# Sheet 3

rev 1 - 20160121

- Beyond Young
- Rough paths
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# 1 Beyond Young

When trying to go beyond Young integral we face the fundamental problem that we cannot expect the integral to be a continuous operations (recall our basic counterexample).

## 1.1 A general existence result: the para-integral

A natural question is that if it is always possible to find a function $I(f,g)$ which satisfies

$$\delta I(f,g)(s,t) = f(s)\delta g(s,t) + O(|t-s|^{\alpha+\beta}) \tag{1}$$

with $f \in C^\alpha$ and $g \in C^\beta$ but with $\alpha+\beta<1$.

**Remark 1.**

i. Uniqueness does not hold anymore since if $I$ is a solution then $\hat{I} = I + \varphi$ is also a solution of (1) for any $\varphi \in C^{\alpha+\beta}$.

ii. Find such a function is equivalent to ask for a solution $J(f,g) \in C_2^{\alpha+\beta}$ of

$$\delta J(f,g)(s,u,t) = \delta f(s,u)\,\delta g(u,t)$$

since then we can let $\delta I(f,g) = f\delta g - J(f,g)$.

iii. We can always consider

$$J_s(f,g)(s,t) = \frac{1}{2}\delta f(s,t)\delta g(s,t)$$

for which we have

$$\delta J_s(f,g) = \frac{\delta f \delta g + \delta g \delta f}{2}$$

and

$$\|J_s(f,g)\|_{\alpha+\beta} \leqslant \|f\|_\alpha \|g\|_\beta.$$

So in case $f = g$ we can always take $J(f,g) = J_s(f,g)$.

iv. If we consider $J_1(f,g)(s,t) = -f(s)\delta g(s,t)$ we have

$$\delta J_1(f,g)(s,u,t) = -f(s)\delta g(s,t) + f(s)\delta g(s,u) + f(u)\delta g(u,t) = \delta f(s,u)\delta g(u,t)$$

as required (indeed it differs from the iterated integral by the increment of a function). However the regularity is not ok, indeed we have only

$$|J_1(f,g)(s,t)| \leqslant \|f\|_\infty \|g\|_\beta |t-s|^\beta.$$

Another possibility is

$$J_2(f,g)(s,t) = \delta f(s,t)g(t) = J_1(f,g)(s,t) + f(t)g(t) - f(s)g(s)$$

since it differs from the previous one by the increment of the function $t \mapsto f(t)g(t)$. Still the regularity is not ok.

A decomposition of the functions $f, g$ into an infinite sequence of blocks living in different scales will allow us to combine the observations contained in the Remark 1 (iv) to produce a map (in general not unique) solving eq. (1).

**Theorem 2.** *(Paraintegral) For any $\alpha, \beta > 0$ such that $\alpha + \beta < 1$ there exists a continuous map $J_<$: $C^\alpha \times C^\beta \to C_2^{\alpha+\beta}$ such that*

$$\delta J_<(f,g)\,(s,u,t) = \delta f(s,u)\,\delta g(u,t), \qquad s < u < t.$$

**Proof.** Let $\rho$: $\mathbb{R} \to \mathbb{R}_+$ smooth, compactly supported around 0 and of integral one. Let $\rho_n(t) = 2^n \rho(2^n t)$ and $f_n = \rho_n * f - \rho_{n-1} * f$ for $n \geqslant 1$ with $f_0 = \rho_0 * f$ and a similar definiton for $g_n$. With these definitions we have $f(t) = \sum_n f_n(t)$. Direct estimates give that

$$|f_n(t)| \lesssim \|f\|_\alpha 2^{-n\alpha}, \qquad |\partial_t f_n(t)| \lesssim \|f\|_\alpha 2^{n-n\alpha},$$

and similar estimates for $g_n$ where the constants depends only on $\rho$. These estimate also show that the sum of the series converges uniformly in $t$. Now take the combination of $J_1$ and $J_2$ given by

$$J_<(f,g) = \sum_{m \leqslant n} J_1(f_n, g_m) + \sum_{m > n} J_2(f_n, g_m)$$

and note that

$$\delta J_<(f,g) = \sum_{m \leqslant n} \delta J_1(f_n, g_m) + \sum_{m > n} \delta J_2(f_n, g_m) = \sum_{m \leqslant n} \delta f_n \delta g_m + \sum_{m > n} \delta f_n \delta g_m = \delta f \delta g$$

as required. But now if $0 < \alpha + \beta < 1$ we can estimate

$$|J_1(f_n, g_m)(s,t)| \lesssim \|f\|_\alpha \|g\|_\beta 2^{-n\alpha - m\beta}(1 \wedge 2^m |t-s|),$$

$$|J_2(f_n, g_m)(s,t)| \lesssim \|f\|_\alpha \|g\|_\beta 2^{-n\alpha - m\beta}(1 \wedge 2^n |t-s|).$$

so

$$|J_{<}(f,g)(s,t)| \lesssim \|f\|_\alpha \|g\|_\beta \sum_{m\leqslant n} 2^{-n\alpha-m\beta}(1\wedge 2^m|t-s|) + \|f\|_\alpha\|g\|_\beta \sum_{m>n} 2^{-n\alpha-m\beta}(1\wedge 2^n|t-s|)$$

$$\lesssim \|f\|_\alpha\|g\|_\beta \sum_m 2^{-m\alpha-m\beta}(1\wedge 2^m|t-s|) + \|f\|_\alpha\|g\|_\beta \sum_n 2^{-n\alpha-n\beta}(1\wedge 2^n|t-s|)$$

$$\lesssim \|f\|_\alpha\|g\|_\beta \sum_{m:2^m|t-s|>1} 2^{-m\alpha-m\beta} + \|f\|_\alpha\|g\|_\beta|t-s| \sum_{m:2^m|t-s|\leqslant 1} 2^{m-m\alpha-m\beta} \lesssim \|f\|_\alpha\|g\|_\beta|t-s|^{\alpha+\beta}$$

which implies $\|J_{<}(f,g)\|_{\alpha+\beta} \lesssim \|f\|_\alpha\|g\|_\beta$. □

