Random matrices and determinantal processes

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1 Introduction

The aim of this work is to explain some connections between random matrices and determinantal processes. First we consider the eigenvalue distributions of the classical Gaussian random matrices ensembles. Of particular interest is the distribution of their largest eigenvalue in the limit of large matrices. For the Gaussian Unitary Ensemble, GUE, it is known as GUE Tracy-Widom distribution [31] and appears in a lot of different models in combinatorics [3], growth models [10, 25, 13, 27], equilibrium statistical mechanics [7], and in non-colliding random walks or Brownian particles [21, 15]. Secondly we introduce the determinantal processes, which are point processes which n-point correlation functions are given by a determinants of a kernel of an integral operator. It turns out that the eigenvalue distribution of the GUE random matrices is a determinantal process which kernel has a particular structure. This is the reason why the GUE Tracy-Widom distribution appears in a lot of models which are not related with random matrices.

2 Classical Random Matrices Ensembles

Initially studied by statisticians in the 20's-30's, random matrices are then introduced in nuclear physics in the 50's to describe the energy levels distribution of heavy nuclei. The reader interested in a short discussion on random matrices in physics can read [16], in mathematics [22], and a good reference for a more extended analysis is [19].

2.1 Gaussian Orthogonal, Unitary, Symplectic Ensembles

Let H be a $N \times N$ matrix. Three important cases of random matrices are the following:

- $\beta = 1$: H is a real symmetric matrix,
- $\beta = 2$: H is a complex hermitian matrix,
- $\beta = 4$: H is a real quaternionic matrix.

Analyzing the consequences of the time-inversion invariance T, Dyson [6] showed that the previous classes of random matrices can describe a system which,

- for $\beta = 1$, is T-invariant and rotational invariant or with integer magnetic moment,
- for $\beta = 2$, is not T-invariant, e.g., with a magnetic field without other discrete symmetries,
- for $\beta = 4$, is T-invariant and with half-integer magnetic moment.

In these three cases the eigenvalues are real and H can be diagonalized by an orthogonal ($\beta = 1$), unitary ($\beta = 2$), or symplectic ($\beta = 4$) transformation. The classical Gaussian ensembles are obtained defining the probability distribution on matrices

$$p(H)\mathrm{d}H = \frac{1}{Z'} e^{-\operatorname{Tr} H^2} \mathrm{d}H \tag{2.1}$$

where $\mathrm{d}H$ is the Lebesgue product measure on the independent elements of H and Z' the normalization. For example, for $\beta = 2$, $\mathrm{d}H = \prod_{i=1}^{N} \mathrm{d}H_{i,i} \prod_{1 \le i < j \le N} \mathrm{dRe} H_{i,j}\ \mathrm{dIm} H_{i,j}$.

The ensembles of random matrices obtained are called **Gaussian Orthogonal** (GOE), **Unitary** (GUE), and **Symplectic** (GSE) **Ensembles** for $\beta = 1$, $\beta = 2$, and $\beta = 4$ respectively.

The distribution (2.1) is also recovered by taking the independent elements of H as Gaussian random variables with mean zero and variance $1/\beta$ for the diagonal terms, $1/2\beta$ for the non-diagonal terms. Another way to obtain (2.1) is to maximize the functional "entropy"

$$S(p) = -\int p(H) \ln p(H) \mathrm{d}H \tag{2.2}$$

under the condition $\mathbb{E}(\operatorname{Tr} H^2) = \mathit{const}$ (like for the canonical and grand-canonical measures in statistical mechanics).

2.2 Eigenvalues distributions

One interesting quantity of random matrices is the eigenvalues distribution. The probability distribution (2.1) is invariant under the orthogonal group $G = O(N)$ for GOE, unitary group $G = U(N)$ for GUE, and symplectic group $G = USp(2N)$ for GSE. This implies that $p(H)$ is a function on the eigenvalues only. It is then possible to factorize (2.1) as

$$p(H)\mathrm{d}H = p(\lambda)\Delta^{\beta}(\lambda)\mathrm{d}\lambda\mathrm{d}G \tag{2.3}$$

where $\mathrm{d}G$ is the Haar measure on G, $\mathrm{d}\lambda$ is the Lebesgue product measure on the eigenvalues, $\prod_{k=1}^{N}\mathrm{d}\lambda_k$, and

