Sheet 2

rev 1 - 20151126

- Young differential equations (YDEs) : existence and uniqueness.
- Euler scheme for YDE and asymptotic error analysis.

1 Young differential equations

Let us now study differential equations driven by irregular signals in the context of Young integration. We consider a family $(F_\alpha)_{\alpha=1,\ldots,m}$ of C_b^2 vector fields on $\mathbb{R}^d$ with which we form the C_b^2 mapping $F:\mathbb{R}^d\to\mathscr{L}(\mathbb{R}^m,\mathbb{R}^d)$ by letting $F(\xi)e_\alpha=F_\alpha(\xi)$ on the collection $(e_\alpha)_{\alpha=1,\ldots,m}$ of canonical basis vectors of $\mathbb{R}^m$. We would like to define solutions of the Young differential equation (YDE)

$$\partial_t y(t)=F(y(t))\partial_t x(t),\qquad y(0)=\xi$$

for a given $x\in C^\gamma([0,1];\mathbb{R}^m)$. When $\gamma\in(0,1)$ the proper way to interpret this differential equation is via the corresponding integral equation

$$y(t)=\xi+\int_0^t F(y(s))\mathrm{d}_s x(s)=\xi+I(F(y(\cdot)),x)(t),\qquad t\in[0,1]. \tag{1}$$

We cannot expect y to have better regularity than x (think about the case where F is constant) so apriori we require $y\in C^\gamma([0,1];\mathbb{R}^d)$. In this situation $F(y(\cdot))\in C^\gamma([0,1];\mathscr{L}(\mathbb{R}^m,\mathbb{R}^d))$ if F is at least Lipshitz and therefore the integral can be defined in the sense of Young under the condition that $\gamma>1/2$. We will assume this all along this section.

Note that eq. (1) is equivalent to the following finite–increment formulation

$$\delta y(s,t)=F(y(s))\delta x(s,t)+o(|t-s|),\qquad y(0)=\xi. \tag{2}$$

We denote by C^n_{lin} the space of C^n functions with linear growth and bounded derivatives.

We first obtain uniform a–priori estimates on YDE, assuming existence.

Lemma 1. *(Apriori estimates) If F is a C^1_{lin} family of vector fields and y is a solution to to eq. (1) then*

$$[\![y]\!]_{\gamma,\tau}\leqslant\tau^{-\gamma}$$

for any $\tau\leqslant 1\wedge(C_{F,\gamma,y(0)}\|x\|_{\gamma,1})^{-1/\gamma}$.

Proof. Consider the map $\Gamma_x: C^\gamma\to C^\gamma$ given by

$$\Gamma_x(y)(t):=\xi+I_Y(F(y(\cdot)),x)(t),\qquad t\in[0,1].$$

By the properties of the Young integral we have

$$[\![\Gamma_x(y)]\!]_{\gamma,\tau} \leqslant C(1 + [\![y - y(0)]\!]_{\infty,\tau} + \tau^{\gamma} [\![y]\!]_{\gamma,\tau}) \|x\|_{\gamma,\tau}.$$

where $C = C_{F,\gamma,y(0)}$. Using the fact that

$$[\![y - y(0)]\!]_{\infty,\tau} \leqslant 2\tau^{\gamma} [\![y]\!]_{\gamma,\tau}$$

and that $\|x\|_{\gamma,\tau} \leqslant \|x\|_{\gamma,1}$ for $\tau \leqslant 1$ we have

$$[\![y]\!]_{\gamma,\tau} = [\![\Gamma_x(y)]\!]_{\gamma,\tau} \leqslant C\|x\|_{\gamma,1} + C\tau^{\gamma} [\![y]\!]_{\gamma,\tau} \|x\|_{\gamma,1}$$

Taking τ such that $C\tau^{\gamma} [\![y]\!]_{\gamma,\tau} \|x\|_{\gamma,1} \leqslant 1/2$ we conclude that $[\![y]\!]_{\gamma,\tau} \leqslant \tau^{-\gamma}$. □

Let us now prove existence of solutions to eq. (1) by an approximation procedure.

