \le \frac{\delta}{2}\,,$$

where Γ_n is the volume of the unit-ball.

Remark:

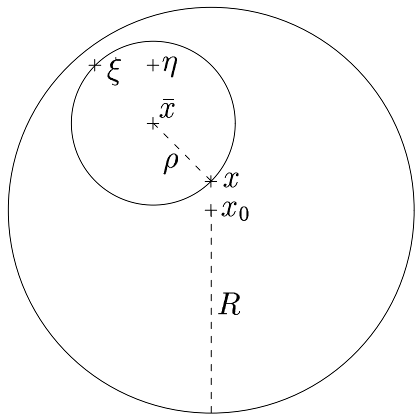
The essential statement is

$$|u(\xi)-u(x)| \le \text{ constant } |\xi - x|^\mu$$

i.e. the *Hölder-continuity for u.*

Condition 3.1 is called a *growth condition* for the integral $\int |\nabla u|^p\,dx$. It restricts the degree of singularity of ∇u.

Proof. It suffices to prove the theorem for C^1-functions.
If $u \in H^{1,p}$, then the mollification $w_h * u$ is defined for every ball $B_{R_1}(x_0)$, $R_1 < R$, if h is small enough and satisfies the same growth condition for the integral $\int |\nabla u|^p\,dx$. From the uniform estimate for $|w_h * u(\xi) - w_h * u(x)|$ we then obtain the theorem as $h \to 0$.
Thus let $u \in C^1(B_R(x_0))$.



Let $x, \xi \in B_R(x_0)$, $\rho = |\xi - x|/2$, $\bar{x} = (\xi + x)/2$ and $\eta \in B_\rho(\bar{x})$.

By the mean-value theorem (in integral-form) $|u(\xi) - u(\eta)| \leq \int_0^1 |\nabla u(\xi + t(\eta - \xi))| \cdot |\eta - \xi| \, dt$.

Note that $B_\rho(\bar{x}) \subset B_R(x_0)$ since $|\xi - x| \leq \delta/2$, $\delta = R - |x - x_0|$.
Since $|\eta - \xi| \leq 2\rho$ we conclude

$$|u(\xi) - u(\eta)| \leq 2\rho \int_0^1 |\nabla u(\xi + t(\eta - \xi))| \, dt \,.$$

Averaging over $B_\rho(\bar{x})$ with respect to the variable η we obtain

$$|u(\xi) - \bar{u}_{B_\rho(\bar{x})}| \leq 2\rho |B_\rho(\bar{x})|^{-1} \int_{B_\rho(\bar{x})} \left[\int_0^1 |\nabla u[\xi + t(\eta - \xi)]| \, dt \right] d\eta \,,$$

where $\bar{u}_{B_\rho(\bar{x})}$ denotes the meanvalue of u taken over $B_\rho(\bar{x})$.
We interchange the order of integration and set $y = \xi + t(\eta - \xi)$; then y ranges over $B_{t\rho}(\bar{x}_t)$, where $\bar{x}_t = (1 - t)\xi + t\bar{x}$. (Indeed, the center is obtained for $\eta = \bar{x}$ which gives the formula for $\bar{x}_t$; the radius $t\rho$ is obtained by taking half of the distance $|y(\xi) - y(x)|$.) Thus we obtain

$$|\bar{u}_{B_\rho(\bar{x})} - u(\xi)| \leq 2\rho |B_\rho(\bar{x})|^{-1} \int_0^1 \left[\int_{B_{t\rho}(\bar{x}_t)} |\nabla u(y)| \, dy \right] t^{-n} \, dt =: A \,.$$

Using Hölder's inequality and then the growth condition (3.1), we obtain

$$A \leq 2\rho|B_\rho|^{-1}\int_0^1 |B_{t\rho}|^{1-1/p} L(\rho t/\delta)^{n/p-1+\mu} t^{-n}\,dt =$$

$$= 2\rho|B_\rho|^{-1/p}\rho^{n/p-1+\mu} L\delta^{1-\mu-n/p}\int_0^1 t^{-1+\mu}\,dt =$$

$$= 2L\delta^{1-\mu-n/p}\Gamma_n^{-1/p}\rho^{\mu}\frac{1}{\mu}\,.$$

Using the same result for x instead of ξ we obtain an estimate for the modulus of continuity and thus the theorem. □

The theorem gives rise to the definition of the *Morrey space* $L^{p,\lambda}(\Omega)$, which consists of all $L^p(\Omega)$-functions such that the *Morrey-norm*

$$|||u|||_{p,\lambda} = \sup_{r;x\in\Omega}\left\{\left(r^{\lambda-n}\int_{B_r\cap\Omega}|u|^p\,dx\right)^{1/p}\Bigg|B_r = \text{ ball of radius } r \text{ and center } x\right\}$$

is finite. The Morrey spaces are also interesting since they have some "invariance-properties" which appear while solving $-\Delta u = f \in L^{p,\lambda}$.

