Sheet 3

- Beyond Young
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1 Beyond Young

When trying to go beyond Young integral we face the fundamental problem that we cannot expect the integral to be a continuous operations (recall our basic counterexample).

1.1 A general existence result: the para-integral

A natural question is that if it is always possible to find a function $I(f,g)$ which satisfies

$$\delta I(f, g)(s,t) = f(s)\delta g(s, t) + O(|t-s|^\alpha \beta)$$  \hspace{1cm} (1)

with $f \in C^\alpha$ and $g \in C^\beta$ but with $\alpha + \beta < 1$.

**Remark 1.**

i. Uniqueness does not hold anymore since if $I$ is a solution then $\tilde{I} = I + \varphi$ is also a solution of (1) for any $\varphi \in C^{\alpha + \beta}$.

ii. Find such a function is equivalent to ask for a solution $J(f, g) \in C_{2}^{\alpha + \beta}$ of

$$\delta J(f, g)(s, u, t) = \delta f(s, u) \delta g(u, t)$$

since then we can let $\delta I(f, g) = f \delta g - J(f, g)$.

iii. We can always consider

$$J_0(f, g)(s, t) = \frac{1}{2} \delta f(s, t) \delta g(s, t)$$

for which we have

$$\delta J_0(f,g) = \frac{\delta f \delta g + \delta g \delta f}{2}$$

and

$$\|J_0(f, g)\|_{\alpha + \beta} \leq \|f\|_{\alpha} \|g\|_{\beta}.$$  

So in case $f = g$ we can always take $J(f, g) = J_0(f, g)$.
iv. If we consider \( J_1(f, g)(s, t) = -f(s)g(s, t) \) we have

\[
\delta J_1(f, g)(s, u, t) = -f(s)\delta g(s, t) + f(s)\delta g(s, u) + f(u)\delta g(u, t) = \delta f(s, u)\delta g(u, t)
\]
as required (indeed it differs from the iterated integral by the increment of a function). However the regularity is not ok, indeed we have only

\[
|J_1(f, g)(s, t)| \lesssim \|f\|_\infty \|g\|_\beta |t - s|^{\beta}.
\]

Another possibility is

\[
J_2(f, g)(s, t) = \delta f(s, t)g(t) = J_1(f, g)(s, t) + f(t)g(t) - f(s)g(s)
\]
since it differs from the previous one by the increment of the function \( t \mapsto f(t)g(t) \). Still the regularity is not ok.

A decomposition of the functions \( f, g \) into an infinite sequence of blocks living in different scales will allow us to combine the observations contained in the Remark 1 (iv) to produce a map (in general not unique) solving eq. (1).

**Theorem 2.** (Paraintegral) For any \( \alpha, \beta > 0 \) such that \( \alpha + \beta < 1 \) there exists a continuous map \( J_\omega : C^\alpha \times C^\beta \to C^{\alpha+\beta} \) such that

\[
\delta J_\omega(f, g)(s, u, t) = \delta f(s, u)\delta g(u, t), \quad s < u < t.
\]

**Proof.** Let \( \rho : \mathbb{R} \to \mathbb{R}_+ \) smooth, compactly supported around 0 and of integral one. Let \( \rho_n(t) = 2^n \rho(2^n t) \) and \( f_n = \rho_n \ast f - \rho_{n-1} \ast f \) for \( n \geq 1 \) with \( f_0 = \rho_0 \ast f \) and a similar definiton for \( g_n \). With these definitions we have \( f(t) = \sum_n f_n(t) \). Direct estimates give that

\[
|f_n(t)| \lesssim \|f\|_a 2^{-na}, \quad |\delta f_n(t)| \lesssim \|f\|_a 2^{-na},
\]

and similar estimates for \( g_n \) where the constants depends only on \( \rho \). These estimate also show that the sum of the series converges uniformly in \( t \). Now take the combination of \( J_1 \) and \( J_2 \) given by

\[
J_\omega(f, g) = \sum_{m \leq n} J_1(f_n, g_m) + \sum_{m > n} J_2(f_n, g_m)
\]

and note that

\[
\delta J_\omega(f, g) = \sum_{m \leq n} \delta J_1(f_n, g_m) + \sum_{m > n} \delta J_2(f_n, g_m) = \sum_{m \leq n} \delta f_n \delta g_m + \sum_{m > n} \delta f_n \delta g_m = \delta f \delta g
\]
as required. But now if \( 0 < \alpha + \beta < 1 \) we can estimate

\[
|J_1(f_n, g_m)(s, t)| \lesssim \|f\|_a \|g\|_\beta 2^{-na - m\beta} (1 \wedge 2^m |t - s|),
\]

\[
|J_2(f_n, g_m)(s, t)| \lesssim \|f\|_a \|g\|_\beta 2^{-na - m\beta} (1 \wedge 2^m |t - s|).
\]