$$\Delta_N(\lambda) = \det(\lambda_i^{j-1})_{i,j=1}^{N} = \prod_{1\leq i<j\leq N}(\lambda_j - \lambda_i) \tag{2.4}$$

is the Vandermonde determinant. The eigenvalues distribution coming from the measure (2.1) is then

$$p(\lambda_1,\dots,\lambda_N)\mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_N = \frac{1}{Z_{\beta,N}}|\Delta_N(\lambda)|^{\beta}\prod_{j=1}^{N}e^{-\lambda_j^2}\mathrm{d}\lambda_j \tag{2.5}$$

where $\lambda_1,\dots,\lambda_N$ are the eigenvalues of H and $Z_{\beta,N}$ is the normalization.

In the case $\beta = 2$, the measure (2.5) is a product of two determinants and a product measure on the eigenvalues. This particular structure leads to a connection with the determinantal process discussed in Section 3.

2.3 Asymptotic distributions F_β: Tracy-Widom distributions

Typically the eigenvalues lies in the interval from $-\sqrt{2N}$ to $\sqrt{2N}$ and the mean distance between eigenvalues scales is $\sim \sqrt{8}/\sqrt{N}$. Let $\mu_j = \lambda_j/\sqrt{2N}$ be the rescaled eigenvalues, then the mean distance between them is $\sim 2/N$. Let $N\cdot\rho_\beta$ be the asymptotic density of the rescaled eigenvalues

$$\rho_\beta(\mu) = \lim_{N\to\infty}\frac{\mathbb{E}_{N,\beta}(\#\{\mu_i\in[\mu,\mu+\mathrm{d}\mu]\})}{N}.$$

Then ρ_β satisfies the **Wigner semi-circle law**

$$\rho_\beta(\mu) = \frac{2}{\pi}\sqrt{(1-\mu^2)_+} \tag{2.6}$$

where $x_+ = \max\{0,x\}$.

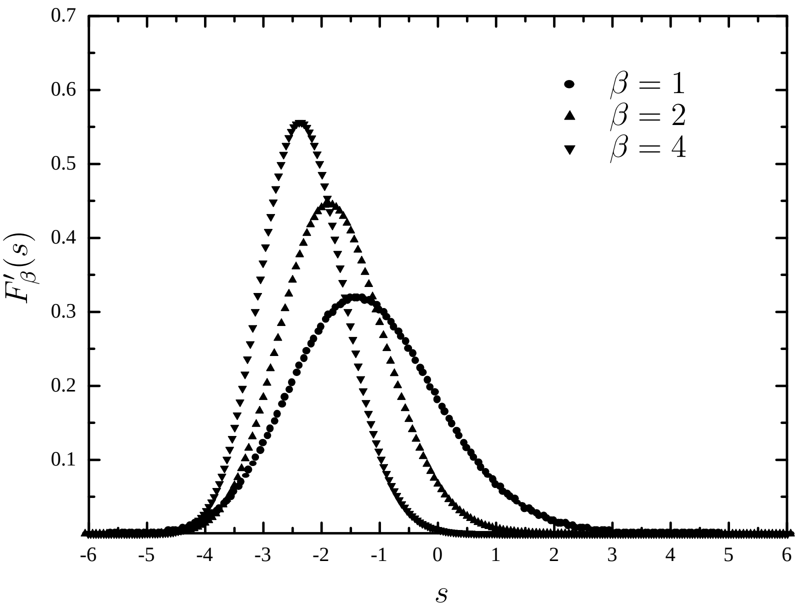

Figure 1: Distribution densities for $\beta = 1, 2, 4$ generated using [24].

The largest eigenvalue $\lambda_{\max}$ is then located close to $\sqrt{2N}$. Tracy and Widom study (see the review paper [32] and references therein) the distribution of $\lambda_{\max}$ in the limit $N \to \infty$ for $\beta = 1, 2, 4$ with the following result. Let $F_{N,\beta}(t) = \mathbb{P}_{N,\beta}(\lambda_{\max} \leq t)$, then $F_\beta(s)$ defined by

$$F_\beta(s) = \lim_{N\to\infty} F_{N,\beta}\left(\sqrt{2N} + s/2^{1/2}N^{1/6}\right) \tag{2.7}$$

exists for $\beta = 1, 2, 4$. They are given by

$$F_2(s) = \exp\left(-\int_s^\infty (x-s)q^2(x)\mathrm{d}x\right) \tag{2.8}$$

where q is the unique solution of the Painlevé II equation

$$q'' = sq + 2q^3$$

satisfying the asymptotic condition $q(s) \sim \mathrm{Ai}(s)$ for $s \to \infty$. F_2 is called the **GUE Tracy-Widom distribution**. In particular, for $x \to \infty$, $F_2(x) \sim$