We now present a technical lemma which gives a criterium that a function z satisfies a Morrey condition. The technique of the proof is called the *"hole-filling method"*; for the reason see the discussion below.

Lemma 3.1. *Let $z : \Omega \to \mathbb{R}^m$ be an L^2-function and suppose that for all concentric balls $B_R \subset \Omega$, $B_{2R} \subset \Omega$, the following condition holds:*

$$\int_{B_R}|z|^2\,dx \leq K\int_{B_{2R}-B_R}|z|^2\,dx + KR^\alpha \tag{3.2}$$

with constants K and $\alpha > 0$. Then there is a $\beta \in]0,1[$ and a K_0 such that

$$\int_{B_R(x)}|z|^2\,dx \leq K_0R^\beta\,, \qquad x\in\Omega_0\subset\subset\Omega\,, \qquad K_0 \text{ depending on } \Omega_0\,.$$

Proof. We add to both parts of inequality (3.2) the quantity $K\int_{B_R}|z|^2\,dx$, i.e. we "fill the hole" in the integraion $\int_{B_{2R}-B_R}$.
This yields

$$(1+K)\int_{B_R}|z|^2\,dx \leq K\int_{B_{2R}}|z|^2\,dx + KR^\alpha\,.$$

Dividing by $(1+K)$ and setting $\theta = \frac{K}{K+1}$ we obtain

$$\int_{B_R} |z|^2\,dx \le \theta \int_{B_{2R}} |z|^2\,dx + KR^\alpha\,.$$

Note that

$$0 < \theta < 1\,. \tag{3.3}$$

We choose $\beta > 0$ so small that $2^\beta\theta < 1$. Since the domain may be assumed to be bounded we may estimate $KR^\alpha \le K_1 R^\beta$ for all R with $B_R \subset \Omega$.
Thus we may write

$$\int_{B_R} |z|^2\,dx \le \theta \int_{B_{2R}} |z|^2\,dx + K_1 R^\beta \tag{3.4}$$

and by recursion we obtain

$$\begin{aligned}\int_{B_R} |z|^2\,dx &\le \theta^2 \int_{B_{4R}} |z|^2\,dx + K_1(R^\beta + \theta 2^\beta R^\beta) \le \\ &\le \theta^3 \int_{B_{8R}} |z|^2\,dx + K_1(R^\beta + \theta 2^\beta R^\beta + \theta^2 2^{2\beta} R^\beta) \le \\ &\dots \\ &\le \theta^N \int_{B_{2^N R}} |z|^2\,dx + K_1 R^\beta \sum_{i=0}^{\infty} (\theta 2^\beta)^i \le \\ &\le \theta^N \int_{B_{2^N R}} |z|^2\,dx + K_2 R^\beta = (*)\end{aligned}$$

since $\theta 2^\beta < 1$. Continuing estimating we get

$$(*) \le \left(\frac{1}{2^N}\right)^\beta \int_{B_{2^N R}} |z|^2\,dx + K_2 R^\beta$$

and setting $R_0 = 2^N R$ we obtain

$$\int_{B_R} |z|^2\,dx \le \left(\frac{R}{R_0}\right)^\beta \int_{B_{R_0}} |z|^2\,dx + K_2 R^\beta \le \bar{K}_0 R^\beta\,, \quad \bar{K}_0 = \left(\frac{1}{\gamma}\right)^\beta \left(\int_{B_{R_0}} |z|^2\,dx + K_2\right)\,,\ R_0 \ge \gamma\,.$$

On the other hand, given R_0 such that $B_{R_0} \subset \Omega$ we obtain for all $R = R_0 2^{-N}$ the above estimate and we conclude the theorem. □

Lemma 3.1 is important because of the fact that condition (3.2) can be checked easily for the gradient of the minimizing function to variational problems. This is expressed by the

Theorem 3.2. *Let $u \in g + H_0^{1,2}$, $g \in H^{1,2}$ given, be a minimum of the variational integral*

$$\int_\Omega F(\nabla u)\,dx$$

where $F : \mathbb{R}^n \to \mathbb{R}$ is continuous and satisfies

$$c|\eta|^2 - K \le F(\eta) \le K|\eta|^2 + K$$

with constants $c > 0$ and $K > 0$. Then there exists a $\beta > 0$ and a K_0 such that

$$\int_{B_R(y)} |\nabla u|^2\,dy \le K_0 R^\beta\,, \qquad y \in \Omega_0 \subset\subset \Omega$$

for all $B_R(y) \subset \Omega$; here K_0 depends on Ω_0.