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so

\[
|J_{\prec}(f, g)(s, t)| \lesssim \|f\|_\alpha \|g\|_\beta \sum_{m \leq n} 2^{-m \alpha - m \beta} (1 \wedge 2^m |t-s|) + \|f\|_\alpha \|g\|_\beta \sum_{n} 2^{-n \alpha - n \beta} (1 \wedge 2^n |t-s|) \\
\lesssim \|f\|_\alpha \|g\|_\beta \sum_{m \geq n} 2^{-m \alpha - m \beta} (1 \wedge 2^m |t-s|) + \|f\|_\alpha \|g\|_\beta \sum_{n} 2^{-n \alpha - n \beta} (1 \wedge 2^n |t-s|) \\
\lesssim \|f\|_\alpha \|g\|_\beta \sum_{m : 2^m \geq |t-s| > 1} 2^{-m \alpha - m \beta} + \|f\|_\alpha \|g\|_\beta |t-s|^{\alpha + \beta} \\
\lesssim \|f\|_\alpha \|g\|_\beta \sum_{m : 2^m \geq |t-s| \leq 1} 2^{-m \alpha - m \beta} \lesssim \|f\|_\alpha \|g\|_\beta |t-s|^{\alpha + \beta}
\]

which implies \(\|J_{\prec}(f, g)\|_{\alpha + \beta} \lesssim \|f\|_\alpha \|g\|_\beta\).  \(\square\)

1.2 Some tools

Regularity of 2-increments.

Lemma 3. Let \(A: \mathbb{D} \times \mathbb{D} \to V\) and

\[
Q_{\alpha, \rho}(A) = \left[ \sum_{n \geq 0} \sum_{k=0}^{2^n-1} \frac{A(d^0_n, d^1_n)}{2^{-n \alpha}} \right]^{1/p}.
\]

Assume that \(\delta \Lambda = \sum_i H_i\) (finite sum). Then if \(\alpha \rho > 2\) we have

\[
\sup_{t, s \in \mathbb{D}} \frac{|A(s, t)|}{|t-s|^\alpha} \leq C Q_{\alpha, \rho}(A) + \sum_i \sup_{s < u < t} \frac{|H_i(s, u, t)|}{|t-u|^\rho |u-s|^\alpha}
\]

for any choice of \(\rho_1, \sigma_i > 0\) such that \(\sigma_i + \sigma_i \geq 2 - 2/p\).

Proof. Recall that \(A(a, b) = A(a, c) + A(c, d) + A(d, b) + \delta A(c, d, b) + \delta \Lambda(a, c, b)\). If \(t, s \in \mathbb{D}\) we have

\[
A(s, t) = A(s^\ell^-, t^\ell^-) + \sum_{k=\ell+1}^{\infty} A(t^{(k-1)-}, t^{k-}) + \sum_{k=\ell+1}^{\infty} A(s^{(k-1)-}, s^{k-}) \\
+ \sum_{k=\ell+1}^{\infty} \delta A(k^-, s^{(k-1)-}, s^{k-}) + \sum_{k=\ell+1}^{\infty} \delta A(t^{(k-1)-}, t^{k-}, s^{(k-1)-}, s^{k-})
\]

where \(\ell\) is the greatest integer which satisfies \(2^{\ell} < |t-s| \leq 2^{\ell+1}\). Then

\[
|A(t^{(k-1)-}, t^{k-})|^p \leq 2^{-k \alpha + 2k} Q_{\alpha, \rho}(A)^p
\]

for all \(k \geq \ell\) and

\[
|\delta A(k^-, s^{(k-1)-}, s^{k-})| \leq \sum_i |H_i(t^-, s^{(k-1)-}, s^{k-})| \leq \sum_i 2^{-\ell \sigma_i} 2^{-2k \nu_i} K_i
\]

where

\[
K_i = \sup_{s < u < t} \frac{|H_i(s, u, t)|}{|t-u|^\rho |u-s|^\alpha}.
\]
If $\alpha > 2/p$ we have
\[
|A(s, t)| \leq Q_{\alpha, p}(A) \left[ 2^{-\ell \alpha + 2\ell/p} + 2 \sum_{k=1}^{\infty} 2^{-k\alpha + 2k/p} \right] + \sum_i K_i \sum_{k=1}^{\infty} (2^{-\ell \alpha - k\ell/p} + 2^{-\ell \alpha - 2k\ell/p})\]
\[
\leq \left( Q_{\alpha, p}(A) + \sum_i K_i \right) |t - s|^{\alpha - 2/p}. \quad \Box
\]

To bound the martingale expectation, we will use the following Burkholder inequality:

**Lemma 4.** Let $m$ be a continuous local martingale with $m_0 = 0$. Then for all $T \geq 0$ and $p > 1$,
\[
E[\sup_{t \leq T} |m|^2] \leq C_p E[\langle m \rangle_T^p].
\]