## 1.2 Some tools

**Regularity of 2-increments.**

**Lemma 3.** *Let $A:\mathbb{D}\times\mathbb{D}\to V$ and*

$$Q_{\alpha,p}(A) = \left[\sum_{n\geqslant 0} 2^{-2n} \sum_{k=0}^{2^n-1} \left|\frac{A(d_i^n, d_{i+1}^n)}{2^{-n\alpha}}\right|^p\right]^{1/p}.$$

*Assume that $\delta A = \sum_i H_i$ (finite sum). Then if $\alpha p > 2$ we have*

$$\sup_{t,s\in\mathbb{D}} \frac{|A(s,t)|}{|t-s|^{\alpha-2/p}} \leqslant C\, Q_{\alpha,p}(A) + \sum_i \sup_{s<u<t\in\mathbb{D}} \frac{|H_i(s,u,t)|}{|t-u|^{\rho_i}|u-s|^{\sigma_i}}$$

*for any choice of $\rho_i, \sigma_i > 0$ such that $\rho_i + \sigma_i \geqslant \alpha - 2/p$.*

**Proof.** Recall that $A(a,b) = A(a,c) + A(c,d) + A(d,b) + \delta A(c,d,b) + \delta A(a,c,b)$. If $t,s\in\mathbb{D}$ we have

$$A(s,t) = A(s^{\ell-}, t^{\ell-}) + \sum_{k=\ell+1}^{\infty} A(t^{(k-1)-}, t^{k-}) + \sum_{k=\ell+1}^{\infty} A(s^{(k-1)-}, s^{k-})$$
$$+ \sum_{k=\ell+1}^{\infty} \delta A(t^{k-}, s^{(k-1)-}, s^{k-}) + \sum_{k=\ell+1}^{\infty} \delta A(t^{(k-1)-}, t^{k-}, s^{k-})$$

where $\ell$ is the greatest integer which satisfies $2^{-\ell-1} < |t-s| \leqslant 2^{-\ell}$. Then

$$|A(t^{(k-1)-}, t^{k-})|^p \leqslant 2^{-k\alpha p+2k} Q_{\alpha,p}(A)^p$$

for all $k \geqslant \ell$ and

$$|\delta A(t^{k-}, s^{(k-1)-}, s^{k-})| \leqslant \sum_i |H_i(t^{k-}, s^{(k-1)-}, s^{k-})| \leqslant \sum_i 2^{-\ell\sigma_i} 2^{-k\rho_i} K_i$$

where

$$K_i = \sup_{s<u<t\in\mathbb{D}} \frac{|H_i(s,u,t)|}{|t-u|^{\rho_i}|u-s|^{\sigma_i}}.$$

If $\alpha > 2/p$ we have

$$|A(s,t)| \leqslant Q_{\alpha,p}(A)\left[2^{-\ell\alpha+2\ell/p} + 2\sum_{k=\ell+1}^{\infty} 2^{-k\alpha+2k/p}\right] + \sum_i K_i \sum_{k=\ell+1}^{\infty} (2^{-\ell\sigma_i}2^{-k\rho_i} + 2^{-\ell\rho_i}2^{-k\sigma_i})$$

$$\lesssim \left(Q_{\alpha,p}(A) + \sum_i K_i\right)|t-s|^{\alpha-2/p}. \qquad \square$$

To bound the martingale expectation, we will use the following Burkholder inequality:

**Lemma 4.** *Let $m$ be a continuous local martingale with $m_0 = 0$. Then for all $T \geqslant 0$ and $p > 1$,*

$$E[\sup_{t\leqslant T} |m_t|^{2p}] \leqslant C_p E[\langle m\rangle_T^p].$$

**Proof.** Start by assuming that $m$ and $\langle m\rangle$ are bounded. Itô's formula yields

$$\mathrm{d}|m_t|^{2p} = (2p)|m_t|^{2p-1}\,\mathrm{d}m_t + \frac{1}{2}(2p)(2p-1)|m_t|^{2p-2}\,\mathrm{d}\langle m\rangle_t,$$

and therefore

$$E[|m_T|^{2p}] = C_p E\left[\int_0^T |m_s|^{2p-2}\,\mathrm{d}\langle m\rangle_s\right] \leqslant C_p E[\sup_{t\leqslant T} |m_t|^{2p-2}\langle m\rangle_T].$$

By Cauchy–Schwartz we get

$$E[|m_T|^{2p}] \leqslant C_p E[\sup_{t\leqslant T} |m_t|^{2p}]^{(2p-2)/2p} E[\langle m\rangle_T^p]^{1/p}.$$

But now Doob's $L^p$ inequality yields $E[\sup_{t\leqslant T} |m_t|^{2p}] \leqslant C_p' E[|m_T|^{2p}]$, and this implies the claim in the bounded case. The unbounded case can be treated with a localization argument. $\square$

## 1.3 Stochastic integrals

Stochastic integrals provide another source of solutions to eq. (1). Let us fix a given filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_t)$ in the following.