$\exp(-\frac{4}{3}x^{3/2})$ and for $x \to -\infty$, $F_2(x) \sim \exp(-\frac{1}{12}|x|^3)$. $F_2(s)$ can also be rewritten as a Fredholm determinant of the Airy operator, see Section 4.1. Finally, for $\beta = 1$

$$F_1(s) = \exp\left(-\frac{1}{2}\int_s^\infty q(x)\mathrm{d}x\right) F_2(s)^{1/2} \tag{2.9}$$

and for $\beta = 4$

$$F_4(s/\sqrt{2}) = \cosh\left(\frac{1}{2}\int_s^\infty q(x)\mathrm{d}x\right) F_2(s)^{1/2}. \tag{2.10}$$

3 Determinantal processes

3.1 Definitions

First we define a point process (or random point field). Let X be a one-particle space like $\mathbb{R}^d$, $\mathbb{Z}^d$, $\mathbb{N}$ or simply $\{0,\dots,M\}$. Let Γ be the space of finite or countable configurations of particles in X, where the particles are *ordered* in some natural way and *each configuration* $\xi = (x_i)$, $x_i \in X$, $i \in \mathbb{Z}$ (or $\mathbb{N}$ if $d > 1$) *is locally finite*, i.e., for every compact $B \subset X$, the number of $x_i \in B$, denoted $\#_\xi(B)$, is finite. Next we define the σ-algebra on Γ via the cylinder sets. For any bounded Borel set $B \subset X$ and $n \geq 0$, $C_n^B = \{\xi \in \Gamma, \#_\xi(B) = n\}$ is a cylinder set. Then we define $\mathcal{F}$ as the σ-algebra generated by all cylinder sets.

Definition 3.1. *A* ***point process*** *(random point field) is a triplet* $(\Gamma, \mathcal{F}, \mathbb{P})$ *where* $\mathbb{P}$ *is a probability measure on* $(\Gamma, \mathcal{F})$.

The second quantity we need to define are the n-point correlation functions.

Definition 3.2. *The* n***-point correlation function*** *of the point process* $(\Gamma, \mathcal{F}, \mathbb{P})$ *is a locally integrable function* $\rho^{(n)} : X^n \to \mathbb{R}_+$ *such that for any disjoint infinitesimally small subsets* $[x_i, x_i + \mathrm{d}x_i]$, $i = 1,\dots,n$,

$$\mathbb{P}(\#_\xi([x_i, x_i+\mathrm{d}x_i]) = 1, i = 1,\dots,n) = \rho^{(n)}(x_1,\dots,x_n)\mu(\mathrm{d}x_1)\dots\mu(\mathrm{d}x_n) \tag{3.1}$$

where μ *is a reference measure on* X, *e.g., the Lebesgue for* $X = \mathbb{R}$ *or the counting measure for* $X = \mathbb{Z}$.

Remark that the n-point correlation functions are symmetric in their arguments. In [28] are given the conditions, found by Lenard [17, 18], for locally integrable functions $\rho_n : X^n \to \mathbb{R}_+$ to be correlation function of some point process.

The correlation functions appears in the computation of expected values of observables. For example, consider a function u with $u(x) \in [0,1]$ for all x. Then

$$\mathbb{E}\biggl(\prod_j (1-u(x_j))\biggr) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{X^n} \rho^{(n)}(x_1,\ldots,x_n) \prod_{j=1}^{n} u(x_j) \mathrm{d}^n\mu(x) \quad (3.2)$$

An interesting class of point processes which will be considered in the rest of the section are the determinantal point processes, also called **fermionic** since the probability that two particles coincide is zero.

Definition 3.3. *A point process is called* ***determinantal point process*** *if the n-point correlation functions are given by*

$$\rho^{(n)}(x_1,\ldots,x_n) = \det(K(x_i,x_j))_{1\le i,j\le n} \quad (3.3)$$

where $K(x,y)$ is a kernel of an integral operator $K : L^2(X,\mu) \to L^2(X,\mu)$, non-negative and locally trace class.