Lemma 2. *If F is a C^1_{lin} family of vector fields then there exists at least one solution $y \in C^{\gamma}$ to eq. (1).*

Proof. Fix $\kappa \in (1/2, \gamma)$ and take a sequence $(x_n \in C^1)_n$ which converges to x in C^{κ}. Let y_n be the unique solution to the ODE $\partial_t y_n = F(y_n) \partial_t x_n$ which exists by standard results on ODEs and the fact that $\partial_t x_n$ is a bounded function. Note that this solution is also a solution of the YDE since it is easy to see that y^n satisfy the increment version of the YDE given in eq. (2). Let $K = 2C_{F,\gamma,\varepsilon} \|x\|_{\gamma,1}$, then eventually for n large we have $\|x_n\|_{\gamma,1} \leqslant 2\|x\|_{\gamma,1}$ so we can choose τ small enough (independent of n) such that $\tau^{\kappa} C_{F,\gamma,\varepsilon} \|x\|_{\gamma,1} \leqslant 1/2$ for n large. Using this τ and Lemma 1 we have $[\![y_n]\!]_{\kappa,\tau} \leqslant \tau^{-\kappa}$ uniformly in n. By Ascoli–Arzela we can take a uniformly convergent subsequence $y_{n_k} \to y \in C^{\kappa}$. Then if we consider the increment equation we have

$$\delta y_n(s,t) = F(y_n(s)) \delta x_n(s,t) + y_n^{\sharp}(s,t)$$

where by the apriori estimates in Lemma 1 we have $|y_n^{\sharp}(s,t)| \lesssim e^{t/\tau} |t-s|^{2\kappa}$ uniformly in n for any $0 < s < t$. Passing to the limit along the chosen subsequence pointwise (in $s < t$) in this relation we obtain that

$$|\delta y(s,t) - F(y(s)) \delta x(s,t)| = \lim_k |\delta y_{n_k}(s,t) - F(y_{n_k}(s)) \delta x_{n_k}(s,t)| = \lim_k |y_{n_k}^{\sharp}(s,t)| \lesssim e^{t/\tau} |t-s|^{2\kappa}$$

for all $0 < s < t$. This immediately shows that y is a solution of the YDE for all times. □

Remark 3. The above proof works only in finite dimensions. However the a–prori estimate holds also in infinite dimensions.

Theorem 4. *Assume that $F \in C^2_{\mathrm{lin}}$ then eq. (1) has only one solution $y = \Phi_F(\xi, x)$.*

Proof. Assume we have two solutions $y, \tilde{y}$ and consider the equation for $\varphi = y - \tilde{y}$ which reads

$$\varphi(t) = I_Y(G(\cdot, \varphi(\cdot)), x)$$

where

$$G(t,\xi)=F(y(t))-F(y(t)-\xi)=-\int_0^1\nabla F(y(t)-\tau\xi)\mathrm{d}\tau\xi.$$

We have

$$[\![G(\cdot,\varphi(\cdot))]\!]_{\gamma,\tau}\leqslant C_{F,\|y\|,\|\tilde{y}\|}([\![\varphi]\!]_{\gamma,\tau}+[\![\varphi]\!]_{\infty,\tau})\leqslant C_{F,\|y\|,\|\tilde{y}\|}([\![\varphi]\!]_{\gamma,\tau}+[\![\varphi]\!]_{\infty,\tau}).$$

The estimate for the Young integral gives

$$[\![\varphi]\!]_{\gamma,\tau}\leqslant C_{F,\|y\|,\|\tilde{y}\|}\tau^{\gamma}[\![\varphi]\!]_{\gamma,\tau}\|x\|_{\gamma,\tau}.$$

so

$$[\![\varphi]\!]_{\gamma,\tau}\leqslant C_{F,\|y\|,\|\tilde{y}\|,\|x\|}\tau^{\gamma}[\![\varphi]\!]_{\gamma,\tau}$$

and taking τ small enough we conclude that $[\![\varphi]\!]_{\gamma,\tau}=0$, that is $y=\tilde{y}$. □

Remark 5. We can weaken substantially the requirements for uniqueness by a localization argument and require only $F\in C^2_{\mathrm{loc}}$.