**Lemma 5.** *Assume that $M$ is a continuous martingale and $h$ an adapted process. Assume that $\mathbb{E}[\|h\|_\alpha^p], +\infty$ for any $p \geqslant 1$ and that $|\mathrm{d}\langle M\rangle_t/\mathrm{d}t| \leqslant L$. Let $I_{\text{Itô}}(h,M)$ be the Itô integral*

$$I_{\text{Itô}}(h,M)(t) = \int_0^t h(s)\mathrm{d}M(s).$$

*Then a.s.*

$$\delta I_{\text{Itô}}(h,M)(s,t) - h(s)\delta M(s,t) = O(|t-s|^{\alpha+\beta})$$

*for any $\beta < 1/2$.*

**Proof.** The hypothesis $|\mathrm{d}\langle M\rangle_t/\mathrm{d}t|\leqslant L$ and Lemma 4 readily give

$$\mathbb{E}|\delta M(s,t)|^{2p}\leqslant C_p L^p|t-s|^p$$

for all $p\geqslant 1$. This implies $\mathbb{E}[Q_{1/2,2p}(M)^{2p}]<\infty$ for all $p\geqslant 1$, giving $\mathbb{E}\|M\|_{1/2-1/p}^{2p}<\infty$ for all $p>2$. Let $J_{\text{Itô}}(h,M)(s,t)=\delta I_{\text{Itô}}(h,M)(s,t)-h(s)\delta M(s,t)=\int_s^t\delta h(s,u)\mathrm{d}M(u)$. Then

$$|\delta J_{\text{Itô}}(h,M)(s,u,t)|=|\delta h(s,u)\delta M(u,t)|\leqslant\|h\|_\alpha\|M\|_\beta|t-u|^\beta|u-s|^\alpha$$

for all $\beta<1/2$. Next, using again Lemma 4, we have

$$\mathbb{E}|J_{\text{Itô}}(h,M)(s,t)|^{2p}\leqslant C_p\,\mathbb{E}\left\{\left[\int_s^t(\delta h(s,u))^2\mathrm{d}\langle M\rangle_u\right]^p\right\}\leqslant C_p L^p\mathbb{E}[\|h\|_\alpha^{2p}]\left[\int_s^t|u-s|^{2\alpha}\mathrm{d}u\right]^p$$

$$\leqslant C_p L^p\mathbb{E}[\|h\|_\alpha^{2p}]|t-s|^{2p(\alpha+1/2)}.$$

So we have that $\mathbb{E}[Q_{\alpha+1/2,2p}(J_{\text{Itô}}(h,M))^{2p}]<\infty$ for all $p\geqslant 1$. Using Lemma 3 we can conclude that $\mathbb{E}\|J_{\text{Itô}}(h,M)\|_{\alpha+\beta-1/p}^{2p}<\infty$ for all $p>1/(\alpha+\beta)$. $\square$

## 1.4 Integration of closed 1-forms

Now consider $x\in C^\gamma([0,1];\mathbb{R}^d)$ and $\varphi:\mathbb{R}^d\to\mathbb{R}^d$ such that $\nabla_i\varphi_j-\nabla_j\varphi_i=0$. This means that the one form $\omega=\varphi_i(x)\mathrm{d}x^i$ is closed: $\mathrm{d}\omega=\nabla_j\varphi_i(x)\mathrm{d}x^j\wedge\mathrm{d}x^i=\frac{1}{2}(\nabla_j\varphi_i(x)-\nabla_i\varphi_j(x))\mathrm{d}x^j\wedge\mathrm{d}x^i=0$. Then

$$-\delta(\varphi_i(x)\delta x^i)=\delta\varphi_i(x)\delta x^i=\nabla_j\varphi_i(x)\delta x^j\delta x^i+C_3^{3\gamma}$$

$$=\frac{1}{2}(\nabla_j\varphi_i(x)+\nabla_i\varphi_j(x))\delta x^j\delta x^i+C_3^{3\gamma}$$

$$=\frac{1}{4}(\nabla_j\varphi_i(x)+\nabla_i\varphi_j(x))(\delta x^j\delta x^i+\delta x^i\delta x^j)+C_3^{3\gamma}$$

$$=\nabla_j\varphi_i(x)\delta S^{i,j}+C_3^{3\gamma}=\delta(\nabla_j\varphi_i(x)S^{i,j})+C_3^{3\gamma}$$

with $S^{i,j}(s,t)=\frac{1}{2}\delta x^i(s,t)\delta x^j(s,t)$. In other words

$$\delta(\varphi_i(x)\delta x^i+\nabla_j\varphi_i(x)S^{i,j})\in C_3^{3\gamma}$$

which means that there exists a unique $y$ such that

$$\delta y=\varphi_i(x)\delta x^i+\nabla_j\varphi_i(x)S^{i,j}+C_2^{3\gamma}.$$

Let us call

$$y(t)=\int_0^t\varphi_i(x(s))\mathrm{d}x^i(s).$$

In $\mathbb{R}^d$ the fact that $\omega$ is closed, implies that it is also exact: there exists $\psi:\mathbb{R}^d\to\mathbb{R}$ such that $\omega=\mathrm{d}\psi$. This means that $\varphi_i=\nabla_i\psi$. Now, Taylor expansion gives