Corollary 3.1. *If $n = 2$ then $u \in C^{0,\beta/2}$, i.e. u is Hölder continuous on interior domains of Ω_0 with exponent $\beta/2$.*

The theorem is also true if u is an r-vector function.

of theorem 3.2: Let $B_{2R}(y) \subset \Omega$ and let τ be a Lipschitz-continuous function such that

$$\tau(x) = \begin{cases} 1 & \text{on} \quad B_R(y) \\ 0 & \text{on} \quad \mathbb{R}^n - B_{2R}(y) \end{cases}$$

and $|\nabla\tau| \le R^{-1}$ on $B_{2R}(y) - B_R(y)$. Let c be a constant which will be defined later. If $g \in H^{1,2}$ and $u \in g + H_0^{1,2}$, then $u + \tau^2(c - u) \in g + H_0^{1,2}$ and since u is a minimum of the variational integral we know that

$$\int_\Omega F(\nabla u)\,dx \le \int_\Omega F(\nabla(u + (c-u)\tau^2))\,dx\,;$$

moreover, since $\nabla(u + (c-u)\tau^2) = \nabla u$ on $\Omega - B_{2R}(y)$ we have

$$\int_{B_{2R}} F(\nabla u)\,dx \le \int_{B_{2R}} F(\nabla(u + (c-u)\tau^2))\,dx\,. \tag{3.5}$$

By the growth condition for F

$$\begin{aligned} F(\nabla(u + (c-u)\tau^2)) &\le K|\nabla(u + (c-u)\tau^2)|^2 + K \le \\ &\le K'|\nabla u|^2(1-\tau^2)^2 + K'|\nabla\tau|^2|c-u|^2 + K \le \\ &\le K'(|\nabla u|^2 + R^{-2}|c-u|^2)\chi_{*R} + K \qquad \text{on } B_{2R} \end{aligned}$$

where χ_{*R} is the characteristic function of $B_{2R} - B_R$. Here we have used the fact that $\tau = 1$ on B_R.
By the coerciveness condition for F $F(\nabla u) \ge c|\nabla u|^2 - K$, and collecting our wisdom, we arrive at the inequality

$$\int_{B_{2R}} [c|\nabla u|^2 - K]\,dx \le K' \int_{B_{2R}-B_R} [|\nabla u|^2 + R^{-2}|c-u|^2]\,dx + \int_{B_{2R}} K\,dx\,,$$

and thus

$$\int_{B_R} |\nabla u|^2\,dx \le \tilde{K} \int_{B_{2R}-B_R} |\nabla u|^2\,dx + \tilde{K}R^{-2} \int_{B_{2R}-B_R} |c-u|^2\,dx + \tilde{K}R^2\,. \tag{3.6}$$

We now choose c to be equal to the mean value of u on $B_{2R} - B_R$. By Poincaré's inequality

$$\int_{B_{2R}-B_R} |c-u|^2\,dx \le \bar{K}R^2 \int_{B_{2R}-B_R} |\nabla u|^2\,dx\,,$$

and we conclude from (3.6)

$$\int_{B_R} |\nabla u|^2\,dx \le K_1 \int_{B_{2R}-B_R} |\nabla u|^2\,dx + K_1 R^2\,.$$

Now the hole-filler strikes again: we apply 3.1 and are happy. □

Note that it is not hard to generalize the theorem to the variational integral

$$\int_\Omega F(x,u,\nabla u)\,dx$$

where F is continuous and satisfies

$$C|\eta|^2 - K \le F(x,u,\eta) \le K|\eta|^2 + K$$
$$|F(x,u,\eta) - F(x,v,\eta)| \le K|u-v|(1+|\eta|^2)\,.$$

However *it is unknown* whether one can prove the Morrey-condition and thus Hölder continuity in the case of two variables, if u is an r-vector function, $r \ge 2$, which is a *relative minimum* of the above variational-integral. "Relative" has to be understood in the L^∞ or $H^{1,2}$-metric i.e. the integral is minimal in a neighbourhood of u.

We want to prove a variant of the theorem 3.2 by considering Euler's equation for the variational problem.

Theorem 3.3. *Let $u \in g + H_0^{1,2}(\Omega)$, $g \in H^{1,2}$, be a solution of the equation*

$$\sum_{i=1}^n \int_\Omega F_i(\nabla u)\partial_i\varphi\,dx = 0\,, \qquad \varphi \in H_0^{1,2}(\Omega) \tag{3.7}$$

where the F_i are given continuous functions $\mathbb{R}^n \to \mathbb{R}$ which satisfy

$$|F_i(\eta)| \le K(1+|\eta|)\,, \qquad i = 1,\dots,n\,, \quad \eta \in \mathbb{R}^n$$

and

$$\sum_{i=1}^n F_i(\eta)\eta_i \ge c|\eta|^2 - K\,, \qquad \eta = (\eta_1,\dots,\eta_n) \in \mathbb{R}^n$$

with positive constants c and K. Then the conclusion of theorem 3.2 holds.