Exercise 1. Under the assumption $F\in C^2_{\mathrm{lin}}$ prove that the map $\Phi_F\colon\mathbb{R}^d\times C^{\gamma}(\mathbb{R}^m)\to C^{\gamma}(\mathbb{R}^d)$ given by $y=\Phi_F(\xi,x)$ is locally Lipshitz.

Lemma 6. *("Ito formula") Let $G\in C^2$ then*

$$G(y(t))=G(\xi)+\int_0^t\nabla G(y(s))\mathrm{d}y(s).=G(\xi)+\int_0^t\nabla G(y(s))F(y(s))\mathrm{d}x(s).$$

Proof. By Taylor expansion we have

$$\delta G(y)(s,t)=\nabla G(y(s))\delta y(s,t)+O(|t-s|^{2\gamma}).$$

By the definition of the Young integral we have

$$\int_s^t\nabla G(y(r))\mathrm{d}y(r)=\nabla G(y(s))\delta y(s,t)+O(|t-s|^{2\gamma}).$$

and

$$\int_s^t\nabla G(y(r))F(y(r))\mathrm{d}x(r)=\nabla G(y(s))F(y(s))\delta x(s,t)+O(|t-s|^{2\gamma}).$$

Now using the YDE (in the increment formulation) we note that

$$\nabla G(y(s))\delta y(s,t)+O(|t-s|^{2\gamma})=\nabla G(y(s))F(y(s))\delta x(s,t)+O(|t-s|^{2\gamma}).$$

So we have proven the claim since all these three objects are integrals of the same germ $\nabla G(y(s))F(y(s))\delta x(s,t)$ and then coincide. □

1.1 An Euler scheme

Consider the YDE with $F\in C^2_{\mathrm{lin}}$ and the corresponding Euler scheme of step $1/n$:

$$y^n(t_{i+1}^n)=y_0+\sum_{0\le k\le i}F(y^n(t_k^n))\,(x(t_{k+1}^n)-x(t_k^n)) \tag{3}$$

where we let $t_k^n=k/n$ for all $n\geqslant 1$, $k\geqslant 0$.

Theorem 7. *The Euler scheme (3) converges pointwise to the solution y of the YDE (1). Moreover $\Delta^n=y^n-y$ satisfies*

$$\sup_{0\le k<m\le n}\frac{|\Delta^n(t_k^n)-\Delta^n(t_m^n)|}{|t_k^n-t_m^n|^\gamma}\le C n^{1-2\gamma}.$$

Proof. We can embed the approximate discrete solution given by the Euler scheme in a continuous path by setting

$$y^n(t)=y(0)+\sum_{0\le k\le\lfloor nt\rfloor}F(y^n(t_k^n))\,(x(t\wedge t_{k+1}^n)-x(t_k^n)) \tag{4}$$

Note that y^n is a C^γ path. Using the identity

$$F(y^n(s))\,(x(t)-x(s))=\int_s^t F(y^n(u))\,dx(u)-\int_s^t\int_s^u d_vF(y^n(v))\,dx(u) \tag{5}$$

we can write a RDE for the extension of the Euler scheme

$$y^n(t)=y(0)+\int_0^t F(y^n(s))\,dx(s)-\psi^n(t) \tag{6}$$

where the driving term ψ^n is given explicitly by a sum of iterated integrals:

$$\psi^n(t)=\sum_{0\le k\le\lfloor nt\rfloor}\int_{t_k^n}^{t\wedge t_{k+1}^n}\int_{t_k^n}^u d_vF(y^n(v))\,dx(u)=\sum_{0\le k\le n}\psi^{n,k}(t)$$

where

$$\psi^{n,k}(t)=\int_{t_k^n}^{(t\wedge t_{k+1}^n)\vee t_k^n}\int_{t_k^n}^u d_vF(y^n(v))\,dx(u)$$