The positivity is required because the n-point correlation functions are positive. The last condition reflect the fact that each configuration is locally finite. For $X = \mathbb{R}^d$ with $\mathrm{d}\mu = \mathrm{d}^d x$ it can be proven, see [28], that the integral kernel of the operator K in (3.3) can be chosen such that

$$\mathrm{Tr}(K\chi_B) = \int_B K(x,x)\mathrm{d}^d x \quad (3.4)$$

with $\chi_B(x)$ the indicator function of B. Since $K(x,x)$ is the particle density at x, (3.4) has to be finite, i.e., K locally trace class.

3.2 Fredholm determinant, hole probability

For a determinantal process, (3.3) in (3.2) leads to

$$\begin{aligned}\mathbb{E}\biggl(\prod_j (1-u(x_j))\biggr) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{X^n} \det(K(x_i,x_j))_{1\le i,j\le n} \prod_{j=1}^{n} u(x_j) \mathrm{d}^n\mu(x) \\ &\equiv \det(\mathbb{1} - uK)_{L^2(X,\mu)} \quad (3.5)\end{aligned}$$

where for each $\varphi \in L^2(X,\mu)$,

$$[(uK)\varphi](x) = \int_X u(x)K(x,y)\varphi(y)\mathrm{d}\mu(y). \quad (3.6)$$

The last determinant in (3.5) is called **Fredholm determinant** of the operator uK on the space $L^2(X,\mu)$. Remark that uK in (3.5) can be replaced by the symmetrized $u^{1/2}Ku^{1/2}$.

A special important case is the **hole probability**. Consider a $B \subseteq X$, then the probability that there are no particles in B is

$$\mathbb{P}(\#_\xi(B)=0) = \mathbb{E}\bigg(\prod_j (1-\chi_B(x_j))\bigg) = \det(\mathbb{1}-K)_{L^2(B,\mu)}. \tag{3.7}$$

Let us consider a determinantal point process on $\mathbb{R}$ or $\mathbb{Z}$ which has a *last particle* and denote its position by $x_{\max}$. Then the distribution of the last particle is given by

$$\mathbb{P}(x_{\max} \leq t) = \mathbb{P}(\#_\xi((t,\infty))=0) = \det(\mathbb{1}-K)_{L^2((t,\infty),\mu)}. \tag{3.8}$$

3.3 When a measure comes from a determinantal process

The structure of the measure (2.5) is a product of two determinants of *one-variable* functions times a product measure $\mathrm{d}^N\mu$. A result of Borodin (Prop. 2.2 of [5]), and Tracy and Widom ([31] for GUE), gives a condition on a measure to be the one of a determinantal process.

Theorem 3.4. [5, 31] *If we have a measure of the form*

$$\frac{1}{Z_N}\det(\varphi_j(x_k))_{j,k=1}^N \det(\psi_j(x_k))_{j,k=1}^N \mathrm{d}^N\mu(x), \tag{3.9}$$

then it is a determinantal process with kernel

$$K_N(x,y) = \sum_{i,j=1}^N \psi_i(x)[A^{-1}]_{i,j}\varphi_j(y) \tag{3.10}$$

where

$$A = [A_{i,j}]_{i,j=1}^N, \quad A_{i,j} = \int_X \psi_j(t)\varphi_i(t)\mathrm{d}\mu(t) \tag{3.11}$$

Unfortunately, although an explicit formula is given, it is not always easy (feasible) to invert the matrix A as $N\to\infty$. A particular case is when $A=\mathbb{1}$ in a particular basis. In this case the kernel $K_N(x,y)$ becomes of simple form and the limiting distribution can be analyzed.

3.4 Some important examples: sine kernel and Airy kernel

Let $x, x' \in \mathbb{R}$, the **sine kernel** is defined by

$$S(x,x') = \frac{\sin(\pi(x-x'))}{\pi(x-x')} \tag{3.12}$$

and the **Airy kernel** by

$$A(x,x') = \frac{\mathrm{Ai}(x)\,\mathrm{Ai}'(x') - \mathrm{Ai}'(x)\,\mathrm{Ai}(x')}{x-x'} \tag{3.13}$$

where $\mathrm{Ai}(x)$ is the Airy function [1]. In some models appears the *discrete sine kernel*, which means only that $x, x' \in \mathbb{Z}$.