Proof. We intend to verify the condition of lemma 3.1 in order to apply the hole-filling technique and then Morrey's lemma.
For this, let τ be defined as in the proof of theorem 3.2 and set in (3.7) $\varphi = (u-c)\tau^2$ where c is defined as in the proof of theorem 3.2 namely c is the mean-value of u taken over $B_{2R} - B_R$. This yields

$$\sum_{i=1}^{n} \int_\Omega F_i(\nabla u)\partial_i[(u-c)\tau^2]\, dx = 0$$

and using the coerciveness and growth condition in a simple way

$$\int_\Omega |\nabla u|^2 \tau^2\, dx \le K' R^n + K' \int_\Omega (u-c)^2 |\nabla \tau|^2\, dx$$

or

$$\int_{B_R} |\nabla u|^2 \tau^2\, dx \le K' R^n + R^{-2} K' \int_{B_{2R}-B_R} (u-c)^2\, dx\,.$$

Using Poincaré's inequality as in the preceding proof we obtain the "hole-filling"-condition of lemma 3.1. □

Finally, let us present an example of a discontinuous solution of a quasi linear system:

Set $u = \sin \ln \ln \frac{1}{|x|}$, $v = \cos \ln \ln \frac{1}{|x|}$, $x \in \mathbb{R}^2$.
Let us calculate Δu and Δv. Set $f = \ln \frac{1}{|x|}$. Then

$$\begin{aligned}\Delta u = \Delta \sin \ln f &= \sum_{i=1}^{2} \partial_i \left(\frac{\partial_i f}{f} \cos \ln f \right) = \\ &= -\frac{|\nabla f|^2}{f^2} \cos \ln f - \frac{|\nabla f|^2}{f^2} \sin \ln f \qquad \text{for } |x| \ne 0\,,\ |x| < e^{-1}\,.\end{aligned}$$

We have used the fact that, in $\mathbb{R}^2 \setminus \{0\}$, the identity $\Delta f = 0$ holds.

Similary,

$$\begin{aligned}\Delta v = \Delta \cos \ln f &= -\sum_{i=1}^{2} \partial_i \left(\frac{\partial_i f}{f} \sin \ln f \right) = \\ &= +\frac{|\nabla f|^2}{f^2} \sin \ln f - \frac{|\nabla f|^2}{f^2} \cos \ln f\,.\end{aligned}$$

Furthermore, $|\nabla u|^2 = \frac{|\nabla f|^2}{f^2} \cos^2 \ln f$, $\qquad |\nabla v|^2 = \frac{|\nabla f|^2}{f^2} \sin^2 \ln f$.
Thus

$$\begin{aligned}\Delta u &= \left(|\nabla u|^2 + |\nabla v|^2\right)(u+v)\,,\\ \Delta v &= \left(|\nabla u|^2 + |\nabla v|^2\right)(u-v)\,.\end{aligned}$$

This holds in the set $\{x \in \mathbb{R}^2 \mid |x| < e^{-1}, x \neq 0\}$, $e = \exp 1$, and since $v, u \in H^{1,2}(B_{e^{-1}})$ it is easy to show that for $\varphi \in C_0^\infty(B_{e^{-1}})$

$$(\nabla u, \nabla \varphi) = +((|\nabla u|^2 + |\nabla v|^2)(u+v), \varphi)\,,$$
$$(\nabla u, \nabla \varphi) = -((|\nabla u|^2 + |\nabla v|^2)(u-v), \varphi)$$

Thus, we have a weak solution of a two dimensional elliptic system which is in $H_0^{1,2}(B_{e^{-1}}) \cap L^\infty$ and is *not continuous.*

It would be very interesting to construct a similar example which comes from a *variational* problem. For n large, such examples have been constructed.

4. The differentiability of weak solutions to Euler equations of variational problems

Under suitable conditions, namely growth conditions and ellipticity one can show that the solutions to Euler equations have one more derivative than follows from the existence theory. We start with the simplest case of the equation

$$-\sum_{i=1}^{n} \partial_i F_i(\nabla u) = 0\,. \tag{4.1}$$

Theorem 4.1. *Let $u \in H^{1,2}(\Omega)$ be a weak solution of equation (4.1) and let $F_i : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable and assume that the matrix $(F_{ik}(\eta))_{i,k=1}^n$ where $F_{ik}(\eta) = \frac{\partial}{\partial \eta_k} F_i(\eta)$ has the properties*

(i) $\sum\limits_{i,k=1}^{n} F_{ik}\xi_i\xi_k \geq c|\xi|^2\,,\qquad \xi \in \mathbb{R}^n$*, with a constant $c > 0$ and*
(ii) $|F_{ik}| \leq K$ *uniformly with a constant $K, i, k = 1, \ldots, n$.*

Then

$$u \in H^{2,2}_{\mathrm{loc}}(\Omega)\,.$$

Remark: We do not treat regularity up to the boundary here. If $\partial\Omega \in C^2$, one can prove $u \in H^{2,2}(\Omega)$.