**Proof.** Start by assuming that $m$ and $\langle m \rangle$ are bounded. Itô’s formula yields
\[
d|m|^2 = (2p)|m|^{2p-1} \, dm + \frac{1}{2} (2p)(2p-1)|m|^{2p-2} \, d\langle m \rangle,
\]
and therefore
\[
E[|m_T|^2] = C_p E \left[ \int_0^T |m_s|^{2p-2} \, d\langle m \rangle \right] \leq C_p E[\sup_{t \leq T} |m_t|^{2p-2} \langle m \rangle_T].
\]

By Cauchy–Schwarz we get
\[
E[|m_T|^2] \leq C_p E[\sup_{t \leq T} |m_t|^{2p}]^{2p-2}/(2p) E[\langle m \rangle_T^p]^{1/p}.
\]

But now Doob’s $L^p$ inequality yields $E[\sup_{t \leq T} |m_t|^{2p}] \leq C'_p E[|m_T|^2]$, and this implies the claim in the bounded case. The unbounded case can be treated with a localization argument. \quad \Box

### 1.3 Stochastic integrals

Stochastic integrals provide another source of solutions to eq. (1). Let us fix a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$ in the following.

**Lemma 5.** Assume that $M$ is a continuous martingale and $h$ an adapted process. Assume that $E[\|h\|^p] < \infty$ for any $p \geq 1$ and that $|d\langle M \rangle_t|/dt \leq L$. Let $I^0_{\text{Itô}}(h, M)$ be the Itô integral
\[
I^0_{\text{Itô}}(h, M)(t) = \int_0^t h(s) \, dM(s).
\]
Then a.s.
\[
\delta I^0_{\text{Itô}}(h, M)(s, t) - h(s) \delta M(s, t) = O(|t - s|^{|\alpha + \beta|})
\]
for any $\beta < 1/2$. 

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Proof. The hypothesis \(|d(M)/dt| \leq L|t-s|^p\) and Lemma 4 readily give
\[ E[|\delta M(s, t)|^2] \leq C_p L^p |t-s|^p \]
for all \(p \geq 1\). This implies \(E[Q_{1/2p}(M)^2] < \infty\) for all \(p \geq 1\), giving \(E[|M|^2_{L^{1/p}}] < \infty\) for all \(p > 2\). Let \(J_{(0)}(h, M)(s, t) = \delta I_{(0)}(h, M)(s, t) - h(s)\delta M(s, t) = \int_s^t \delta h(u) dM(u)\). Then
\[ |\delta I_{(0)}(h, M)(s, u)| = |\delta h(s, u)\delta M(u, t)| \leq \|h\|_\alpha\|M\|_\beta |t-u| |u-s|^\alpha \]
for all \(\beta < 1/2\). Next, using again Lemma 4, we have
\[ E[J_{(0)}(h, M)(s, t) |^2] \leq C_p E\left[ \left( \int_s^t (\delta h(s, u))^2 d\langle M \rangle_u \right)^2 \right] \leq C_p L^p E[\|h\|^2_\alpha] \left( \int_s^t |u-s|^{2\alpha} d\mu \right)^p \leq C_p L^p E[\|h\|^2_\alpha] |t-s|^{2p\alpha + 1/2}. \]
So we have that \(E[Q_{(1/2p)}(J_{(0)}(h, M))^2] < \infty\) for all \(p \geq 1\). Using Lemma 3 we can conclude that \(E[|J_{(0)}(h, M)|^2_{\frac{1}{\alpha} + 1/p}] < \infty\) for all \(p > 1/\alpha + \beta\). \(\square\)

1.4 Integration of closed 1-forms

Now consider \(x \in C(\mathbb{R}; \mathbb{R})^d \) and \(\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^d \) such that \(\nabla_x \varphi | = -\nabla_x \varphi | = 0\). This means that the one form \(\omega = \varphi_1(x) dx^1 \) is closed: \(d \omega = \nabla_x \varphi_1(x) dx^1 \wedge dx^1 = \frac{1}{2} (\nabla_x \varphi_1(x) - \nabla_x \varphi_1(x)) dx^1 \wedge dx^1 = 0\). Then
\[ -\delta(\varphi_1(x) dx^1) = \delta(\varphi_1(x) dx^1) = \nabla_x \varphi_1(x) dx^1 \wedge dx^1 + C_3^{\gamma} \]
\[ = \frac{1}{2} (\nabla_x \varphi_1(x) + \nabla_x \varphi_1(x)) dx^1 \wedge dx^1 + C_3^{\gamma} \]
\[ = \frac{1}{4} (\nabla_x \varphi_1(x) + \nabla_x \varphi_1(x))(\delta x^1 \delta x^1 + \delta x^1 \delta x^1) + C_3^{\gamma} \]
\[ = \nabla_x \varphi_1(x) \delta S^{1/j} + C_3^{\gamma} = \delta(\nabla_x \varphi_1(x) S^{1/j}) + C_3^{\gamma} \]
with \(S^{1/j}(s, t) = \frac{1}{2} \delta x^1(s, t) \delta x^1(s, t) \). In other words
\[ \delta(\varphi_1(x) \delta x^1 + \nabla_x \varphi_1(x) S^{1/j}) \in C_3^{\gamma} \]
which means that there exists a unique \(y \) such that
\[ \delta y = \varphi_1(x) \delta x^1 + \nabla_x \varphi_1(x) S^{1/j} + C_3^{\gamma} \]
Let us call
\[ y(t) = \int_0^t \varphi_1(x(s)) dx^1(s). \]
In \(\mathbb{R}^d\) the fact that \(\omega \) is closed, implies that it is also exact: there exists \(\psi : \mathbb{R}^d \rightarrow \mathbb{R} \) such that \(\omega = dy\). This means that \(\varphi_1 = \nabla_x \psi\). Now, Taylor expansion gives
\[ \delta \psi(x) = \nabla_x \psi(x) \delta x^1 + \nabla_x \psi S^{1/j} + O(|t-s|^{\gamma/3}) \]
so we can identify $\delta y = \delta \psi(x)$ and we have

