Sheet 4

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- Rough differential equations
- Wong–Zakai theorem

1 Rough differential equations

"Data aequatione quotcunque fluentes quantitates involvente, fluxiones invenire; et vice versa" (I. Newton)

1.1 Rough apriori estimate on ODEs

Consider the *controlled* ODE

$$\dot{y}(t) = F(y(t))\dot{x}(t), \qquad y(0) = y_0$$

where $x \in C^1([0, T]; \mathbb{R}^m)$, $y \in C^1([0, T]; \mathbb{R}^d)$ and $F \in C^2(\mathbb{R}^d; \mathscr{L}(\mathbb{R}^m; \mathbb{R}^d))$. Let Φ: $x \mapsto y$ describe the mapping from x to y. We are interested in understanding this mapping in relation to an Holder topology on the control x. In order to do so we need a different description of the ODE, descriprion which does not make reference to the differentiable caracter of the trajectories. To achieve this goal we expand the solution y aroung a given time s to obtain an equation describing approximatively how y behaves for times t near s. Standard Taylor formula gives

$$\delta y(s,t) = F(y(s)) \int_s^t \mathrm{d}x_u + F_2(y(s)) \int_s^t \int_s^u \mathrm{d}x_v \otimes \mathrm{d}x_u + \int_s^t \int_s^u \int_s^v \mathrm{d}_r F_2(y(r)) \mathrm{d}x_v \otimes \mathrm{d}x_u$$

where $F_2^i(\xi)(u \otimes v) = F_a^j(\xi) \nabla_j F_b^i(\xi) u^a v^b$ (Einstein summation convention). The last term can be estimated very easily to be of order $|t-s|^3 \|\dot{x}\|_\infty^2$. Of course this estimate uses the diffentiability of x so it is not very good, however it says to us that if we denote by $\mathbb{X}$ the canonical rough path associated to x the ODE can be recast as the finite-increment relation

$$\delta y = F(y)\mathbb{X}^1 + F_2(y)\mathbb{X}^2 + C_2^{1+}, \qquad y(0) = y_0. \tag{1}$$

Where only the rough path $\mathbb{X}$ appears and there is no (direct) reference to the differentiability of x (or y). The question is if this new description is as powerful as the original ODE. This is indeed the case since uniqueness holds under the condition that $F \in C^3_{\mathrm{loc}}$ and $\mathbb{X} \in \mathscr{C}^\gamma$ for $\gamma > 1/3$.

Theorem 1. *(Uniqueness). Assume $\mathbb{X} \in \mathscr{C}^\gamma$ with $\gamma > 1/3$ and $F \in C^3_{\mathrm{loc}}$. Then there exists only one function $y \in C^\gamma$ solving (1).*

Proof. Let $y, \tilde{y}$ be two solutions. Then $z = \tilde{y} - y$ is a solution to another RDE in $\mathbb{R}^d \times \mathbb{R}^d$:

$$\delta z = G(y,z)\mathbb{X}^1 + G_2(y,z)\mathbb{X}^2 + z^\natural,$$

where $G(\xi,\eta) = F(\xi+\eta) - F(\xi)$ and $G_2(\xi,\eta) = F_2(\xi+\eta) - F_2(\xi)$ and where $z^\natural \in C_2^{1+}$ at least. Note that $G(\xi,0) = G_2(\xi,0) = 0$ so $[\![G(y,z)]\!]_{\infty,\tau} + [\![G_2(y,z)]\!]_{\infty,\tau} \lesssim_{F,y,\tilde{y}} [\![z]\!]_{\infty,\tau}$. Now using again the equation we estabilish

$$[\![z]\!]_{\gamma,\tau} \lesssim_{F,y,\tilde{y},\mathbb{X}} [\![z]\!]_{\infty,\tau} + \tau^{2\gamma} [\![z^\natural]\!]_{3\gamma,\tau}$$

Moreover by a direct computation we show

$$[\![\delta G(y,z) - G_2(y,z)\mathbb{X}^1]\!]_{2\gamma,\tau} \lesssim_{F,y,\tilde{y},\mathbb{X}} ([\![z]\!]_{\infty,\tau} + \tau^{\gamma} [\![z^\natural]\!]_{3\gamma,\tau})$$

and then since

$$\delta z^\natural = (\delta G(y,z) - G_2(y,z)\mathbb{X}^1)\mathbb{X}^1 + \delta G_2(y,z)\mathbb{X}^2$$

we have

$$[\![z^\natural]\!]_{3\gamma,\tau} \lesssim_{F,y,\tilde{y},\mathbb{X}} [\![z]\!]_{\infty,\tau} + \tau^{\gamma} [\![z^\natural]\!]_{3\gamma,\tau}$$