Proof. of theorem 4.1: Let $D_i^{\pm h} w(x) = \pm h^{-1}(w(x \pm h e_i) - w(x))$, i.e. $D_i^{\pm h}$ are the forward and backward difference quotients; e_i is the i-th unitvector. Let $\tau \in C_0^\infty(\Omega)\,,\ \tau \geq 0$. Then, for $h \leq h_0(\tau)\,,\ D_j^{-h}(\tau^2 D_j^h u) \in H_0^{1,2}(\Omega)$ and since u is a weak solution to the Euler equation we have

$$\sum_{i=1}^{n} \int_\Omega F_i(\nabla u)\partial_i D_j^{-h}(\tau^2 D_j^h u)\, dx = 0\,.$$

Using the formula

$$\int_\Omega w D_j^{-h} v\, dx = -\int_\Omega D_j^h w v\, dx$$

if one of the L^2-functions w or v has compact support T and $h < \operatorname{dist}(\partial\Omega, T)$, we obtain

$$\sum_{i=1}^{n} \int_\Omega D_j^h F_i(\nabla u)\partial_i(\tau^2 D_j^h u)\, dx = 0\,. \tag{4.2}$$

We use the formula

$$D_j^h F_i(\nabla u) = \sum_{k=1}^{n} \int_0^1 F_{ik}(t\nabla u(x + h e_j) + (1-t)\nabla u(x))\partial_k D_j^h u\, dt$$

and obtain the estimate

$$\sum_{i=1}^{n} D_j^h F_i(\nabla u)\partial_i D_j^h u \geq c|\nabla D_j^h u|^2$$

by using condition (i) of theorem 4.1.
Calculating the term $\partial_i(\tau^2 D_j^h u)$ in (4.2) by Leibniz' rule we conclude

$$c\int_\Omega |\nabla D_j^h u|^2\tau^2\,dx \leq K'\int_\Omega |\nabla D_j^h u|\;|\nabla\tau|\tau|D_j^h u|\,dx$$

with a constant K' depending on the constant K in (ii).
Using Young's inequality in order to estimate

$$|\nabla D_j^h u|\;|\nabla\tau|\tau|D_j^h u| \leq \varepsilon|\nabla D_j^h u|^2\tau^2 + K(\varepsilon)|\nabla\tau|^2|D_j^h u|^2$$

we arrive at the inequality

$$\int |\nabla D_j^h u|^2\tau^2\,dx \leq K_0\int |\nabla\tau|^2|D_j^h u|^2\,dx\,. \tag{4.3}$$

The right hand side of (4.3) is uniformly bounded as $h \to 0$. This follows from a theorem that if $u \in H^{1,2}$ then the functions $D_j^h u$ are uniformly bounded in the L^2-norm on interior domains (in fact, the convergence in L^2_{loc} to $\partial_j u$). This follows from the fact that for $C^1(\bar\Omega)$-functions v

$$\int_{\Omega_0} |D_j^h v|^2\,dx \leq \int_\Omega |\partial_j v|^2\,dx\,,\ \Omega_0 \subset\subset \Omega\,,\ h < \operatorname{dist}(\partial\Omega_0, \partial\Omega)$$

and the statement on the boundedness of $D_j^h u$ in L^2_{loc} follows by a closure argument. The inequality above follows by representing

$$D_j^h v = h^{-1}\int_0^h \partial_j v(x + te_j)\,dt\,.$$

We may choose τ such that $\tau = 1$ on some domain $\Omega_0 \subset\subset \Omega$ and obtain that

$$\int_\Omega |\nabla D_j^h u|^2\,dx \leq K_1$$

uniformly as $h \to 0$. This holds for any $\Omega_0 \subset\subset \Omega$. From the following lemma we then conclude the statement of the theorem. □

Lemma 4.1. *Let* $z \in L^2(\Omega)$ *and* $\int_{\Omega_0} |D_j^h z|^2\,dx \leq K_0\,,\ j = 1, \dots, n,$ *uniformly as* $h \to 0$ *on compact subdomains of* Ω. *Then* $z \in H^{1,2}_{\text{loc}}(\Omega)$.

Proof. From the weak compactness of bounded sets in $L^2(\Omega_0)$ it follows that there exists a subsequence Λ of numbers $h \to 0$ such that

$$D_j^h z \rightharpoonup z_j\,, \qquad h \to 0\,, \quad h \in \Lambda$$

weakly in $L^2(\Omega_0)$. Now, if $\varphi \in C_0^\infty(\Omega_0)$

$$\int_\Omega \varphi D_j^h z \, dx = -\int_\Omega D_j^{-h} \varphi z \, dx \, ,$$

and passing to the limit $h \to 0$ we obtain

$$\int_\Omega \varphi z_j \, dx = -\int_\Omega \partial_j \varphi z \, dx$$

i.e. z_j is the j-th derivative of z in the sense of distributions and is contained in L^2, that means $z \in W^{1,2}_{\text{loc}}(\Omega)$. Using Meyer-Serrin's theorem that

$$W^{1,2}_{\text{loc}}(\Omega) = H^{1,2}_{\text{loc}}(\Omega)$$

we obtain the statement. □

The difference-quotient procedure with which we proved theorem 4.1 is a classical tool in partial differential equations and is due to Bernstein (maybe even Liechtenstein). For linear partial differential equations one can carry out a similar procedure using higher order difference-quotients $(D_j^h)^m$ and thus getting $H^{1+m,2}_{\text{loc}}(\Omega)$-differentiability. This does not work in the case of non-linear equations. The technique "stops" with $H^{2,2}_{\text{loc}}$-differentiability.