Step 1 Out first step will be to bound the γ–Hölder weigthed norm of ψ^n. We are interested in the limit $n\to\infty$ with τ fixed so we can assume $\tau>1/n$. In the Hölder norm $[\![\psi^n]\!]_{\gamma,\tau}$ there are three kind of contributions: when both times belong to the same interval of size $1/n$, when they belongs to adjacent intervals, and when both these conditions are not satisfied. When $t,s\in[t_k^n,t_{k+1}^n]$ for some k and $t>s$ such that $|t-s|\leqslant\tau\leqslant 1$, we have

$$|\psi^n(t)-\psi^n(s)|=|\psi^{n,k}(t)-\psi^{n,k}(s)|\le|t-s|^\kappa\|\psi^{n,k}\|_{\kappa,[s,t]}\le|t-s|^\kappa C\|F'\|_\infty\|y^n\|_{\gamma,[s,t]}\|x\|_{\gamma,[s,t]}n^{-\gamma}.$$

Where we exploited the following fact to obtain the factor $n^{-\gamma}$: write $\psi^{n,k}(t)-\psi^{n,k}(s)$ as

$$\psi^{n,k}(t)-\psi^{n,k}(s)=\int_s^t\int_s^u d_v F(y^n(v))\,dx(u)+[F(y^n(s))-F(y^n(t_k^n))]\,[x(t)-x(s)]$$

then

$$\left|\int_s^t\int_s^u d_v F(y^n(v))\,dx(u)\right|\leq C\|F'\|_\infty\|y^n\|_{\gamma,[s,t]}\|x\|_{\gamma,[s,t]}\,|t-s|^{2\gamma}$$

and, more easily,

$$|[F(y^n(s))-F(y^n(t_k^n))]\,[x(t)-x(s)]|\leq\|F'\|_\infty\|y^n\|_{\gamma,[s,t]}\|x\|_{\gamma,[s,t]}\,|t-s|^\gamma\,|s-t_k^n|^\gamma.$$

When $s\in[t_k^n,t_{k+1}^n]$ and $t\in[t_{k+1}^n,t_{k+2}^n]$:

$$\begin{aligned}|\psi^n(t)-\psi^n(s)|&\leqslant|\psi^n(t)-\psi^n(t_{k+1}^n)|+|\psi^n(t_{k+1}^n)-\psi^n(s)|\\&\leqslant C\|F'\|_\infty\|y^n\|_{\gamma,[s,t]}\|x\|_{\gamma,[s,t]}\,n^{-\gamma}\,(|t-t_{k+1}^n|^\gamma+|t_{k+1}^n-s|^\gamma).\\&\leqslant C\,2^{1-\kappa}\|F'\|_\infty\|y^n\|_{\gamma,[s,t]}\|x\|_{\gamma,[s,t]}\,n^{-\gamma}\,|t-s|^\gamma\end{aligned}$$

Otherwise, we have

$$\begin{aligned}|\psi^n(t)-\psi^n(s)|&\leqslant|\psi^n(t)-\psi^n(t_{k+1}^n)|+|\psi^n(t_m^n)-\psi^n(s)|+\sum_{m\leq q<k}|\psi^n(t_{q+1}^n)-\psi^n(t_q^n)|\\&\leqslant C\|F'\|_\infty\|y^n\|_{\gamma,[s,t]}\|x\|_{\gamma,[s,t]}\,n^{-\gamma}\,(|t-t_{k+1}^n|^\gamma+|t_m^n-s|^\gamma+(k-m)\,n^{-\gamma})\\&=C\|F'\|_\infty\|y^n\|_{\gamma,[s,t]}\|x\|_{\gamma,[s,t]}\,n^{-\gamma}\,(|t-t_{k+1}^n|^\gamma+|t_m^n-s|^\gamma+[(k-m)/n]\,n^{1-\gamma})\\&\leqslant C\|F'\|_\infty\|y^n\|_{\gamma,[s,t]}\|x\|_{\gamma,[s,t]}\,n^{1-2\gamma}\,(|t-t_{k+1}^n|^\gamma+|t_m^n-s|^\gamma+[(k-m)/n]^\gamma)\\&\leqslant C\,3^{1-\kappa}\|F'\|_\infty\|y^n\|_{\gamma,[s,t]}\|x\|_{\gamma,[s,t]}\,n^{1-2\gamma}\,|t-s|^\gamma\end{aligned}$$

and since $\kappa<1$ we obtain

$$[\![\psi^n]\!]_{\gamma,\tau}\leq C\,[\![y^n]\!]_{\gamma,\tau}\,n^{1-2\gamma}\leq C\,[\![y^n]\!]_{\gamma,\tau}\,n^{1-2\gamma}$$

where the constant C depends only on $\gamma,F,\|x\|_{\gamma,1}$.