Taking τ small enough we obtain $[\![z^\natural]\!]_{3\gamma,\tau} \lesssim_{F,y,\tilde{y},\mathbb{X}} [\![z]\!]_{\infty,\tau}$ from which

$$[\![z]\!]_{\infty,\tau} \leqslant |z(0)| + \tau^{\gamma} [\![z]\!]_{\gamma,\tau} \leqslant |z(0)| + C_{F,y,\tilde{y},\mathbb{X}} \tau^{\gamma} [\![z]\!]_{\infty,\tau}$$

and taking τ smaller if necessary we conclude

$$[\![z]\!]_{\infty,\tau} \leqslant 2|z(0)|.$$

In particular $[\![z]\!]_{\infty,\tau} = 0$ if $z(0) = 0$.

□

This shows that in quite generality eq. (1) describe time evolution of y as well as the corresponding ODE. The advantage of course is that we do not have to assume that x is smooth but only that the rough path $\mathbb{X}$ is of sufficient regularity. However if we start from a rough path $\mathbb{X}$ which is not the canonical lift of a smooth path x we still have to determine whether eq. (1) admits a solution or not.

One possible strategy applies if $\mathbb{X} \in \mathscr{C}^{\gamma}_{\mathrm{wg}}$. In this case we take $1/3 < \rho < \gamma$ and a sequence $\mathbb{X}^n \in \mathscr{C}^1_{g,x^n}$ converging to $\mathbb{X}$ in $\mathscr{C}^{\rho}$ and let y^n the solution of the ODE driven by x^n. Then

$$\delta y^n = F(y^n)\mathbb{X}^{n,1} + F_2(y^n)\mathbb{X}^{n,2} + y^{n,\natural}, \qquad y^n(0) = y_0.$$

And we would like to pass to the limit as $n \to \infty$ under minimal conditions on F. To do so we need the following apriori estimates.

Lemma 2. *Apriori bounds for solutions. Assume that* $\|\nabla F_2\|_{\infty} + \|\nabla F\|_{\infty} < +\infty$ *and that* y *is a solution on* $[0,T]$ *of*

$$\delta y = F(y)\mathbb{X}^1 + F_2(y)\mathbb{X}^2 + y^\natural$$

then

$$[\![y^\natural]\!]_{3\gamma,\tau} \lesssim C_F \|\mathbb{X}\|_{\mathscr{C}^\gamma,1}([\![y]\!]_{\gamma,\tau} + [\![y^\sharp]\!]_{2\gamma,\tau}).$$

Moreover there exists τ *small such that*

$$[\![y]\!]_{\gamma,\tau} + [\![y^\sharp]\!]_{2\gamma,\tau} \leqslant \|\mathbb{X}\|_{\mathscr{C}^\gamma,1}(|F(y(0))| + |F_2(y(0))|).$$

Proof. The equation for $y^\natural$ reads

$$\delta y^\natural = (\delta F(y) - F_2(y)\mathbb{X}^1)\mathbb{X}^1 + \delta F_2(y)\mathbb{X}^2.$$

Now

$$\delta F(y)(s,t) - F_2(y(s))\mathbb{X}^1(s,t) = \int_0^1 \mathrm{d}\tau \nabla F(y(s) + \tau\delta y(s,t))\delta y(s,t) - F_2(y(s))\mathbb{X}^1(s,t)$$

$$= \int_0^1 \mathrm{d}\tau (\nabla F(y(s) + \tau\delta y(s,t))F(y(s)) - F_2(y(s)))\mathbb{X}^1(s,t) + \int_0^1 \mathrm{d}\tau \nabla F(y(s) + \tau\delta y(s,t))y^\sharp(s,t)$$

$$= \int_0^1 \mathrm{d}\tau (F_2(y(s) + \tau\delta y(s,t)) - F_2(y(s)))\mathbb{X}^1(s,t)$$