In the asymptotic limit when the bulk of the system (or spectrum) is considered, one can find the sine kernel, see Section 4.1 for the GUE random matrices case. The Airy kernel arises at the edge of the system (or spectrum), see Section 5 for a discussion.

4 GUE and determinantal processes

Let us consider the case of $N \times N$ hermitian matrices and let $V(x)$ be an *even degree* polynomial with positive leading coefficient. Then we define a measure on the random matrices by

$$p(H)\mathrm{d}H = \frac{1}{Z'_N} e^{-\operatorname{Tr} V(H)} \mathrm{d}H \tag{4.1}$$

with $\mathrm{d}H = \prod_{i=1}^N \mathrm{d}H_{i,i} \prod_{1\leq i<j\leq N} \mathrm{dRe}H_{i,j}\ \mathrm{dIm}H_{i,j}$.

The GUE ensemble is recovered by setting $V(x) = x^2$. Then for the eigenvalue distribution of H one obtains

$$p(\lambda_1,\ldots,\lambda_N)\mathrm{d}\lambda_1\cdots\mathrm{d}\lambda_N = \frac{1}{Z_N}\Delta_N(\lambda)^2 \prod_{j=1}^N e^{-V(\lambda_j)}\mathrm{d}\lambda_j. \tag{4.2}$$

The determinantal nature of the eigenvalues is a result of Gaudin, Mehta, and Dyson works, see Chapter 5 of [19] for the GUE case. Let $p_k(x)$, $k = 0,1,\ldots$ be the orthogonal polynomials with respect to $e^{-V(x)}\mathrm{d}x$, normalized as $\int_{\mathbb{R}} p_i(x)p_j(x)e^{-V(x)}\mathrm{d}x = \delta_{i,j}$. Then

Theorem 4.1. *The eigenvalue process is a determinantal process with correlation kernel*

$$K_N(x,y) = \sum_{k=0}^{N-1} p_k(x)p_k(y)e^{-\frac{1}{2}(V(x)+V(y))}. \tag{4.3}$$

(4.3) can be rewritten using Christoffel-Darboux formula [30] as

$$K_N(x,y) = \frac{u_{N-1}}{u_N}\frac{p_N(x)p_{N-1}(y) - p_{N-1}(x)p_N(y)}{x-y}. \tag{4.4}$$

where u_k is the leading coefficient of p_k.

The connection with Theorem 3.4 is as follows. Originally one has $\varphi_j(x) = \psi_j(x) = x^{j-1}$. After a change of basis (Gram-Schmidt orthonormalization) one obtains (3.9) with $\varphi_j(x) = \psi_j(x) = p_{j-1}(x)e^{-x^2/2}$, $A = \mathbb{1}$, and the p_j's being the Hermite polynomials. Then (3.10) leads to (4.3) with $V(x) = x^2$.

4.1 Asymptotics for GUE

In the limit $N \to \infty$, the sine and Airy kernels appears depending on the focused region: bulk or edge.

The sine kernel arises in the bulk of the spectrum as follows. Let $a \in (-1,1)$, then

$$\lim_{N\to\infty} \frac{1}{u(a)\sqrt{2N}} K_N\left(a\sqrt{2N} + \frac{x}{u(a)\sqrt{2N}}, a\sqrt{2N} + \frac{x'}{u(a)\sqrt{2N}}\right) = S(x,x'). \tag{4.5}$$

The Airy kernel arises in the edge-scaling:

$$\lim_{N\to\infty} \frac{1}{2^{1/2}N^{1/6}} K_N\left(\sqrt{2N} + \frac{x}{2^{1/2}N^{1/6}}, \sqrt{2N} + \frac{x'}{2^{1/2}N^{1/6}}\right) = A(x,x'). \tag{4.6}$$

From the asymptotic at the edge of the spectrum, the GUE Tracy-Widom distribution $F_2(s)$ can be expressed as the Fredholm determinant

$$F_2(s) = \det(\mathbb{1} - A)_{L^2((t,\infty),\mathrm{d}x)} \tag{4.7}$$

with A the Airy kernel.

4.2 A note on GOE and GSE: Pfaffian processes

The eigenvalue point process for GOE and GSE are not determinantal but **Pfaffian processes**, which are generalization of the determinantal processes. They were introduced in [26], see also the introduction of [29].