$$\psi(x(t)) - \psi(x(0)) = \int_0^t \phi_i(x(s))dx^i(s).$$

Valid until $\gamma > 1/3$. Similar results hold for any $\gamma > 0$. When $d = 1$ any one-form is exact so this result allow to integrate an arbitrary function along an arbitrary Hölder path. When $d > 1$ the closedness condition is non–trivial and only particular one–forms can be integrated, namely those which are differentials of scalar functions.

2 Rough paths

In the following we will fix $\gamma > 1/3$ and an interval $I \subseteq \mathbb{R}$. All the Hölder spaces will considered on this interval unless specified otherwise.

**Definition 6.** A $\gamma$-Hölder rough path $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ in $\mathbb{R}^d$ is a pair

$$\mathbb{X}^1 \in C^2_r(\mathbb{R}^d), \quad \mathbb{X}^2 \in C^{2\gamma}_r(\mathbb{R}^d \otimes \mathbb{R}^d)$$

with $\mathbb{X}^1 = \delta x$ for some $x \in C^r(I; \mathbb{R}^d)$ and satisfying the Chen relation :

$$\delta \mathbb{X}^2(s, u, t) = \mathbb{X}^1(s, u) \mathbb{X}^1(u, t), \quad s < u < t.$$

We say that $\mathbb{X}$ lies above $x$. We denote collectively $(X^i)_{i=1,...,d}$ and $(\dot{X}^{i,j})_{i,j=1,...,d}$ the components of $\mathbb{X}^1$ and $\mathbb{X}^2$ with respect to the canonical basis of $\mathbb{R}^d$ and $\mathbb{R}^d \otimes \mathbb{R}^d$. We denote by $\mathcal{C}^r(\mathbb{R}^d)$ the space of the $\gamma$-Hölder rough paths in $\mathbb{R}^d$ and we let

$$\|\mathbb{X}\|_{\mathcal{C}^r} = \|\mathbb{X}^1\|_r + \|\mathbb{X}^2\|_{2\gamma}.$$

On $\mathcal{C}^r$ we consider the distance $d_{\mathcal{C}^r}(\mathbb{X}, \mathbb{Y}) = \|\mathbb{X} - \mathbb{Y}\|_{\mathcal{C}^r}$. With $\mathcal{C}^r_2(\mathbb{R}^d) \subseteq \mathcal{C}^r(\mathbb{R}^d)$ we denote the subset of rough paths lying above a given $x \in C^r(I; \mathbb{R}^d)$, the “fiber” at $x$.

i. The space of rough paths is not a linear space since the Chen relation is non–linear.

ii. We can interpret the data $X^i, X^{i,j}$ as the given of the (abstract) iterated integrals

$$X^i(s, t) = \int_{s < u < t} dx^i(u), \quad X^{i,j}(s, t) = \int_{s < u < v < t} dx^i(u)dx^j(v)$$

together with suitable regularity as elements in $C_2$.

iii. When $\gamma > 1/2$ there can be at most only one rough path above a given path $x$. It is given by

$$\mathbb{X}^1(s, t) = x(t) - x(s), \quad \mathbb{X}^2(s, t) = \int_s^t \mathbb{X}^1(s, u) \otimes d_u x(u)$$

where the integral is understood in Young sense (or as a classical Lebesgue integral if $x \in C^1$). This rough path is called the canonical lift of $x$.  