$$- \int_0^1 \mathrm{d}\tau \int_0^\tau \mathrm{d}\sigma \nabla F(y(s) + \tau\delta y(s,t))\nabla F(y(s) + \sigma\delta y(s,t))\delta y(s,t)\mathbb{X}^1(s,t)$$

$$+ \int_0^1 \mathrm{d}\tau \nabla F(y(s) + \tau\delta y(s,t))y^\sharp(s,t)$$

so

$$\|\delta F(y) - F_2(y)\mathbb{X}^1\|_{2\gamma} \lesssim (\|\nabla F_2\|_\infty + \|\nabla F\|_\infty^2)\|y\|_\gamma \|\mathbb{X}^1\|_\gamma + \|\nabla F\|_\infty \|y^\sharp\|_{2\gamma}$$

Finally

$$[\![y^\natural]\!]_{3\gamma,\tau} \lesssim C_F \|\mathbb{X}\|_{\mathscr{C}^\gamma,1}([\![y]\!]_{\gamma,\tau} + [\![y^\sharp]\!]_{2\gamma,\tau})$$

so

$$[\![y^\sharp]\!]_{2\gamma,\tau} \leqslant [\![F_2(y)]\!]_{\infty,\tau} \|\mathbb{X}\|_{\mathscr{C}^\gamma,1} + C_F \tau^\gamma \|\mathbb{X}\|_{\mathscr{C}^\gamma,1}([\![y]\!]_{\gamma,\tau} + [\![y^\sharp]\!]_{2\gamma,\tau})$$

and

$$[\![y]\!]_{\gamma,\tau} \leqslant [\![F(y)]\!]_{\infty,\tau} \|\mathbb{X}\|_{\mathscr{C}^\gamma,1} + \tau^\gamma [\![F_2(y)]\!]_{\infty,\tau} \|\mathbb{X}\|_{\mathscr{C}^\gamma,1} + C_F \tau^{2\gamma} \|\mathbb{X}\|_{\mathscr{C}^\gamma,1}([\![y]\!]_{\gamma,\tau} + [\![y^\sharp]\!]_{2\gamma,\tau})$$

But now since

$$[\![F(y)]\!]_{\infty,\tau} + [\![F_2(y)]\!]_{\infty,\tau} \leqslant |F(y(0))| + |F_2(y(0))| + \tau^\gamma C_F [\![y]\!]_{\gamma,\tau}$$

we have

$$[\![y]\!]_{\gamma,\tau} + [\![y^\sharp]\!]_{2\gamma,\tau} \leqslant (1 + \tau^\gamma)\|\mathbb{X}\|_{\mathscr{C}^\gamma,1}(|F(y(0))| + |F_2(y(0))|)$$

$$+ C_F(\tau^\gamma + \tau^{2\gamma} + \tau^{3\gamma})\|\mathbb{X}\|_{\mathscr{C}^\gamma,1}^2([\![y]\!]_{\gamma,\tau} + [\![y^\sharp]\!]_{2\gamma,\tau})$$

and taking τ small enough we have

$$[\![y]\!]_{\gamma,\tau}+[\![y^{\sharp}]\!]_{2\gamma,\tau}\leqslant\|\mathbb{X}\|_{\mathscr{C}^{\gamma},1}(|F(y(0))|+|F_2(y(0))|).$$

□

This lemma shows that for $\tau>0$ small enough (can be choosen uniformly in n) we have

$$[\![y^n]\!]_{\rho,\tau}\leqslant\|\mathbb{X}^n\|_{\mathscr{C}^{\rho},1}(|F(y(0))|+|F_2(y(0))|).$$

and since $\|\mathbb{X}^n\|_{\rho,1}$ is bounded we have uniform apriori Hölder estimates for $(y^n)_n$. By Ascoli–Arzela we can find a converging subsequence (always called $(y^n)_n$). Note moreover that the apriori estimates gives also

$$[\![y^{n,\natural}]\!]_{3\rho,\tau}\lesssim C_F\|\mathbb{X}^n\|_{\rho,1}^2(|F(y(0))|+|F_2(y(0))|)$$

so in the end we have that

$$y^{n,\natural}(s,t)=\delta y^n(s,t)-F(y^n(s))\mathbb{X}^{n,1}(s,t)-F_2(y^n(s))\mathbb{X}^{n,2}(s,t)$$

converges pointwise to

$$y^{\natural}(s,t)=\delta y(s,t)-F(y(s))\mathbb{X}^1(s,t)-F_2(y(s))\mathbb{X}^2(s,t)$$

and that $[\![y^{\natural}]\!]_{3\rho,\tau}<\infty$. This shows existence of solutions in the case of a geometric $\mathbb{X}$. The apriori estimates can then be used to show that $[\![y^{\natural}]\!]_{3\gamma,\tau}<\infty$.