$$\delta\psi(x)=\nabla_i\psi(x)\delta x^i+\nabla_j\nabla_i\psi S^{i,j}+O(|t-s|^{3\gamma})$$

so we can identify $\delta y = \delta\psi(x)$ and we have

$$\psi(x(t)) - \psi(x(0)) = \int_0^t \varphi_i(x(s))\mathrm{d}x^i(s).$$

Valid until $\gamma > 1/3$. Similar results hold for any $\gamma > 0$. When $d = 1$ any one-form is exact so this result allow to integrate an arbitrary function along an arbitrary Hölder path. When $d > 1$ the closedness condition is non–trivial and only particular one–forms can be integrated, namely those which are differentials of scalar functions.

# 2 Rough paths

In the following we will fix $\gamma > 1/3$ and an interval $I \subseteq \mathbb{R}$. All the Hölder spaces will considered on this interval unless specified otherwise.

**Definition 6.** *A $\gamma$-Hölder rough path $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ in $\mathbb{R}^d$ is a pair*

$$\mathbb{X}^1 \in C_2^{\gamma}(\mathbb{R}^d), \qquad \mathbb{X}^2 \in C_2^{2\gamma}(\mathbb{R}^d \otimes \mathbb{R}^d)$$

*with $\mathbb{X}^1 = \delta x$ for some $x \in C^{\gamma}(I; \mathbb{R}^d)$ and satisfying the Chen relation :*

$$\delta\mathbb{X}^2(s, u, t) = \mathbb{X}^1(s, u)\mathbb{X}^1(u, t), \qquad s < u < t.$$

*We say that $\mathbb{X}$ lies above $x$. We denote collectively $(X^i)_{i=1,\ldots,d}$ and $(X^{i,j})_{i,j=1,\ldots,d}$ the components of $\mathbb{X}^1$ and $\mathbb{X}^2$ with respect to the canonical basis of $\mathbb{R}^d$ and $\mathbb{R}^d \otimes \mathbb{R}^d$. We denote by $\mathscr{C}^{\gamma}(\mathbb{R}^d)$ the space of the $\gamma$-Hölder rough paths in $\mathbb{R}^d$ and we let*

$$\|\mathbb{X}\|_{\mathscr{C}^{\gamma}} = \|\mathbb{X}^1\|_{\gamma} + \|\mathbb{X}^2\|_{2\gamma}.$$

*On $\mathscr{C}^{\gamma}$ we consider the distance $\mathrm{d}_{\mathscr{C}^{\gamma}}(\mathbb{X}, \tilde{\mathbb{X}}) = \|\mathbb{X} - \tilde{\mathbb{X}}\|_{\mathscr{C}^{\gamma}}$. With $\mathscr{C}_x^{\gamma}(\mathbb{R}^d) \subseteq \mathscr{C}^{\gamma}(\mathbb{R}^d)$ we denote the subset of rough paths lying above a given $x \in C^{\gamma}(I; \mathbb{R}^d)$, the "fiber" at $x$.*

i. The space of rough paths is not a linear space since the Chen relation is non–linear.

ii. We can interpret the data $X^i, X^{i,j}$ as the given of the (abstract) iterated integrals

$$X^i(s, t) = \int_{s<u<t} \mathrm{d}x^i(u), \qquad X^{i,j}(s, t) = \int_{s<u<v<t} \mathrm{d}x^i(u)\mathrm{d}x^j(v)$$

together with suitable regularity as elements in $C_2$.

iii. When $\gamma > 1/2$ there can be at most only one rough path above a given path $x$. It is given by

$$\mathbb{X}^1(s, t) = x(t) - x(s), \qquad \mathbb{X}^2(s, t) = \int_s^t \mathbb{X}^1(s, u) \otimes \mathrm{d}_u x(u)$$

where the integral is understood in Young sense (or as a classical Lebesgue integral if $x \in C^1$). This rough path is called the *canonical lift* of $x$.

iv. Take $\gamma < 1/2$. If $\mathbb{X} \in \mathscr{C}_x^\gamma$ and $\varphi \in C^{2\gamma}(I; \mathbb{R}^d \otimes \mathbb{R}^d)$ then $\tilde{\mathbb{X}} = (\mathbb{X}^1, \mathbb{X}^2 + \delta\varphi)$ is also an element of $\mathscr{C}_x^\gamma$ and all of them have this form. In particular there are infinitely many rough paths above the same path if $\gamma < 1/2$ (or none). The fiber $\mathscr{C}_x^\gamma$ is an affine space with vector space $C^{2\gamma}(I; \mathbb{R}^d \otimes \mathbb{R}^d)$ and action $(\mathbb{X}, \varphi) \mapsto (\mathbb{X}^1, \mathbb{X}^2 + \delta\varphi)$. Elements in $\mathscr{C}_0^\gamma \simeq C^{2\gamma}(I; \mathbb{R}^d \otimes \mathbb{R}^d)$ are called *pure area* rough paths.