Step 2 Now we will show that the sequence $(y^n)_n$ is uniformly bounded in the norm $[\![\cdot]\!]_{\gamma,\tau}$. Using eq. (6) we have

$$[\![y^n]\!]_{\gamma,\tau}\leq C[1+\tau^\gamma\,[\![y^n]\!]_{\gamma,\tau}]\,\|x\|_{\gamma,1}+[\![\psi^n]\!]_{\gamma,\tau}$$

So choosing τ small enough we obtain

$$[\![y^n]\!]_{\gamma,\tau}\leq 2(C+[\![\psi^n]\!]_{\gamma,\tau}).$$

for some constant C depending only on F,x,γ. Moreover, recalling the bound (1) we have

$$[\![y^n]\!]_{\gamma,\tau}\leq C\,(C+n^{1-2\gamma}\,[\![y^n]\!]_{\gamma,\tau})$$

and when $n \geqslant n_0$ where n_0 is such that $C n_0^{1-2\gamma} \leq 1/2$ (and does not depend on y_0) we end up with

$$[\![y^n]\!]_{\gamma,\tau} \leq C \qquad [\![\psi^n]\!]_{\gamma,\tau} \leq C n^{1-2\gamma} \tag{7}$$

uniformly in $n \geqslant n_0$ and y_0.

By compactness of the ball $\{z \in C^\gamma\colon [\![z]\!]_{\gamma,\tau} \leq C|y_0|, z_0 = y_0\}$ in the uniform topology we have that there exists a uniformly converging subsequence $(y^{n_k})_k$ of $(y^n)_n$. It is easy to see that any limit point should satisfy eq. (1) since the correction term in eq. (6) go to zero in the topology of C^γ. Given uniquenes of the YDE (1) under the condition $F \in C^2$ we know that the limit point of any converging subsequence is the unique solution y, this implies that all the sequence converges to y in the uniform topology (but by interpolation also in any κ–Holder norm for $\kappa < \gamma$).

Step 3 We look for an explicit bound for the convergence rate in C^γ. We are interested in comparing y^n with the true solution y:

$$y^n(t) - y(t) = \int_0^t [F(y^n(s)) - F(y(s))]\,\mathrm{d}x(s) - \psi^n(t) \tag{8}$$

$$F(y^n(t)) - F(y(t)) = \int_0^1 \mathrm{d}r\, F'(y(t) + r\,(y^n(t) - y(t)))\,(y^n(t) - y(t))$$

$$[\![F(y^n) - F(y)]\!]_{\gamma,\tau} \leq \|F'\|_\infty\, [\![y^n - y]\!]_{\gamma,\tau} + \|F''\|_\infty\, [\![y^n - y]\!]_{\infty,\tau}(2\|y\|_\gamma + \|y^n\|_\gamma)$$

so we have

$$[\![y^n - y]\!]_{\gamma,\tau} \leq C\tau^\gamma\, [\![y^n - y]\!]_{\gamma,\tau} \|x\|_{\gamma,1} + [\![\psi^n]\!]_{\gamma,\tau}$$

where $C = C(a, \|y\|_\gamma, \|y^n\|_\gamma)$. Note that we need the un-weighted norms of y and y^n in these bounds. So in order to proceed we have to restricts all the considerations to a bounded interval $[0, T]$. Taking τ small enough we get

$$[\![y^n - y]\!]_{\gamma,\tau} \leq 2\,[\![\psi^n]\!]_{\gamma,\tau}.$$

Recall that we have proven that $[\![\psi^n]\!]_{\gamma,\tau} \leq C n^{1-2\gamma}$, so we are done. □