If $\mathbb{X}$ is not geometric we proceed slightly differently. In particular we know that there exists a sequence $(\tilde{\mathbb{X}}^n\in\mathscr{C}_g^1)_n$ and a sequence of functions $(\varphi^n\in C^1)_n$ such that $\mathbb{X}^n=\tilde{\mathbb{X}}^n+(0,\delta\varphi^n)$ converges in $\mathscr{C}^{\rho}$ to $\mathbb{X}$. Then we define y^n as the solution to the ODE

$$\dot{y}^n=F(y^n)\dot{x}^n+F_2(y^n)\dot{\varphi}^n$$

and it is easy to check that even in this case

$$\delta y^n=F(y^n)\mathbb{X}^1+F_2(y^n)\mathbb{X}^2+y^{n,\natural}.$$

From which the proof proceed as above. In conclusion we have proven that

Theorem 3. *Assume that $\|\nabla F_2\|_{\infty}+\|\nabla F\|_{\infty}<+\infty$ and $\mathbb{X}\in\mathscr{C}^{\gamma}$. Then there exists a global solution to the RDE (1). If $F\in C_{\mathrm{loc}}^3$ this solution is unique.*

Without proceeding via approximations it is possible to prove existence via a fixpoint argument in the space of controlled paths. This goes as follows.

Consider the map $\Gamma\colon(y,y^X)\mapsto(z,z^X)$ given by

$$\delta z=F(y)\mathbb{X}^1+\nabla F(y)y^X\mathbb{X}^2+z^{\natural},\qquad z^X=F(y)$$

with $z^\natural \in C_2^{3\gamma}$. By Theorem 12 and Lemma 13 we have that

$$[\![(F(y),F(y)^X)]\!]_{\mathscr{D}_{\mathbb{X}}^{2\gamma},\tau} \lesssim C_F\big[1+[\![(y,y^X)]\!]_{\mathscr{D}_{\mathbb{X}}^{2\gamma},\tau}\big]^2.$$

$$[\![F(y)^X]\!]_{\infty,\tau} \leqslant |F(y)^X(0)|+\tau^\gamma [\![F(y)^X]\!]_{\gamma,\tau} \leqslant |F(y)^X(0)|+\tau^\gamma C_F\big[1+[\![(y,y^X)]\!]_{\mathscr{D}_{\mathbb{X}}^{2\gamma},\tau}\big]^2.$$

$$[\![(z,z^X)]\!]_{\mathscr{D}_{\mathbb{X}}^{2\gamma},\tau} \lesssim \|\mathbb{X}\|_{\mathscr{C}^\gamma,1}([\![F(y)^X]\!]_{\infty,\tau}+\tau^\gamma [\![(F(y),F(y)^X)]\!]_{\mathscr{D}_{\mathbb{X}}^{2\gamma},\tau})$$

$$\lesssim \|\mathbb{X}\|_{\mathscr{C}^\gamma,1}|F(y)^X(0)|+\tau^\gamma C_F \|\mathbb{X}\|_{\mathscr{C}^\gamma,1}\big[1+[\![(y,y^X)]\!]_{\mathscr{D}_{\mathbb{X}}^{2\gamma},\tau}\big]^2.$$

Fix $L > \|\mathbb{X}\|_{\mathscr{C}^\gamma,1}|F(y)^X(0)|$ and take τ small enough so that

$$\|\mathbb{X}\|_{\mathscr{C}^\gamma,1}|F(y)^X(0)|+\tau^\gamma C_F \|\mathbb{X}\|_{\mathscr{C}^\gamma,1}[1+L]^2 \leqslant L.$$

Then Γ maps the ball

$$B_L = \big\{(y,y^X)\in \mathscr{D}_{\mathbb{X}}^{2\gamma}: [\![(y,y^X)]\!]_{\mathscr{D}_{\mathbb{X}}^{2\gamma},\tau} \leqslant L\big\} \subseteq \mathscr{D}_{\mathbb{X}}^{2\gamma}$$

into itself. It is not difficult to prove that Γ is also continuous in the norm $[\![(y,y^X)]\!]_{\mathscr{D}_{\mathbb{X}}^{2\gamma},\tau}$ and that B_L is convex and compact. By Schauder fixed point theorm there exists at least one y satisfying $y=\Gamma(y)$.