**Lemma 7.** *Let $\gamma < 1/2$, then $\mathscr{C}_x^\gamma$ is not empty.*

**Proof.** Let $x \in C^\gamma$. Fix $\rho < \gamma$ and consider a sequence $(y_n \in C^1)_{n \geqslant 1}$ such that $x = \sum_n y_n$ in $C^\rho$ and $\|y_n\|_\infty + 2^{-n}\|\dot{y}_n\|_\infty \lesssim 2^{-n\gamma}\|x\|_\gamma$ (such a sequence always exists). Let $\mathbb{X}_n^1 = \sum_{k \leqslant n} \delta y_k$ and define recursively $\mathbb{X}_n^2$ as

$$\mathbb{X}_{n+1}^2(s,t) = \mathbb{X}_n^2(s,t) + \mathbb{X}_n^1(s,t) y_{n+1}(t) - y_{n+1}(s)\mathbb{X}_n^1(s,t) - y_{n+1}(s)\delta y_{n+1}(s,t)$$

then if $\delta\mathbb{X}_n^2 = \mathbb{X}_n^1\mathbb{X}_n^1$ we have also

$$\delta\mathbb{X}_{n+1}^2(s,u,t) = \mathbb{X}_n^1(s,u)\mathbb{X}_n^1(u,t) + \mathbb{X}_n^1(s,u)\delta y_{n+1}(u,t) + \delta y_{n+1}(s,u)\mathbb{X}_n^1(u,t) + \delta y_{n+1}(s,u)\delta y_{n+1}(u,t)$$

$$= \mathbb{X}_{n+1}^1(s,u)\mathbb{X}_{n+1}^1(u,t).$$

Moreover

$$|\mathbb{X}_{n+1}^2(s,t) - \mathbb{X}_n^2(s,t)| \leqslant |\mathbb{X}_n^1(s,t) y_{n+1}(t)| + |y_{n+1}(s)\mathbb{X}_n^1(s,t)| + |y_{n+1}(s)\delta y_{n+1}(s,t)|$$

$$\lesssim \|x\|_\gamma^2 2^{-\gamma(n+1)}|t-s|^{2\rho} 2^{(2\rho-\gamma)n} \lesssim \|x\|_\gamma^2 |t-s|^{2\rho} 2^{(2\rho-2\gamma)n}$$

so the sequence $(\mathbb{X}_n^2)_n$ converges in $C_2^{2\rho}$ to an element which we call $\mathbb{X}^2$ and such that $\delta\mathbb{X}^2 = \mathbb{X}^1\mathbb{X}^1$. We have

$$|\mathbb{X}^2(s,t)| \leqslant \sum_n |\mathbb{X}_{n+1}^2(s,t) - \mathbb{X}_n^2(s,t)|$$

$$= \sum_{n: 2^n|t-s| \leqslant 1} |\mathbb{X}_{n+1}^2(s,t) - \mathbb{X}_n^2(s,t)| + \sum_{n: 2^n|t-s| > 1} |\mathbb{X}_{n+1}^2(s,t) - \mathbb{X}_n^2(s,t)|$$

$$\lesssim \|x\|_\gamma^2 \sum_{n: 2^n|t-s| \leqslant 1} 2^{n-2\gamma n}|t-s| + \|x\|_\gamma \sum_{n: 2^n|t-s| > 1} 2^{-\gamma n}|t-s|^\gamma \lesssim \|x\|_\gamma^2 |t-s|^{2\gamma}$$

Setting $\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2)$ we have $\mathbb{X} \in \mathscr{C}_x^\gamma$ as required.

□

When $\gamma > 1/2$ rough paths satisfy an additional algebraic relation, called the shuffle relation:

$$X^i(s,t) X^j(s,t) = X^{ij}(s,t) + X^{ji}(s,t). \tag{2}$$

**Definition 8.** *We call weakly geometric rough paths satisfying the relation eq. (2) and denote them collectively with $\mathscr{C}_{\mathrm{wg}}^\gamma$. Moreover we denote by $\mathscr{C}_g^\gamma$ the closure of $\mathscr{C}^1$ in $\mathscr{C}^\gamma$ and call them geometric rough paths.*

If $\gamma > 1/2$ we have $\mathscr{C}^\gamma = \mathscr{C}^\gamma_{\text{wg}} = \mathscr{C}^\gamma_g$. Since elements of $\mathscr{C}^1$ satisfy the shuffle relation, this will remain valid also for all elements of $\mathscr{C}^\gamma_g$ so $\mathscr{C}^\gamma_g \subseteq \mathscr{C}^\gamma_{\text{wg}} \subseteq \mathscr{C}^\gamma$ for any $\gamma$. As far as the relation between $\mathscr{C}^\gamma_g$ and $\mathscr{C}^\gamma_{\text{wg}}$ is concerned we have the following result

**Theorem 9.** *For every $\mathbb{X} \in \mathscr{C}^\gamma_{\text{wg}}$ there exists a sequence $(\mathbb{X}_n)_{n \geqslant 1}$ in $\mathscr{C}^1$ such that $\mathbb{X}_n \to \mathbb{X}$ in $\mathscr{C}^\rho$ for any $\rho < \gamma$. In particular $\mathscr{C}^\gamma_g \subseteq \mathscr{C}^\gamma_{\text{wg}} \subseteq \mathscr{C}^\rho_g$.*

As a preliminary to a proof of this theorem let us discuss a particular case, the approximation theory for pure area rough paths.