Note that it is not hard to generalize theorem 4.1 to the case of the general equation

$$-\sum_{i=1}^n \partial_i F_i(x, u, \nabla u) + F_0(x, u, \nabla u) = 0$$

under ellipticity and sufficiently *strong* growth conditions for F_i.

Note that in the case of our counterexample in chapter 3 these growth conditions are *not* satisfied since $\sin \ln \ln \frac{1}{|x|} \notin H^{2,2}_{\text{loc}}$ for $n = 2$. (The reason is that the lower order term $F_0(x, u, \nabla u)$ in our counterexample has quadratic growth in ∇u.)

By more refined techniques one can treat this in the scalar case or when one has C^∞-regularity for u.

We intend to refine theorem 4.1 by showing that the second derivatives of the solution satisfy a local Morrey-condition and thus $\nabla u \in C^\alpha$ for $n = 2$.

5. The Moser-technique

If $u \in g + H_0^{1,2}(\Omega)$, $g \in H^{1,2}(\Omega)$, is a minimum for the variational integral $\int_\Omega F(\nabla u)\,dx$ where $F \in C^2(\mathbb{R}^n)$ and the matrix $(F_{ik}(\eta))_{i,k=1}^n$ has positive eigenvalues which are bounded by positive constants from above and below, then we know already that $u \in H_{\text{loc}}^{2,2}(\Omega)$, and the following equation holds:

$$\sum_{i,k=1}^n \int_\Omega F_{ik}(\nabla u)\partial_k\partial_j u \partial_i\varphi\,dx = 0\,, \qquad \varphi \in H_0^{1,2}\,, \quad j = 1,\dots,n\,. \tag{5.1}$$

Setting $F_{ik}(\nabla u) = a_{ik}$, we have

$$a_{ik} \in L^\infty(\Omega)$$

and

$$\sum_{i,k=1}^n a_{ik}(x)\xi_i\xi_k \geq c|\xi|^2\,, \qquad x \in \mathbb{R}^n\,, \quad \xi \in \mathbb{R}^n$$

with some constant $c > 0$.

From (5.1) we conclude that the function $z = \partial_j u$ satisfies the equation

$$\sum_{i,k=1}^n \int_\Omega a_{ik}\partial_k z \partial_i\varphi\,dx = 0\,. \tag{5.2}$$

In order to obtain regularity results for $\partial_j u$ it suffices to study (5.2) and to prove regularity theorems for solutions $z \in H_{\text{loc}}^{1,2}(\Omega)$ of (5.2).

By the famous theorem of De Giorgi-Nash the solutions $z \in H_{\text{loc}}^{1,2}(\Omega)$ of (5.2) are *Hölder-continuous* under the above hypotheses. The De Giorgi-Nash proof of this theorem was later simplified by Jürgen Moser. We present here the first part of his proof, namely the proof that $z \in L_{\text{loc}}^\infty(\Omega)$.

Theorem 5.1. *Let $a_{ik} \in L^\infty(\Omega)$ and*

$$\sum_{i,k=1}^n a_{ik}(x)\xi_i\xi_k \geq c|\xi|^2\,, \qquad x \in \mathbb{R}^n\,, \quad \xi \in \mathbb{R}^n\,,$$

with a constant $c > 0$. Let $z \in H_{\text{loc}}^{1,2}(\Omega)$ be a weak solution[3] *to the equation*

$$-\sum_{i,k=1}^n \partial_i\left(a_{ik}(x)\partial_k z\right) = 0 \text{ in } \Omega\,,$$

then $z \in L_{\text{loc}}^\infty(\Omega)$ and the inequality

$$\|z\|_{L^\infty(B_R)}^2 \leq \frac{K}{R^n}\int_{B_{2R}} |z|^2\,dx$$

[3] i.e. $\sum_{i,k=1}^n \int_\Omega a_{ik}\partial_k z\partial_i\varphi\,dx = 0\,,\ \varphi \in H_0^{1,2}(\Omega)$

holds for all balls B_R such that $B_{2R} \subset \Omega$ is concentric to B_R.

Remark: Under the above hypotheses, $u \in C^\alpha(\Omega)$ with some $\alpha \in]0,1[$.