Remark 4. With a bit more of regularity on F, e.g. bounded with 2 bounded derivatives one can actually show that the map Γ is a contraction for τ small enough. This ensures existence via Banach fix point theorem. This has the advantage of being more elementary and of not requiring a compact image for Γ. As a result this strategy works quite easily also for RDEs in infinite dimension.

Remark 5. The condition $\|\nabla F\|_\infty + \|\nabla^2 F\|_\infty < \infty$ is not sufficient to guarantee global existence of solutions. Indeed there exists F with linear growth such that $\|\nabla F_2\|_\infty = +\infty$ and for which we have explosion for a particular $\mathbb{X}$. For example, one can take pure area rough path $\mathbb{X}=(0,\delta\varphi)$ with $\varphi(t)=Ct$. In this case the RDE is a standard ODE of the form

$$\mathrm{d}y = F_2(y)C\mathrm{d}t$$

and we can arrange things such that $F_2(y)=O(y^2)$.

2 The Wong–Zakai theorem

In this section we want to sketch the proof of the Wong–Zakai theorem for Stratonovich SDE. Let $(B(t))_{t\in[0,1]}$ be a m–dimensional Brownian motion and F a C_b^3 family of vectorfields in $\mathbb{R}^d$ as above. We denote by B^n piecewise linear approximations of B on the dyadic partition $D_n = \{t_k^n = k\,2^{-n}: k=0,\ldots,2^n\}$ and by Y^n the solution of the random ODE

$$\partial_t Y^n(t) = F(Y^n(t))\partial_t B^n(t), \qquad Y^n(0)=y_0.$$

Then the Wong–Zakai theorem states that:

Theorem 6. (Wong–Zakai) *The family* $(Y^n)_{n\geqslant 0}$ *converges a.s. in* $C([0,1];\mathbb{R}^d)$ *to the solution* Y *of the Stratonovich SDE*

$$\mathrm{d}Y(t) = F(Y(t)) \circ \mathrm{d}B(t), \qquad Y(0) = y_0.$$

In order to prove this result with rough path techniques we need several steps

1. Identify Y^n with the solution y^n of the RDE driven by the canonical lift $\mathbb{B}^n$ of B^n in $\mathscr{C}^\gamma$ for some $\gamma \in (1/3, 1/2)$.

2. Prove that $\mathbb{B}^n \to \mathbb{B}_{\mathrm{Strat}}$ in $\mathscr{C}^\gamma$ almost surely.

3. Prove that y^n converges to the solution y of the RDE driven by $\mathbb{B}$ in C^γ (for example).

4. Identify the rough integral controlled by $\mathbb{B}_{\mathrm{Stat}}$ with the standard Stratonovich integral.

Let us start with the last point. Recall that if H is a Brownian semi–martingale with decomposition $\mathrm{d}H = h\mathrm{d}B + k\mathrm{d}t$ and B a Brownian motion (both adapted to the same filtration) then the Stratonovich integral of H wrt. B can be expressed via the Ito integral by

$$\int_0^t H(s) \circ \mathrm{d}B(s) = \int_0^t H(s)\mathrm{d}B(s) + \frac{1}{2}\int_0^t h(s)\mathrm{d}s.$$

Lemma 7. *Let* (H, H^B) *be an adapted and bounded process which belongs a.s. to* $\mathscr{D}_{\mathbb{B}}^{2\gamma}$. *Then the Ito integral of H against B equals a.s. the analogous controlled integral against the Ito Rough Path over B.*

Proof. Recall that the Ito integral $\int_0^t H(s)\mathrm{d}B(s)$ is the L^2 limit of the Riemann sums

$$\sum_i H(t_i)\delta B(t_i, t_{i+1})$$

over a family of partitions of $[0, t]$ while the rough integral can be computed as the limit of the *compensated* Riemman sums

$$\sum_i H(t_i)\delta B(t_i, t_{i+1}) + \sum_i H^B(t_i)\mathbb{B}^2(t_i, t_{i+1}).$$