**Theorem 10.** *Assume $\gamma < 1/2$ and let $\varphi \in C^{2\gamma}(I; \mathbb{R}^d \otimes_a \mathbb{R}^d)$ then there exists $x_n \in C^1$ such that the canonical lift $\mathbb{X}_n$ converges in $\mathscr{C}^\rho$ to the pure area path $\mathbb{X} = (0, \delta\varphi)$ for any $\rho < \gamma$.*

**Proof.** Let $(\varphi_n)_n$ a sequence in $C^1$ converging to $\varphi$ in $C^{2\rho}$ for some $\rho < \gamma$ and such that $\|\dot{\varphi}_n\|_\infty \lesssim 2^{n - n\gamma} \|\varphi\|_\gamma$. Fix sufficiently large positive numbers $(L_{ij})_{i,j=1,\ldots,d}$ all different one from the other and let

$$x_n^i(t) = \frac{1}{2} \sum_j \dot{\varphi}_n^{ij}(t) 2^{-L_{ij} n/2} \sin(2^{L_{ij} n} t) + \sum_j 2^{-L_{ji} n/2} \cos(2^{L_{ji} n} t).$$

By a long but direct estimation we can show that

$$\left\| \int_s^t (x_n^i(u) - x_n^i(s)) \mathrm{d}_u x_n^j(u) - \frac{1}{2} \int_s^t (\dot{\varphi}_n^{ji}(u) - \dot{\varphi}_n^{ij}(u)) \mathrm{d}u \right\|_{C_2^{1-}} \to 0$$

as $n \to \infty$. Moreover

$$\frac{1}{2} \int_s^t (\dot{\varphi}_n^{ji}(u) - \dot{\varphi}_n^{ij}(u)) \mathrm{d}u \to \delta\varphi^{ij}(s,t)$$

in $C_2^{2\rho}$. Since $x_n \to 0$ in $C^{1/2-}$ the claim follows. In order to show the main estimate note that

$$\int_s^t (x_n^i(u) - x_n^i(s)) \mathrm{d}_u x_n^j(u) =$$

$$= -\sum_{k,m} \int_s^t 2^{-L_{ik} n/2} 2^{L_{mj} n/2} [\dot{\varphi}_n^{ik}(u) \sin(2^{L_{ik} n} u) - \dot{\varphi}_n^{ik}(s) \sin(2^{L_{ik} n} s)] \sin(2^{L_{mj} n} u) \mathrm{d}u$$

$$+ \sum_{k,m} \int_s^t 2^{-L_{ik} n/2} 2^{L_{mj} n/2} [\dot{\varphi}_n^{jm}(u) \cos(2^{L_{ik} n} u) - \dot{\varphi}_n^{jm}(s) \cos(2^{L_{ik} n} s)] \cos(2^{L_{mj} n} u) \mathrm{d}u$$

$$- \sum_{k,m} \int_s^t 2^{-L_{ik} n/2} 2^{L_{mj} n/2} [\cos(2^{L_{ik} n} u) - \cos(2^{L_{ik} n} s)] \sin(2^{L_{mj} n} u) \mathrm{d}u$$

$$+ \sum_{k,m} \int_s^t 2^{-L_{ik} n/2} 2^{L_{mj} n/2} [\dot{\varphi}_n^{ik}(u) \sin(2^{L_{ik} n} u) - \dot{\varphi}_n^{ik}(s) \sin(2^{L_{ik} n} s)] \dot{\varphi}_n^{jm}(u) \cos(2^{L_{mj} n} u) \mathrm{d}u$$

and that by trigonometric identities all the integrals contains oscillating factors with frequencies of the form $2^{L_{ik}n} \pm 2^{L_{mj}n}$ which are different from zero unless $k = m$. Moreover the only terms which produce non-oscillating factors are those of the form sin − sin or cos − cos which are then linear in $\dot{\varphi}_n$. Using integration by parts and the fact that the numbers $L$ are large enough to beat the growth of $\dot{\varphi}_n$ all the oscillating terms can be shown to go to zero (one use also the fact that boundary terms vanishes). The claim follows directly.

□

**Proof.** (of Thm. 9) The general case is similar to the pure area case. Let $\mathbb{X} \in \mathscr{C}^{\gamma}_{\mathrm{wg},x}$ for some $x \in C^{\gamma}$. Fix $\rho < \gamma$ and consider a sequence $(y_n \in C^1)_{n \geqslant 1}$ such that $x = \sum_n y_n$ in $C^{\rho}$ and $\|y_n\|_{\infty} + 2^{-n}\|\dot{y}_n\|_{\infty} \lesssim 2^{-n\gamma}\|x\|_{\gamma}$ (such a sequence always exists). Let $y_{\leqslant n} = \sum_{k \leqslant n} y_k$ and

$$x^i_n(t) = y^i_{\leqslant n} + \frac{1}{2}\sum_j \dot{\varphi}^{ij}_n(t) 2^{-L_{ij}n/2} \sin(2^{L_{ij}n} t) + \sum_j 2^{-L_{ji}n/2} \cos(2^{L_{ji}n} t).$$

(notations as in the previous theorem) where $\varphi_n$ is for the moment an indeterminate sequence. Using the same ideas as above we can show that

$$\left\| \int_s^t (x^i_n(u) - x^i_n(s)) \mathrm{d}_u x^j_n(u) - \int_s^t (y^i_{\leqslant n}(u) - y^i_{\leqslant n}(s)) \mathrm{d}_u y^j_{\leqslant n}(u) - \frac{1}{2}\int_s^t (\dot{\varphi}^{ji}_n(u) - \dot{\varphi}^{ij}_n(u)) \mathrm{d}u \right\|_{C_2^{1-}} \to 0.$$

Now the point is that we can choose the sequence $\varphi$ such that it cancels the contribution of the antisymmetric part of $\int_s^t (y^i_{\leqslant n}(u) - y^i_{\leqslant n}(s)) \mathrm{d}_u y^j_n(u)$ and replaces it with an approximation (converging in $C^{2\gamma}$) of the antisymmetric part of $\mathbb{X}^2$. We leave the details to the reader. □

Weakly geometric rough paths can be approximated by lifts of smooth paths by loosing just a bit regularity in the convergence statement. Approximation of general rough path is less clear. In particular we cannot hope to approximate a general rough path with smooth canonical lifts since the shuffle relation is not true in the limit. But as we will now see, this is the only obstruction.