1.2 Asymptotic error analysis for the Euler scheme

We would like now to characterize the asymptotic behavior of the Euler scheme for which we have obtained an upper bound on the rate of convergence (in some Hölder norm). We have already seen that the Euler scheme behaves like the YDE (cfr. (6)) with a small perturbation given by the function ψ^n introduced in the proof of theorem 7. Moreover this function is small when n is large, since essentially $[\![\psi^n]\!]_{\gamma,\tau} \lesssim n^{1-2\kappa}$. Our aim is to blow up the difference $\Delta^n = y^n - y$ between the Euler scheme and the true solution on a scale of the order of n^α for some $\alpha > 0$ and obtain a finite limit under some reasonable assumption. This will identify a criterion for the determination of the exact convergence rate of the Euler scheme. Ideally this criterion should depend only on properties of the function x and not on the vectorfield F or the solution y.

Hypothesis (Φ). Define the following approximated path

$$\Phi^n(t) = \sum_{0 \leq k \leq \lfloor nt \rfloor} \int_{t_k^n}^{t \wedge t_{k+1}^n} \int_{t_k^n}^{u} dx(v)\, dx(u) = \sum_{0 \leq k \leq n} \Phi^{n,k}(t) \tag{9}$$

and assume that $n^\alpha \Phi^n \to \phi \neq 0$ in C^γ for some $\alpha > 0$

Actually is not difficult to see that we should have $\alpha \geqslant 2\gamma - 1$. Then we will be able to prove the following result:

Theorem 8. *Under Hyp. (Φ) and if $\alpha < 3\gamma - 1$, the rescaled error $e^n = n^\alpha (y^n - y)$ converges in C^γ to the function e which is the unique solution of the RDE*

$$e(t) = \int_0^t F'(y(s))\, e(s)\, \mathrm{d}x(s) + \int_0^t F'(y(s))\, F(y(s))\, \mathrm{d}\phi(s)$$

Proof. Write an equation for e^n:

$$e^n(t) = n^\alpha \int_0^t [F(y^n(s)) - F(y(s))]\, dx(s) - n^\alpha \psi^n(t) \tag{10}$$

Step 1 The first term we will handle is $n^\alpha \psi^n(t)$.

First note that

$$\int_s^t \int_s^u d_v F(y^n(v))\, dx(u) = \int_s^t \int_s^u d_v F(y(v))\, dx(u) + O(n^{1-2\gamma} |t-s|^{2\gamma})$$

and then that

$$\int_s^t \int_s^u d_v F(y(v))\, dx(u) = \int_s^t \int_s^u \nabla F(y(v)) F(y(v)) \mathrm{d}x(v)\, \mathrm{d}x(u)$$

$$= \nabla F(y(s)) F(y(s)) \int_s^t \int_s^u \mathrm{d}x(v)\, \mathrm{d}x(u) + O(|t-s|^{3\gamma}).$$

These two bounds can be estabilished with straighforward computations (using the sewing map, for example). Then

$$\psi^n(t) = \sum_{0 \leq k \leq \lfloor nt \rfloor} \nabla F(y(t_k^n))\, F(y(t_k^n)) \int_{t_k^n}^{t \wedge t_{k+1}^n} \int_{t_k^n}^{u} dx(v)\, dx(u) + \sum_{0 \leq k \leq \lfloor nt \rfloor} r(t_k^n, t \wedge t_{k+1}^n)$$

Using the fact that $[\![y^n - y]\!]_{\gamma,\tau} \leq C n^{1-2\kappa}$ and arguments like the one used to estimante $[\![\psi^n]\!]_{\gamma,\tau}$ in the proof of Thm. ? it is not difficult to show that the function

$$R^n(t) = \sum_{0 \leq k \leq \lfloor nt \rfloor} r(t_k^n, t \wedge t_{k+1}^n)$$

can be bounded as $[\![R^n]\!]_{\gamma,\tau} \le C n^{1-3\gamma}$ where the constant depends only on a and y, and that