So we need to show that the difference is going to zero (in probability or L^2 for example). But since H^B is bounded and adapted and $\mathbb{B}^2(t_i, t_{i+1})$ is independent of $\mathscr{F}_{t_i}$ we have

$$\mathbb{E}\left[\left(\sum_i H^B(t_i)\mathbb{B}^2(t_i, t_{i+1})\right)^2\right] \lesssim \sum_i \mathbb{E}|H^B(t_i)|^2 |t_{i+1} - t_i|^2 \to 0.$$

$\square$

Theorem 8. $\mathbb{B}^n \to \mathbb{B}_{\text{Strat}}$ *almost surely in* $\mathscr{C}^\gamma$ *for any* $\gamma < 1/2$.

Proof. Fix $\gamma < \rho < 1/2$. Let $(\mathscr{G}_n = \sigma(B_t;\, t \in D_n))_{n \geqslant 0}$ the σ-field of observations of B along the dyadic partitions of [0, 1]. A simple Gaussian computation shows that $B^n(t) = \mathbb{E}(B(t)|\mathscr{G}_n)$ for $t \in [0, 1]$. In this way we can look at the piecewise linear approximations as conditional expectations of B along $(\mathscr{G}_n)_n$. Then, for each $t \in [0, 1]$ the L^2 martingale $(B^n(t))_n$ converges a.s. to $B(t)$ and recalling the (dyadic) Garcia–Rodemich–Rumsey inequality we have $\|B^n\|_\rho \leqslant C_p Q_p(B^n) \leqslant C_p Q_p(\mathbb{E}(B(\cdot)|\mathscr{G}_n))$ for some $1 < p < +\infty$ (which depends on ρ). Taking expectations and using Jensen's inequality we get

$$\mathbb{E}\|B^n\|_\rho^p \leqslant C_p \mathbb{E} Q_p(\mathbb{E}(B(\cdot)|\mathscr{G}_n))^p \leqslant C_p \mathbb{E}[\mathbb{E}[Q_p(B)^p|\mathscr{G}_n]] \leqslant C_p C_p \mathbb{E}[Q_p(B)^p] < +\infty$$

using a standard argument. Now observe that choosing $0 < \varepsilon < 1$ such that $(1-\varepsilon)\rho = \gamma$ we have (with $\Delta^N = B^n - B$)

$$|\delta\Delta^n(s,t)| \leqslant |\delta\Delta^n(s,t)|^\varepsilon |\delta\Delta^n(s,t)|^{1-\varepsilon} \leqslant |\delta\Delta^n(s_n,t_n) + \delta\Delta^n(s_n,s) + \delta\Delta^n(t_n,t)|^\varepsilon \|\Delta^n\|_\rho^{1-\varepsilon} |t-s|^{(1-\varepsilon)\rho}$$

$$\lesssim |\delta\Delta^n(s_n,t_n)|^\varepsilon \|\Delta^n\|_\rho^{1-\varepsilon} |t-s|^{(1-\varepsilon)\rho} + 2\|\Delta^n\|_\rho 2^{-n\rho\varepsilon} |t-s|^{(1-\varepsilon)\rho}$$

where s_n, t_n are the points in D_n nearer to s, t respectively. Since D_n is a finite set and for all $t \in D_n$ we have $\Delta^n(t) \to 0$ a.s. Then

$$\sup_{s<t} \frac{|\delta B^n(s,t)|}{|t-s|^\gamma} \leqslant \sup_{s,t \in D_n} |\delta\Delta^n(s,t)|^\varepsilon \|\Delta^n\|_\rho^{1-\varepsilon} + 2\|\Delta^n\|_\rho 2^{-n\rho\varepsilon} \to 0$$

since $\|\Delta^n\|_\rho < \infty$ a.s.. It remains to prove the equivalent convergence statement for $\mathbb{B}^{n,2}$. First we note that the symmetric part $S\mathbb{B}^{n,2}(s,t)$ of $\mathbb{B}^{n,2}(s,t)$ is equal to $\frac{1}{2}\mathbb{B}^{n,1}(s,t) \otimes \mathbb{B}^{n,1}(s,t)$ so convergence of $S\mathbb{B}^{n,2}$ to $\frac{1}{2}\mathbb{B}^1 \otimes \mathbb{B}^1 = S\mathbb{B}^2_{\text{Strat}}$ in $C_2^{2\gamma}$ follows by the convergence of $\mathbb{B}^{n,1} = \delta B^n$. In order to deal with the antisymmetric component it is enough to prove convergence for $(\mathbb{B}^{n,2})^{i,j}$ for $i \neq j$. In this case we have