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iv. Take \( \gamma < 1/2 \). If \( \mathbb{X} \in \mathcal{C}_\gamma \) and \( \varphi \in C^{2\gamma}(I; \mathbb{R}^d \otimes \mathbb{R}^d) \) then \( \mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2 + \delta \varphi) \) is also an element of \( \mathcal{C}_\gamma \) and all of them have this form. In particular there are infinitely many rough paths above the same path if \( \gamma < 1/2 \) (or none). The fiber \( \mathcal{C}_\gamma \) is an affine space with vector space \( C^{2\gamma}(I; \mathbb{R}^d \otimes \mathbb{R}^d) \) and action \((\mathbb{X}, \varphi) \mapsto (\mathbb{X}^1, \mathbb{X}^2 + \delta \varphi)\). Elements in \( \mathcal{C}_0 \approx C^{2\gamma}(I; \mathbb{R}^d \otimes \mathbb{R}^d) \) are called pure area rough paths.

Lemma 7. Let \( \gamma < 1/2 \), then \( \mathcal{C}_\gamma \) is not empty.

Proof. Let \( x \in C^\gamma \). Fix \( p < \gamma \) and consider a sequence \((y_n \in C^1)_{n \geq 1}\) such that \( x = \sum_n y_n \) in \( C^\gamma \) and 
\[
\|y_n\|_\infty + 2^{-p}\|y_n\|_\gamma \lesssim 2^{-\gamma p}\|x\|_\gamma
\]
(such a sequence always exists). Let \( \mathbb{X}_n = \sum_{k \leq n} \delta y_k \) and define recursively \( \mathbb{X}_n^2 \) as
\[
\mathbb{X}_n^2(s, t) = \mathbb{X}_n(s, t) + \mathbb{X}_n^1(s, t)y_{n+1}(t) - y_{n+1}(s)\mathbb{X}_n^1(s, t) - y_{n+1}(s)\delta y_{n+1}(s, t)
\]
then if \( \delta \mathbb{X}_n^2 = \mathbb{X}_n^1 \mathbb{X}_n^1 \) we have also
\[
\delta \mathbb{X}_n^2(s, u) = \mathbb{X}_n^1(s, u)\mathbb{X}_n^1(u, t) + \mathbb{X}_n^1(s, u)\delta y_{n+1}(u, t) + \delta y_{n+1}(s, u)\mathbb{X}_n^1(u, t) + \delta y_{n+1}(s, u)\delta y_{n+1}(u, t)
\]
Moreover
\[
|\mathbb{X}_n^2(s, t) - \mathbb{X}_n^2(s, t)| \leq |\mathbb{X}_n^2(s, t)y_{n+1}(t)| + |y_{n+1}(s)\mathbb{X}_n^1(s, t)| + |y_{n+1}(s)\delta y_{n+1}(s, t)|
\]
\[
\lesssim \|x\|^2 2^{-\gamma(n+1)}|t-s|^{2n \gamma} \lesssim \|x\|^2|t-s|^{2n \gamma}
\]
so the sequence \((\mathbb{X}_n^2)_{n \geq 1}\) converges in \( C^{2\gamma} \) to an element which we call \( \mathbb{X}^2 \) and such that \( \delta \mathbb{X}^2 = \mathbb{X}^1 \mathbb{X}^1 \). We have
\[
|\mathbb{X}^2(s, t)| \leq \sum_n \|\mathbb{X}_n^2(s, t) - \mathbb{X}_n^2(s, t)\|
\]
\[
= \sum_{n:2^n|t-s| \leq 1} \|\mathbb{X}_n^2(s, t) - \mathbb{X}_n^2(s, t)\| + \sum_{n:2^n|t-s| > 1} \|\mathbb{X}_n^2(s, t) - \mathbb{X}_n^2(s, t)\|
\]
\[
\lesssim \|x\|^2 \sum_{n:2^n|t-s| \leq 1} 2^{-2n}m|t-s| + \|x\|^2 \sum_{n:2^n|t-s| > 1} 2^{-m}|t-s|^\gamma \lesssim \|x\|^2|t-s|^{\gamma}
\]
Setting \( \mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2) \) we have \( \mathbb{X} \in \mathcal{C}_\gamma \) as required.

When \( \gamma > 1/2 \) rough paths satisfy an additional algebraic relation, called the shuffle relation:
\[
\mathbb{X}(s, t)\mathbb{X}'(s, t) = \mathbb{X}^{ij}(s, t) + \mathbb{X}^{ij}(s, t).
\]

Definition 8. We call weakly geometric rough paths satisfying the relation eq. (2) and denote them collectively with \( \mathcal{C}_\gamma^w \). Moreover we denote by \( \mathcal{C}_\gamma^g \) the closure of \( \mathcal{C}_\gamma^1 \) in \( \mathcal{C}_\gamma^w \) and call them geometric rough paths.
If $\gamma > 1/2$ we have $\mathcal{C}^\gamma = \mathcal{C}_{wg}^\gamma = \mathcal{C}^\gamma$. Since elements of $\mathcal{C}^1$ satisfy the shuffle relation, this will remain valid also for all elements of $\mathcal{C}_{g}^\gamma$, so $\mathcal{C}_{g}^\gamma \subseteq \mathcal{C}_{wg}^\gamma \subseteq \mathcal{C}^\gamma$ for any $\gamma$. As far as the relation between $\mathcal{C}_{g}^\gamma$ and $\mathcal{C}_{wg}^\gamma$ is concerned we have the following result

**Theorem 9.** For every $\chi \in \mathcal{C}_{wg}^\gamma$ there exists a sequence $(\chi_n)_{n \geq 1}$ in $\mathcal{C}^1$ such that $\chi_n \to \chi$ in $\mathcal{C}^\rho$ for any $\rho < \gamma$. In particular $\mathcal{C}_{g}^\gamma \subseteq \mathcal{C}_{wg}^\gamma \subseteq \mathcal{C}^\rho$.