$$\begin{aligned}\sum_{0\le k\le \lfloor nt\rfloor} & \nabla F(y(t_k^n))\, F(y(t_k^n)) \int_{t_k^n}^{t\wedge t_{k+1}^n} \int_{t_k^n}^{u} dx(v)\, dx(u) \\ &= \int_0^t F'(y(s))\, F(y(s))\, d\,\phi^n(s) \\ &+ \sum_{0\le k\le \lfloor nt\rfloor} \int_{t_k^n}^{t\wedge t_{k+1}^n} \int_{t_k^n}^{u} \int_{t_k^n}^{v} d_w [F'(y(w))\, F(y(w))]\, dx(v)\, dx(u)\end{aligned}$$

where again the last term can be shown to go to zero in C^γ as $n^{1-3\gamma}$. Gathering all these observations we obtain that

$$n^\alpha \psi^n \xrightarrow{C^\gamma} \int_0^{\cdot} F'(y(s))\, F(y(s))\, \mathrm{d}\,\phi(s) =: \psi(t)$$

as $n\to\infty$ whenever $\alpha < 3\gamma - 1$.

Step 2 Let us return to eq. (10) and bound as follows:

$$[\![e^n]\!]_{\gamma,\tau} \leqslant C\tau^\gamma\, [\![n^\alpha (F(y^n) - F(y))]\!]_{\gamma,\tau} \|x\|_{\tau,1} + [\![n^\alpha \psi^n]\!]_{\gamma,\tau}$$

Now observe that

$$n^\alpha [F(y^n(t)) - F(y(t))] = \int_0^1 dr\, \nabla F\,(y(t) + r\,(y^n(t) - y(t)))\, e^n(t)$$

so

$$\begin{aligned}[\![n^\alpha (F(y^n) - F(y))]\!]_{\gamma,\tau} &\leqslant \|\nabla F\|_\infty [\![e^n]\!]_{\gamma,\tau} + \|\nabla^2 F\|_\infty [\![e^n]\!]_{\infty,\tau} (2\|y\|_{\gamma,\tau} + \|y^n\|_{\gamma,\tau}) \\ &\leqslant \|\nabla F\|_\infty [\![e^n]\!]_{\gamma,\tau} + \|\nabla^2 F\|_\infty \tau^\gamma [\![e^n]\!]_{\gamma,\tau} (2\|y\|_{\gamma,\tau} + \|y^n\|_{\gamma,\tau})\end{aligned}$$

Then

$$[\![e^n]\!]_{\gamma,\tau} \leqslant C\tau^\gamma [\![e^n]\!]_{\gamma,\tau} + [\![n^\alpha \psi^n]\!]_{\gamma,\tau}$$

Then as usual, when τ is small we obtain the bound $[\![e^n]\!]_{\gamma,\tau} \leqslant C\, \|n^\alpha \psi^n\|_\kappa$. Moreover, for n large enough we have $\|e^n\| \leqslant 2C\|\psi\|_\kappa$.

Step 3 Now that we have a uniform bound in C^γ for the sequence $(e^n)_n$ we can estimate its distance form e using the equation

$$e^n(t) - e(t) = \int_0^t [n^\alpha (F(y^n(s)) - F(y(s))) - F'(y(s))\, e(s)]\, dx(s) + \psi^n(t) - \psi(t)$$

which we can rewrite as

$$e^n(t) - e(t) = \int_0^t F'(y(s))\, (e^n(s) - e(s))\, dx(s) + \int_0^t [n^\alpha (F(y^n(s)) - F(y(s))) - F'(y(s))\, e^n(s)]\, dx(s) + \psi^n(t) - \psi(t)$$

Since $\psi^n \to \psi$ in C^γ, it is not difficult, using standard arguments and this equation, to prove that $e^n \to e$ in C^γ, as required. $\square$

An improved scheme

Given the results of the previous section we can design an improved Euler scheme by subtracting the limiting asymptotic error in the following way:

$$\hat{y}^n(t_{i+1}^n) = y_0 + \sum_{0 \le k \le i} F(\hat{y}^n(t_k^n))\,\delta x(t_k^n, t_{k+1}^n) + n^{-\alpha} \sum_{0 \le k \le i} (F\nabla F)(\hat{y}^n(k/n))\,\delta\phi(t_k^n, t_{k+1}^n). \qquad (11)$$

Note that in this scheme it is not required to compute the double iterated integral of the function x, but only to know its limiting behavior encoded in the process ϕ.