Proof. We introduce the truncated function

$$z_l(x) = \begin{cases} z(x) & \text{if} \quad |z(x)| \le l\,, \\ l \operatorname{sign} z(x) & \text{if} \quad |z(x)| > l\,. \end{cases}$$

By an approximation argument one can show that if $z \in H^{1,2}_{\text{loc}}$ then $z_l \in H^{1,2}_{\text{loc}}$. Furthermore, $z_l \in L^\infty$ and $[z_l]^p \in H^{1,2}_{\text{loc}}(\Omega)$ where $[\xi]^p = \xi|\xi|^{p-1}$, $p \ge 1$. Thus, if $\tau \in C_0^\infty(\Omega)$ we may use $\tau^2[z_l]^p$ as a testfunction and obtain

$$\sum_{i,k=1}^n \int_\Omega a_{ik}\partial_k z \partial_i([z_l]^p\tau^2)\,dx = 0$$

and

$$p\sum_{i,k=1}^n \int_{\Omega_l} a_{ik}\partial_k z\partial_i z|z|^{p-1}\tau^2\,dx \le K\int_\Omega |\nabla z|\;|z_l|^p|\nabla\tau^2|\,dx\,,$$

where Ω_l denotes the set defined by $|z| < l$. Using the definiteness of (a_{ik}) we obtain

$$\boxed{cp\int_{\Omega_l} |\nabla z|^2|z|^{p-1}\tau^2\,dx \le K\int_\Omega |\nabla z|\;|z_l|^p|\nabla\tau^2|\,dx\,.} \tag{5.3}$$

We confine ourselves to the case $n \ge 3$.

Since $z \in H^{1,2}_{\text{loc}}$ we have

$$z \in L^{\frac{2n}{n-2}}_{\text{loc}}(\Omega)$$

by Sobolev's inequality; setting $p = \frac{n}{n-2}$ we conclude that the L^2_{loc}-norms of $|z_l|^p$ are uniformly bounded as $l \to \infty$ since

$$\int_{\Omega_0} |z_l|^{2p}\,dx \le \int_{\Omega_0} |z|^{2p}\,dx = \int_{\Omega_0} |z|^{\frac{2n}{n-2}}\,dx$$

for any $\Omega_0 \subset\subset \Omega$.

Thus, the right hand side of (5.3) is uniformly bounded as $l \to \infty$, and passing to the limit $l \to \infty$ we obtain by Fatou's lemma that $\int_\Omega |\nabla z|^2|z|^{p-1}\tau^2\,dx$ exists and is finite.

We rewrite this in the form

$$\int_\Omega |\nabla z|^2|z|^{p-1}\tau^2\,dx = \frac{4}{(p+1)^2}\int_\Omega \left|\nabla|z|^{\frac{p}{2}+\frac{1}{2}}\right|^2\tau^2\,dx$$

and obtain that

$$|z|^{\frac{p}{2}+\frac{1}{2}} \in H^{1,2}_{\text{loc}}\,, \qquad \text{where} \quad p = \frac{n}{n-2}\,.$$

Again using Sobolev's theorem, we obtain

$$|z|^{\frac{p}{2}+\frac{1}{2}} \in L_{\mathrm{loc}}^{\frac{2n}{n-2}}$$

or

$$z \in L_{\mathrm{loc}}^{\frac{pn}{n-2}+\frac{n}{n-2}} = L^{(\frac{n}{n-2})^2+\frac{n}{n-2}} .$$

Repeating the argument above - i.e. using (5.3) with $p = (\frac{n}{n-2})^2 + \frac{n}{n-2}$ we obtain

$$z \in L_{\mathrm{loc}}^{q} \text{ with } q = \left(\frac{n}{n-2}\right)^3 + \left(\frac{n}{n-2}\right)^2 + \frac{n}{n-2}$$

and finally for $q_i = \sum_{k=1}^{i} (\frac{n}{n-2})^k$.

Since $\frac{n}{n-2} > 1$ we obtain $z \in L_{\mathrm{loc}}^{q}$ for any $q < \infty$.

Thus, we may pass to the limit $l \to \infty$ in equation (5.3) for any $p < \infty$ and obtain

$$p \int_{\Omega} |\nabla z|^2 |z|^{p-1} \tau^2 \, dx \le K \int_{\Omega} |\nabla z| \; |z|^p |\nabla \tau^2| \, dx , \qquad \tau \ge 0$$

and, $p \ge 1$,

$$\frac{p}{2} \int_{\Omega} |\nabla z|^2 |z|^{p-1} \tau^2 \, dx \le K' \int_{\Omega} |z|^{p+1} |\nabla \tau|^2 \, dx .$$

(We have used $|\nabla z| \; |z|^p = |\nabla z| \; |z|^{\frac{p-1}{2}} |z|^{\frac{p+1}{2}}$ and Young's inequality.)