Pfaffians

First we define the Pfaffian. Let $A = [A_{i,j}]_{i,j=1}^{2N}$ be an *antisymmetric* matrix, then its Pfaffian is defined by

$$\mathrm{Pf}(A) = \sum_{\substack{\sigma \in S_{2N} \\ \sigma_{2i-1} < \sigma_{2i}}} (-1)^{|\sigma|} \prod_{i=1}^{N} A_{\sigma_{2i-1},\sigma_{2i}}. \tag{4.8}$$

The Pfaffian is the square root of the determinant: $\mathrm{Pf}(A) = \sqrt{\det A}$ if A is antisymmetric.

Pfaffian processes

Let (X, μ) be a measure space, $f_1, \ldots, f_{2N}$ complex-valued functions on X and $\varepsilon(x, y)$ be an *antisymmetric kernel* such that

$$p(x_1, \ldots, x_{2N}) = \frac{1}{Z_{2N}} \det[f_j(x_k)]_{j,i=1}^{2N} \, \mathrm{Pf}[\varepsilon(x_j, x_k)]_{j,k=1}^{2N} \tag{4.9}$$

defines the density of a $2N$-dimensional probability distribution on X^{2N} with respect to the product measure generated by μ.

The normalization constant Z_{2N} equals $(2N)! \, \mathrm{Pf}[M]$ where the matrix $M = [M_{i,j}]_{i,j=1}^{2N}$ is defined as

$$M_{i,j} = \int_{X^2} f_i(x) \varepsilon(x, y) f_j(y) \mathrm{d}\mu(x) \mathrm{d}\mu(y). \tag{4.10}$$

The *n-point correlation functions $\rho^{(n)}(x_1, \ldots, x_n)$ are given by Pfaffians:*

$$\rho^{(n)}(x_1, \ldots, x_n) = \mathrm{Pf}[\tilde{K}(x_i, x_j)]_{i,j=1}^{n} \tag{4.11}$$

where $\tilde{K}(x, y)$ is the antisymmetric kernel

$$\tilde{K}(x, y) = \begin{pmatrix} \tilde{K}_1(x, y) & \tilde{K}_2(x, y) \\ \tilde{K}_3(x, y) & \tilde{K}_4(x, y) \end{pmatrix} \tag{4.12}$$

with

$$\begin{aligned} \tilde{K}_1(x, y) &= \textstyle\sum_{i,j=1}^{2N} f_i(x) M_{i,j}^{-T} f_j(y) \\ \tilde{K}_2(x, y) &= \textstyle\sum_{i,j=1}^{2N} f_i(x) M_{i,j}^{-T} (\varepsilon f_j)(y) \\ \tilde{K}_3(x, y) &= \textstyle\sum_{i,j=1}^{2N} (\varepsilon f_i)(x) M_{i,j}^{-T} f_j(y) \\ \tilde{K}_4(x, y) &= -\varepsilon(x, y) + \textstyle\sum_{i,j=1}^{2N} (\varepsilon f_i)(x) M_{i,j}^{-T} (\varepsilon f_j)(y) \end{aligned} \tag{4.13}$$

provided that M is invertible, and $(\varepsilon f_i)(x) = \int_X \varepsilon(x, y) f_i(y) \mathrm{d}\mu(y)$.

If the kernel has the form $\begin{pmatrix} \varepsilon & K_0 \\ -K_0 & 0 \end{pmatrix}$, then it is a *determinantal process.*

GOE and GSE as Pfaffian processes

As explained in [29], the GOE and GSE are Pfaffian processes.

The GOE case is recovered as follows. Let $X = \mathbb{R}$, $f_j(x) = x^{j-1}$, $j = 1, \ldots, 2N$, $\varepsilon(x, y) = \frac{1}{2}\operatorname{sgn}(y - x)$, and $\mathrm{d}\mu(x) = e^{-x^2}\mathrm{d}x$. Then the probability density (4.9) is the one of (2.5) for $\beta = 1$.