As a preliminary to a proof of this theorem let us discuss a particular case, the approximation theory for pure area rough paths.

**Theorem 10.** Assume $\gamma < 1/2$ and let $\varphi \in C^{2\gamma}(I; \mathbb{R}^d \otimes \mathbb{R}^d)$ then there exists $x_n \in C^{1}$ such that the canonical lift $\chi_n$ converges in $\mathcal{C}^\rho$ to the pure area path $\chi = (0, \delta \varphi)$ for any $\rho < \gamma$.

**Proof.** Let $(\varphi_n)_n$ a sequence in $C^1$ converging to $\varphi$ in $C^{2\rho}$ for some $\rho < \gamma$ and such that $\|\varphi_n\|_\infty \leq 2^{n-\rho n} \|\varphi\|_\rho$. Fix sufficiently large positive numbers $(L_{ij})_{i,j=1,\ldots,d}$ all different one from the other and let

$$x_n^i(t) = \frac{1}{2} \sum_j \int_s^t (\varphi_n^i(u) - \varphi_n^i(s)) du x_n^j(u) - \frac{1}{2} \sum_j \int_s^t (\varphi_n^j(u) - \varphi_n^j(s)) du x_n^i(u).$$

By a long but direct estimation we can show that

$$\left\| \int_s^t (x_n^i(u) - x_n^i(s)) du x_n^j(u) - \frac{1}{2} \sum_j \int_s^t (\varphi_n^j(u) - \varphi_n^j(s)) du x_n^i(u) \right\|_{C^1_{2\rho}} \to 0$$

as $n \to \infty$. Moreover

$$\frac{1}{2} \int_s^t (\varphi_n^i(u) - \varphi_n^i(s)) du \to \delta \varphi^i(s, t)$$

in $C_{2\rho}^{2\gamma}$. Since $x_n \to 0$ in $C^{1/2-}$ the claim follows. In order to show the main estimate note that

$$\int_s^t (x_n^i(u) - x_n^i(s)) du x_n^j(u) =$$

$$= - \sum_{k,m} \int_s^t 2^{-L_{ik}n/2} 2^{L_{km}n/2} [\varphi_n^k(u) \sin(2L_{ik} u) - \varphi_n^k(s) \sin(2L_{ik} s)] \sin(2L_{km} u) du$$

$$+ \sum_{k,m} \int_s^t 2^{-L_{ik}n/2} 2^{L_{km}n/2} [\varphi_n^m(u) \cos(2L_{ik} u) - \varphi_n^m(s) \cos(2L_{ik} s)] \cos(2L_{km} u) du$$

$$- \sum_{k,m} \int_s^t 2^{-L_{ik}n/2} 2^{L_{km}n/2} [\cos(2L_{ik} u) - \cos(2L_{ik} s)] \sin(2L_{km} u) du$$

$$+ \sum_{k,m} \int_s^t 2^{-L_{ik}n/2} 2^{L_{km}n/2} [\varphi_n^k(u) \sin(2L_{ik} u) - \varphi_n^k(s) \sin(2L_{ik} s)] \varphi_n^m(u) \cos(2L_{km} u) du$$
and that by trigonometric identities all the integrals contains oscillating factors with frequencies of the form $2^{m+1}n \pm 2^{m-j}n$ which are different from zero unless $k = m$. Moreover the only terms which produce non-oscillating factors are those of the form $\sin \phi \cos \phi$ which are then linear in $\phi$. Using integration by parts and the fact that the numbers $L$ are large enough to beat the growth of $\phi_n$, all the oscillating terms can be shown to go to zero (one use also the fact that boundary terms vanishes). The claim follows directly.