Let $\mathbb{X} \in \mathscr{C}^{\gamma}$ and consider the defect in the shuffle relation

$$D^{ij}(s,t) = X^i(s,t) X^j(s,t) - X^{ij}(s,t) - X^{ji}(s,t).$$

A simple computation shows that $\delta D^{ij} = 0$, indeed:

$$\delta D^{ij}(s,u,t) = X^i(s,u) X^j(u,t) + X^j(s,u) X^i(u,t) - X^i(s,u) X^j(u,t) - X^j(s,u) X^i(u,t) = 0.$$

Moreover $D \in C_2^{2\gamma}(\mathbb{R}^d \otimes_s \mathbb{R}^d)$ where $\mathbb{R}^d \otimes_s \mathbb{R}^d$ denotes the symmetric tensor product. Then there exists a function $d \in C^{2\gamma}(I; \mathbb{R}^d \otimes_s \mathbb{R}^d)$ such that $D = \delta d$. We can define now $\mathbb{X}_g = (\mathbb{X}^1, \mathbb{X}^2 + \delta d/2)$ and check that $\mathbb{X}_g \in \mathscr{C}^{\gamma}_{\mathrm{wg}}$:

$$X^i_g(s,t) X^j_g(s,t) = X^i(s,t) X^j(s,t) = X^{ij}(s,t) + X^{ji}(s,t) + D^{ij}(s,t)$$

$$= X^{ij}(s,t) + \frac{1}{2}\delta d^{ij}(s,t) + X^{ji}(s,t) + \frac{1}{2}\delta d^{ji}(s,t) = X^{ij}_g(s,t) + X^{ji}_g(s,t).$$

So to every rough path $\mathbb{X} \in \mathscr{C}_x^\gamma$ lying above $x$ we can associate a geometric rough path $\mathbb{X}_g \in \mathscr{C}_{g,x}^\gamma$ by modifying its the symmetric part of its second order component. Note that this projection is not unique since there are a priori many weakly geometric rough paths above the same path, differing one from the other by an antisymmetic increment in the second order component: indeed $\tilde{\mathbb{X}}_g = (\mathbb{X}_g^1, \mathbb{X}_g^2 + \delta\varphi)$ with $\varphi \in C^{2\gamma}(I; \mathbb{R}^d \otimes_a \mathbb{R}^d)$ is again in $\mathscr{C}_{g,x}^\gamma$.

This construction shows the existence of an isomorphism of metric spaces :

$$\mathscr{C}^\gamma(\mathbb{R}^d) \simeq \mathscr{C}_{\mathrm{wg}}^\gamma(\mathbb{R}^d) \times C^{2\gamma}(I; \mathbb{R}^d \otimes_s \mathbb{R}^d).$$

(see Hairer–Kelly for a generalisation of these considerations)

# 3 Controlled paths

**Definition 11.** *A pair $(h, h^X)$ where $h \in C^\gamma([0,1]; V)$ and $h^X \in C^\gamma([0,1]; \mathscr{L}(\mathbb{R}^d; V))$ is a path controlled by $x$ if*

$$h^\sharp(s,t) = \delta h(s,t) - h^X(s)\mathbb{X}^1(s,t) \in C_2^{2\gamma}(V).$$

*We denote by $\mathscr{D}_{\mathbb{X}}^{2\gamma}(V)$ the linear space space of paths controlled by $\mathbb{X}$ and on $\mathscr{D}_{\mathbb{X}}^{2\gamma}$ we consider the semi–norm*

$$\|(h, h^X)\|_{\mathscr{D}_{\mathbb{X}}^{2\gamma}} = \|h^X\|_\gamma + \|h^\sharp\|_{2\gamma}.$$

Given a rough path $\mathbb{X}$ and path $(h, h^X) \in \mathscr{D}_{\mathbb{X}}^{2\gamma}(\mathscr{L}(\mathbb{R}^d; V))$ controlled by $\mathbb{X}$ we can define a new controlled path $(z, z^X) \in \mathscr{D}_{\mathbb{X}}^{2\gamma}(V)$ by letting $z^X = h$ and $z$ the unique solution to

$$\delta z(s,t) = h(s)\mathbb{X}^1(s,t) + h^X(s)\mathbb{X}^2(s,t) + z^\natural(s,t)$$

with $z^\natural \in C_2^{3\gamma}(V)$. We call it the integral of $h$ with respect to $\mathbb{X}$.