We rewrite this in the form

$$\int_{\Omega} \left| \nabla |z|^{\frac{p+1}{2}} \right|^2 \tau^2 \, dx \le \frac{\bar{K}(p+1)^2}{p} \int_{\Omega} |z|^{p+1} |\nabla \tau|^2 \, dx$$

$$\le K \cdot p \int_{\Omega} |z|^{p+1} |\nabla \tau|^2 \, dx .$$

Furthermore

$$\int \left| \nabla \left(\tau |z|^{\frac{p+1}{p}} \right) \right|^2 dx \le K_1 p \int |z|^{p+1} |\nabla \tau|^2 \, dx ,$$

and passing from $p+1$ to p, $p \ge 2$

$$\int |\nabla \left(\tau |z|^{p/2} \right)|^2 \, dx \le K_2 p \int |z|^p |\nabla \tau|^2 \, dx .$$

Using Sobolev's inequality we obtain

$$\boxed{\left(\int \tau^{\frac{2n}{n-2}} |z|^{\frac{pn}{n-2}} \, dx \right)^{\frac{n-2}{n}} \le K_2 p \int |z|^p |\nabla \tau|^2 \, dx} \tag{5.4}$$

and this is the starting inequality for the Moser technique.

By approximation, we may assume τ be a Lipschitz function with compact support. We define a sequence of balls B_{R_i} with radius R_i defined by

$$R_0 = 2R$$
$$R_i = \left(2 - \frac{6}{\pi^2}\sum_{k=1}^{i}\frac{1}{k^2}\right) R\,, \qquad i = 1, 2, \dots .$$

Obviously, $R_i > R$ and $R_i \to R$ for $i \to \infty$.

We then define a sequence of cut-off functions τ_i with the property

$$\operatorname{supp}\tau_i \subset B_{R_i}\,, \tau_i = 1 \text{ on } B_{R_{i+1}}\,,$$

$$|\nabla\tau_i| \le \frac{K}{R_i - R_{i+1}} = \frac{K'}{R} i^2\,.$$

(The B_{R_i} are concentric to each other.)

Thus we obtain from (5.4)

$$\left(\int_{B_{i+1}} |z|^{\frac{pn}{n-2}}\,dx\right)^{\frac{n-2}{n}} \le Kpi^4R^{-2}\int_{B_i} |z|^p\,dx\,, \qquad p \ge 2\,, \tag{5.5}$$

where we used the abbreviation $B_i = B_{B_{R_i}}$.

We now define p_i by

$$p_0 = 2$$
$$p_i = 2\left(\frac{n}{n-2}\right)^i$$

and (5.5) yields

$$\left(\int_{B_{i+1}} |z|^{p_{i+1}}\,dx\right)^{\frac{n-2}{n}} \le Kp_i\frac{i^4}{R^2}\int_{B_i} |z|^{p_i}\,dx$$

and because of $i^4 \le \hat{K}p_i$

$$\left(\int_{B_{i+1}} |z|^{p_{i+1}}\,dx\right)^{\frac{n-2}{n}} \le \frac{K_*}{R^2}p_i^2\int_{B_i} |z|^{p_i}\,dx\,,$$

and finally

$$\boxed{\|z\|_{L^{p_{i+1}}(B_{i+1})} \le \left(\frac{K_*}{R^2}\right)^{\frac{1}{p_i}} \cdot p_i^{2/p_i}\|z\|_{L^{p_i}(B_i)}\,.}$$

By recursion, we obtain

$$\|z\|_{L^{p_i}(B_i)} \le \prod_{k=0}^{i-1} \left(\frac{K_*}{R^2}\right)^{\frac{1}{p_k}} \prod_{k=0}^{i-1} p_k^{2/p_k} \|z\|_{\underbrace{L^{p_0}(B_{i_0})}_{= L^2(B_{2R})}}$$

Passing to the limit $i \to \infty$ it turns out that the right hand side of the above inequality remains finite (see below) and we obtain

$$\|z\|_{L^\infty(B_R)} \le \prod_{k=0}^{\infty} \left(\frac{K_*}{R^2}\right)^{\frac{1}{p_k}} \prod_{k=0}^{\infty} p_k^{2/p_k} \|z\|_{L^2(B_{2R})} .$$

One sees easily that

$$\prod_{k=0}^{\infty} \left(\frac{K_*}{R^2}\right) = \left(\frac{K_*}{R^2}\right)^{\sum_{k=0}^{\infty} \frac{1}{p_k}} = \frac{K'}{R^{n/2}}$$

(recall $p_k = 2(\frac{n}{n-2})^k)$). Furthermore

$$\prod_{k=0}^{\infty} p_k^{2/p_k} = \exp\left(\sum_{k=0}^{\infty} \frac{2}{p_k} \ln p_k\right) ,$$

and this converges since $\ln p_k \sim k$

$$\frac{1}{p_k} \sim \beta^k , \qquad \beta = \frac{n-2}{n} < 1 .$$

This completes the proof of the theorem. □