\qed

**Proof.** (of Thm. 9) The general case is similar to the pure area case. Let $X \in \mathcal{C}_{\text{wg},t}$ for some $x \in C^\gamma$. Fix $\rho < \gamma$ and consider a sequence $(y_n \in C^1)_{n \geq 1}$ such that $x = \sum_n y_n$ in $C^\rho$ and $\|y_n\|_\infty + 2^{-n}\|y_n\|_\infty \leq 2^{-nr}\|x\|_f$ (such a sequence always exists). Let $y_{\leq n} = \sum_{k \leq n} y_k$ and

$$x_n(t) = y_{\leq n} + \frac{1}{2} \sum_j \phi_n^{ij}(t) 2^{-L_{ij}n^2} \sin(2^{2L_{ij}}t) + \sum_j 2^{-L_{ij}n^2} \cos(2^{2L_{ij}}t).$$

(notations as in the previous theorem) where $\phi_n$ is for the moment an indeterminate sequence. Using the same ideas as above we can show that

$$\left\| \int_t^\tau (x_n^j(s) - x_n^j(u))ds + \int_t^\tau (y_{\leq n}^j(s) - y_{\leq n}^j(u))ds - \frac{1}{2} \int_t^\tau (\phi_n^{ij}(u) - \phi_n^{ij}(us))du \right\|_{C^2_{\text{f}}} \to 0.$$

Now the point is that we can choose the sequence $\phi$ such that it cancels the contribution of the antisymmetric part of $\int_t^\tau (y_{\leq n}^j(s) - y_{\leq n}^j(u))ds$ and replaces it with an approximation (converging in $C^{2r}$) of the antisymmetric part of $\mathcal{X}$, we leave the details to the reader.

Weakly geometric rough paths can be approximated by lifts of smooth paths by loosing just a bit regularity in the convergence statement. Approximation of general rough path is less clear. In particular we cannot hope to approximate a general rough path with smooth canonical lifts since the shuffle relation is not true in the limit. But as we will now see, this is the only obstruction.

Let $X \in \mathcal{C}^\gamma$ and consider the defect in the shuffle relation

$$D^ij(s,t) = X^i(s,t)X^j(s,t) - X^ij(s,t) - X^j(s,t).$$

A simple computation shows that $\delta D^ij = 0$, indeed:

$$\delta D^ij(s,u,t) = X^i(s,u)X^j(u,t) + X^i(s,u)X^j(u,t) - X^i(s,u)X^j(u,t) - X^i(s,u)X^j(u,t) = 0.$$

Moreover $D \in C^\infty_\delta(R^d \otimes R^d)$ where $R^d \otimes R^d$ denotes the symmetric tensor product. Then there exists a function $d \in C^\infty(I; R^d \otimes R^d)$ such that $D = \delta d$. We can define now $\mathcal{X}_d = (\mathcal{X}^1, \mathcal{X}^2 + \delta d/2)$ and check that $\mathcal{X}_d \in \mathcal{C}_{\text{wg}}$,

$$X^i_d(s,t)X^j_d(s,t) = X^i(s,t)X^j(s,t) = X^ij(s,t) + X^j(s,t) + D^ij(s,t)$$

$$= X^ij(s,t) + \frac{1}{2} \delta d^ij(s,t) + X^j(s,t) + \frac{1}{2} \delta d^ij(s,t) = X^ij_d(s,t) + X^j_d(s,t).$$
So to every rough path $\mathcal{X} \in \mathcal{C}_2^T$ lying above $x$ we can associate a geometric rough path $\mathcal{X}_g \in \mathcal{C}_g^{T,x}$ by modifying its the symmetric part of its second order component. Note that this projection is not unique since there are a priori many weakly geometric rough paths above the same path, differing one from the other by an antisymmetric increment in the second order component: indeed $\mathcal{X}_g = (\mathcal{X}_g^1, \mathcal{X}_g^2 + \delta \varphi)$ with $\varphi \in C^{2z}(I; \mathbb{R}^d \otimes \mathbb{R}^d)$ is again in $\mathcal{C}_g^{T,x}$.

This construction shows the existence of an isomorphism of metric spaces:

$$\mathcal{C}_r(\mathbb{R}^d) \cong \mathcal{C}_r^{w_g}(\mathbb{R}^d) \times C^{2z}(I; \mathbb{R}^d \otimes \mathbb{R}^d).$$

(see Hairer–Kelly for a generalisation of these considerations)

### 3 Controlled paths

**Definition 11.** A pair $(h, h^X)$ where $h \in C^r([0, 1]; V)$ and $h^X \in C^r([0, 1]; \mathcal{L}(\mathbb{R}^d; V))$ is a path controlled by $X$ if

$$h^X(s,t) = \delta h(s,t) - h^X(s)\mathcal{X}^1(s,t) \in C_2^{2z}(V).$$

We denote by $\mathcal{D}_X^2(V)$ the linear space space of paths controlled by $X$ and on $\mathcal{D}_X^2$ we consider the semi–norm

$$\|(h, h^X)\|_{\mathcal{D}_X^2} = \|h^X\|_r + \|h^2\|_{2z}.$$

Given a rough path $\mathcal{X}$ and path $(h, h^X) \in \mathcal{D}_X^2(\mathcal{L}(\mathbb{R}^d; V))$ controlled by $\mathcal{X}$ we can define a new controlled path $(z, z^X) \in \mathcal{D}_X^2(V)$ by letting $z^X = h$ and $z$ the unique solution to

$$\delta z(s,t) = h(s)\mathcal{X}^1(s,t) + h^X(s)\mathcal{X}^2(s,t) + z^2(s,t)$$

with $z^X \in C_2^{2z}(V)$. We call it the integral of $h$ with respect to $\mathcal{X}$.