**Theorem 12.** *Given a rough path $\mathbb{X}$ and path $(h, h^X) \in \mathscr{D}_{\mathbb{X}}^{2\gamma}(\mathscr{L}(\mathbb{R}^d; V))$ controlled by $\mathbb{X}$ we can define a new controlled path $(z, z^X) \in \mathscr{D}_{\mathbb{X}}^{2\gamma}(V)$ by letting $z^X = h$ and $z$ the unique solution to*

$$\delta z(s,t) = h(s)\mathbb{X}^1(s,t) + h^X(s)\mathbb{X}^2(s,t) + z^\natural(s,t)$$

*with $z^\natural \in C_2^{3\gamma}(V)$. We call it the integral of $h$ with respect to $\mathbb{X}$ and we have*

$$\|z^\natural\|_{3\gamma} \lesssim \|(h, h^X)\|_{\mathscr{D}_{\mathbb{X}}^{2\gamma}}(1 + \|\mathbb{X}\|_{\mathscr{C}^\gamma})$$

*and*

$$[\![(z, z^X)]\!]_{\mathscr{D}_{\mathbb{X}}^{2\gamma},\tau} \lesssim \|\mathbb{X}\|_{\mathscr{C}^\gamma,1}([\![h^X]\!]_{\infty,\tau} + \tau^\gamma [\![(h, h^X)]\!]_{\mathscr{D}_{\mathbb{X}}^{2\gamma},\tau})$$

**Proof.** Such a path is clearly unique and is well defined since if we let

$$A(s,t) = h(s)\mathbb{X}^1(s,t) + h^X(s)\mathbb{X}^2(s,t)$$

we have

$$\delta A(s,u,t) = -\delta h(s,u)\mathbb{X}^1(u,t) - \delta h^X(s,u)\mathbb{X}^2(u,t) + h^X(s)\delta\mathbb{X}^2(s,u,t)$$

$$= -(\delta h(s,u) - h^X(s)\mathbb{X}^1(s,u))\mathbb{X}^1(u,t) - \delta h^X(s,u)\mathbb{X}^2(u,t) = -h^\sharp(s,u)\mathbb{X}^1(u,t) - \delta h^X(s,u)\mathbb{X}^2(u,t)$$

and by assumption $\delta A \in C_3^{3\gamma}(V)$ so that we can apply the sewing map to obtain

$$z^\natural = \Lambda(h^\sharp\mathbb{X}^1 + \delta h^X\mathbb{X}^2) \in C_2^{3\gamma}(V).$$

Then

$$[\![z^X]\!]_{\gamma,\tau} \leqslant [\![h^X]\!]_{\infty,\tau}\|\mathbb{X}^1\|_{\gamma,\tau} + [\![h^\sharp]\!]_{2\gamma,\tau}, \qquad [\![z^\sharp]\!]_{2\gamma,\tau} \leqslant [\![h^X]\!]_{\infty,\tau}\|\mathbb{X}^2\|_{2\gamma,\tau} + \tau^\gamma [\![z^\natural]\!]_{3\gamma,\tau}$$

and the final bound follows. □

**Lemma 13.** *Let* $(f, f^X) \in \mathscr{D}_{\mathbb{X}}^{2\gamma}(V)$ *and let* $\varphi \in C^2(V; W)$ *then* $(\varphi(f), \varphi(f)^X) \in \mathscr{D}_{\mathbb{X}}^{2\gamma}(W)$ *where* $\varphi(f)(t) = \varphi(f(t))$ *and* $\varphi(f)^X = \nabla\varphi(f) f^X \in C^\gamma(\mathscr{L}(\mathbb{R}^d; W))$. *Moreover*

$$\|(\varphi(f), \varphi(f)^X)\|_{\mathscr{D}_{\mathbb{X}}^{2\gamma}} \lesssim C_\varphi\big(1 + \|(f, f^X)\|_{\mathscr{D}_{\mathbb{X}}^{2\gamma}}\big)^2.$$

**Proof.** Taylor expansion gives

$$\delta\varphi(f)(s,t) = \int_0^1 \mathrm{d}\tau \nabla\varphi(f(s) + \tau\delta f(s,t))\delta f(s,t)$$

$$= \nabla\varphi(f(s))\delta f(s,t) + \int_0^1 (1-\tau)\mathrm{d}\tau \nabla^2\varphi(f(s) + \tau\delta f(s,t))(\delta f(s,t) \otimes \delta f(s,t))$$

$$= \nabla\varphi(f(s))\delta f(s,t) + O(|t-s|^{2\gamma})$$

Using the controlled hypothesis on $f$ we get

$$\delta\varphi(f)(s,t) = \nabla\varphi(f(s)) f^X(s)\mathbb{X}^1(s,t) + \varphi(f)^\sharp(s,t)$$

where

$$\varphi(f)^\sharp(s,t) = \nabla\varphi(f(s)) f^\sharp(s,t) + \int_0^1 (1-\tau)\mathrm{d}\tau \nabla^2\varphi(f(s) + \tau\delta f(s,t))(\delta f(s,t) \otimes \delta f(s,t)).$$

Then we can let $\varphi(f)^X(s) = \nabla\varphi(f(s)) f^X(s)$ and observe that

$$\|\varphi(f)^\sharp\|_{2\gamma} \lesssim \|\nabla\varphi\|_\infty \|f^\sharp\|_{2\gamma} + \|\nabla^2\varphi\|_\infty \|f\|_\gamma^2 \lesssim C_\varphi(1 + \|(f, f^X)\|_{\mathscr{D}_{\mathbb{X}}})^2.$$

□