The GSE case is slightly more complicated. Let $X = Y \cup Z$ with $Y = Z = \mathbb{R}$, $\mathrm{d}\mu(x) = e^{-x^2}\mathrm{d}x$. The configuration of $2N$ particles $x_1, \ldots, x_{2N}$ in X consists into two identical copies of N particles $y_1, \ldots, y_N$ in Y and $z_1, \ldots, z_N$ in Z. Then define $f_j(y) = y^j$ for $y \in Y$, $f_j(z) = jz^{j-1}$ for $z \in Z$, and the antisymmetric kernel as $\varepsilon(y_1, y_2) = \varepsilon(z_1, z_2) = 0$, $\varepsilon(y, z) = -\varepsilon(z, y) = \delta_{y,z}$. With this setting, the probability density (4.9) is the one of (2.5) for $\beta = 4$.

In [9] (section 4), the eigenvalue distribution for GOE and GSE was also studied and they showed that (3.5) becomes

$$\mathbb{E}\bigg(\prod_j (1 - u(x_j))\bigg) = \sqrt{\det(\mathbb{1} - Ku)} \tag{4.14}$$

for some 2×2 matrix kernels K. These kernels are closely related to the one introduced above. K can indeed be taken to be $K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tilde{K}$.

5 Some models where F_β arises

The F_2 distribution arises together with the Airy kernel. For example at the edge of the system (or spectrum), like for the GUE random matrices [20, 8]. Other examples where the Airy kernel and the GUE Tracy-Widom distribution appear are the reported below. These models are not directly connected with GUE random matrices, but they have the same limiting distribution because, at least in the asymptotic limit, they are determinantal processes in the same class of GUE eigenvalues.

1) **The 3D Ising corner**. In [7] we analyze a simplified model of a crystal corner at low temperature which is equivalent to the following problem. Let us consider a ferromagnetic three-dimensional Ising model on $\mathbb{Z}^3$ and a starting configuration where all the spins σ_x in the positive octant are -1 and the others are $+1$, i.e., $\sigma_x = -1$ for $x \in \mathbb{Z}^3_+$ and $\sigma_x = +1$ for $x = (x_1, x_2, x_3) \in \mathbb{Z}^3 \setminus \mathbb{Z}^3_+$. At zero temperature, spins can flip if the number of nearest neighbour with opposite spins is conserved. Let $V(\sigma) = \sum_{x \in \mathbb{Z}^3_+} (\sigma_x + 1)/2$ be the number of spin which have flipped with respect to the starting configuration. To have an equilibrium state, we add

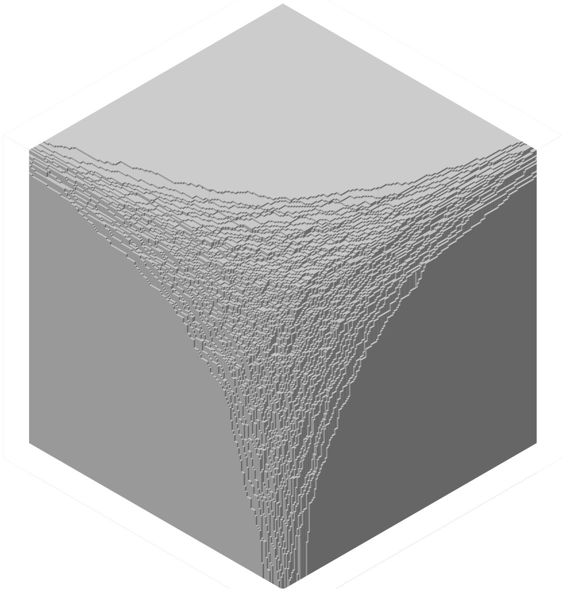

Figure 2: *3D Ising corner viewed from the* (111)*-direction for* $L = 50$.

a binding chemical potential $\mu = -\frac{1}{L}$, which implies that the weight of a configuration is $\exp(-\frac{1}{L}V(\sigma))$. A computer generated realization is shown in Figure 2. We denote by k^{th} level line the line bordering the $+$ and $-$ phase at height k above the $1-2$ plane. Then we consider the projection on the $(1\,1\,1)$ plane of these lines. For fixed $t = x_2 - x_1$, the positions of the lines is a determinantal process. We are interested in the statistics of the line bordering the flat facet and the rounder piece, i.e., the position of the 1^{st} level line. Let then b_L the position of the 1^{st} level line at $t = 0$. It grows like cL, $c = \ln 4$ plus some fluctuations of order $L^{1/3}$. We prove that, as $L \to \infty$, for $\kappa = 4^{-1/3}$, $(b_L - cL)/(\kappa L^{1/3}) \to \zeta_2$, a random variable F_2 distributed. The same result, with other values of c and κ, holds for any $t = \tau L$, τ fixed.