$$\mathbb{B}^{n,2}(s,t)^{i,j} = \int_s^t (B^{n,i}(u) - B^{n,i}(s)) \mathrm{d}B^{n,j}(u)$$

and a direct computation using the Riemman sums approximation of the r.h.s. gives $\mathbb{B}^{n,2}(s,t)^{i,j} = \mathbb{E}[\mathbb{B}^2(s,t)^{i,j}|\mathscr{G}_n]$. Then a.s. convergence holds as above by the L^2 martingale convergence theorem and also we have a.s. uniform boundedness in $C_2^{2\rho}$ of $(\mathbb{B}^{n,2})_n$. An interpolation argument as above concludes the proof that $\|\mathbb{B}^{n,2} - \mathbb{B}^2_{\text{Strat}}\|_{2\gamma} \to 0$ a.s. □

Lemma 9. *If* $\mathbb{X}^n \to \mathbb{X}$ *in* $\mathscr{C}^\gamma$ *then the solution* y^n *of the RDE driven by* $\mathbb{X}^n$ *converges in* C^γ *to the solution* y *of the RDE driven by* $\mathbb{X}$.

Proof. The idea is to use the sewing lemma to compare the two solutions. Let $z = y^n - y$ then

$$\delta z = F(y)\mathbb{X}^1 - F(y^n)\mathbb{X}^{n,1} + F_2(y)\mathbb{X}^2 - F_2(y^n)\mathbb{X}^{n,2} + z^\natural$$

and

$$\delta z^\natural = (\delta F(y) - F_2(y)\mathbb{X}^1)\mathbb{X}^1 - (\delta F(y^n) - F_2(y^n))\mathbb{X}^{n,1}\mathbb{X}^{n,1} + \delta F_2(y)\mathbb{X}^2 - \delta F_2(y^n)\mathbb{X}^{n,2}.$$

A straigforward estimation of the various terms leads to ($\tau \leqslant 1$)

$$[\![z^\natural]\!]_{3\gamma,\tau} \lesssim ([\![z]\!]_{\gamma,\tau} + [\![z]\!]_{\infty,\tau}) + \tau^\gamma [\![z^\natural]\!]_{3\gamma,\tau} + \|\mathbb{X}^1 - \mathbb{X}^{n,1}\|_\gamma + \|\mathbb{X}^2 - \mathbb{X}^{n,2}\|_{2\gamma}$$

where the implicit constant can depend on $y, y^n, \mathbb{X}, \mathbb{X}^n$ but can be checked to be uniformly bounded in n. Using the equation we have also

$$[\![z]\!]_{\gamma,\tau} \lesssim [\![z]\!]_{\infty,\tau} + \|\mathbb{X}^1 - \mathbb{X}^{n,1}\|_\gamma + \|\mathbb{X}^2 - \mathbb{X}^{n,2}\|_{2\gamma} + \tau^{2\gamma} [\![z^\natural]\!]_{3\gamma,\tau}$$

Taking τ small enough we obtain

$$[\![z]\!]_{\gamma,\tau} + [\![z^\natural]\!]_{3\gamma,\tau} \lesssim [\![z]\!]_{\infty,\tau} + \|\mathbb{X}^1 - \mathbb{X}^{n,1}\|_\gamma + \|\mathbb{X}^2 - \mathbb{X}^{n,2}\|_{2\gamma}$$

But now we have also

$$[\![z]\!]_{\infty,\tau} \lesssim |z(0)| + \tau^\gamma [\![z]\!]_{\gamma,\tau}$$

so finally (for τ sufficiently small)

$$[\![z]\!]_{\infty,\tau} + [\![z]\!]_{\gamma,\tau} + [\![z^\natural]\!]_{3\gamma,\tau} \lesssim |z(0)| + \|\mathbb{X}^1 - \mathbb{X}^{n,1}\|_\gamma + \|\mathbb{X}^2 - \mathbb{X}^{n,2}\|_{2\gamma}$$

and since $z(0) = 0$ this quantity goes to zero as $n \to \infty$.

$\square$