**Theorem 12.** Given a rough path $\mathcal{X}$ and path $(h, h^X) \in \mathcal{D}_X^2(\mathcal{L}(\mathbb{R}^d; V))$ controlled by $\mathcal{X}$ we can define a new controlled path $(z, z^X) \in \mathcal{D}_X^2(V)$ by letting $z^X = h$ and $z$ the unique solution to

$$\delta z(s,t) = h(s)\mathcal{X}^1(s,t) + h^X(s)\mathcal{X}^2(s,t) + z^2(s,t)$$

with $z^X \in C_2^{2z}(V)$. We call it the integral of $h$ with respect to $\mathcal{X}$ and we have

$$\|z^X\|_{2z} \leq \|(h, h^X)\|_{\mathcal{D}_X^2}(1 + \|\mathcal{X}\|_{2z})$$

and

$$\|(z, z^X)\|_{\mathcal{D}_X^2} \leq \|\mathcal{X}\|_{2z}(1(\|h^X\|_{2z} + \|\mathcal{X}\|_{2z})).$$
**Proof.** Such a path is clearly unique and is well defined since if we let

\[ A(s, t) = h(s)X^1(s, t) + h^X(s)X^2(s, t) \]

we have

\[
\delta A(s, u, t) = -\delta h(s, u)X^1(u, t) - \delta h^X(s, u)X^2(u, t) + h^X(s)\delta X^2(s, u, t)
\]

\[
= -(\delta h(s, u) - h^X(s)X^1(s, u))X^1(u, t) - \delta h^X(s, u)X^2(u, t) = -h^X(s, u)X^1(u, t) - \delta h^X(s, u)X^2(u, t)
\]

and by assumption \( \delta A \in C^2_\gamma(V) \) so that we can apply the sewing map to obtain

\[ z^3 = \Lambda(h^X + \delta h^X)X^2) \in C_2^\gamma(V). \]

Then

\[
\|[z^X]_{\gamma, r} \leq \|[h^X]_{\infty, \gamma}\|X^1_{\gamma, r} + \|[h^X]_{2\gamma, \gamma} + \|[z^X]_{2\gamma, \gamma} \leq \|[h^X]_{\infty, \gamma}\|X^2_{2\gamma, \gamma} + \tau\|[z^X]_{3\gamma, \gamma}
\]

and the final bound follows. \( \Box \)

**Lemma 13.** Let \( (f, f^X) \in D^2_{\gamma} (V) \) and let \( \varphi \in C^2(V; W) \) then \( (\varphi(f), \varphi(f)^X) \in D^2_{\gamma} (W) \) where \( \varphi(f)(t) = \varphi(f(t)) \) and \( \varphi(f)^X = \nabla \varphi(f)f^X \in C^2(\mathbb{R}^d, W) \). Moreover

\[ \|(\varphi(f), \varphi(f)^X)\|_{D^2_{\gamma}} \leq C_\varphi(1 + \|(f, f^X)\|_{D^2_{\gamma}})^2. \]

**Proof.** Taylor expansion gives

\[
\delta \varphi(f)(s, t) = \int_0^1 \frac{d}{dt} \varphi(f(s) + t\delta f(s, t))\delta f(s, t)
\]

\[
= \nabla \varphi(f(s))\delta f(s, t) + \int_0^1 (1 - \tau)\frac{d}{dt} \nabla^2 \varphi(f(s) + \tau\delta f(s, t))\delta f(s, t) \otimes \delta f(s, t)
\]

\[
= \nabla \varphi(f(s))\delta f(s, t) + O(|t - s|^2\tau)
\]

Using the controlled hypothesis on \( f \) we get

\[
\delta \varphi(f)(s, t) = \nabla \varphi(f(s))f^X(s)X^1(s, t) + \varphi(f)^X(s, t)
\]

where

\[
\varphi(f)^X(s, t) = \nabla \varphi(f(s))f^X(s) + \int_0^1 (1 - \tau)\frac{d}{dt} \nabla^2 \varphi(f(s) + \tau\delta f(s, t))\delta f(s, t) \otimes \delta f(s, t).
\]

Then we can let \( \varphi(f)^X(s) = \nabla \varphi(f(s))f^X(s) \) and observe that

\[
\|\varphi(f)^X\|_{2\gamma} \leq \|\nabla \varphi\|_{\infty}\|f^X\|_{2\gamma} + \|\nabla^2 \varphi\|_{\infty}\|f\|_{2\gamma}^2 \leq C_\varphi(1 + \|(f, f^X)\|_{D^2_{\gamma}})^2.
\]

\( \Box \)