2) **The longest increasing subsequences of random permutations**. Let us consider the set of all permutations of N numbers. Let σ be a permutation. An increasing subsequence of length k is a sequence of numbers $1 \leq j_1 < \ldots < j_k \leq N$ such that $\sigma(j_n) \leq \sigma(j_{n+1})$ for $n = 1, \ldots, k-1$. The problem of finding the behavior of the length of the longest increasing subsequence, ℓ_N, in the limit $N \to \infty$, was introduced by Ulam in 1961 [33]. Baik, Deift, and Johansson in [3] prove a previous conjecture, i.e., $\ell_N \sim 2\sqrt{N}$ for large N, and more importantly find the law of the fluctuations. More precisely, they show that, as $N \to \infty$, $(\ell_N - 2\sqrt{N})/(2^{1/2}N^{1/6}) \to \zeta_2$ where ζ_2 is

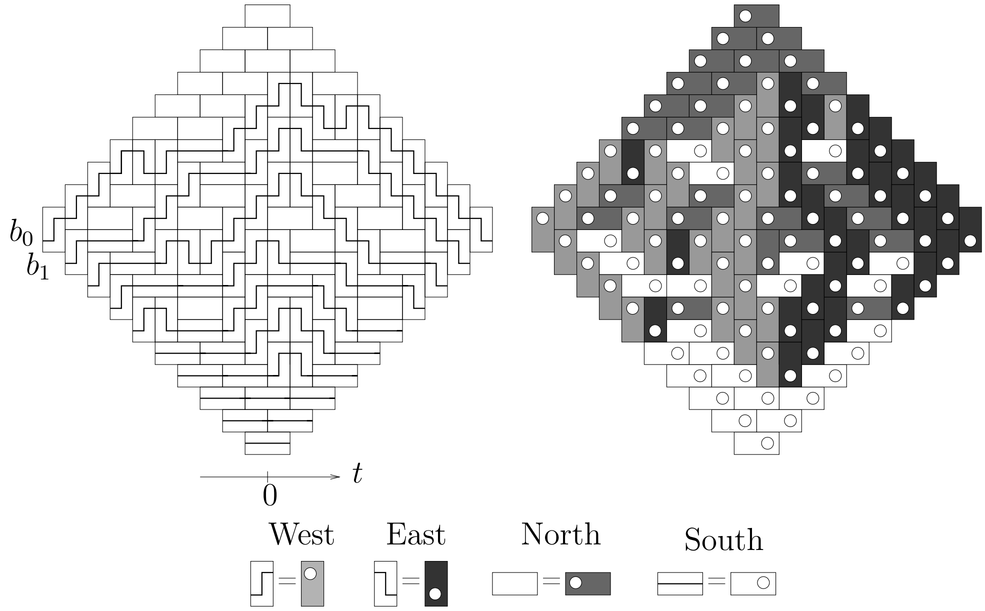

Figure 3: *Aztec diamond with* $N = 10$.

a random variable F_2-distributed. A nice review of this problem is [2].

3) **The polynuclear growth model (PNG)**. In [23, 25] a 1+1 dimensional growth model is studied. An initial flat one-dimensional substrate is considered where nucleations generate a droplet with the following rule. There is a first island which grows with unit speed laterally. Nucleations can occurs only above the first island, independently and with unit rate. Each nucleation generates an island, of height one, which start spreading with unit speed. When two islands meet, they simply merge. Consider the distribution of the height above the origin h_t after time t. Then as $t \to \infty$, $(h_t - 2t)/t^{1/3} \to \zeta_2$, a random variable F_2 distributed. For the height above a position different from the origin a similar result holds. A discretized version of this model is studied in [13]. In [27] a half-space version of the PNG is also analyzed.

4) Some other **corner growth models** where F_2 appears are studied in [10].

5) **The arctic circle of the Aztec diamond**. The Aztec diamond is defined as follows. Let us consider the diamond shaped subset containing $2N^2$ squares of a checkerboard table, see Figure 3 for an example with $N = 10$. It is called Aztec diamond. Then one considers the set of random tilings

with dominos of the Aztec diamond with uniform weight. For large N the Aztec diamond divides into five regions delimited by the arctic circle [11]. Inside the arctic circle there is a disordered region of the tiling. Outside this boundary the tilings forms a completely regular brick wall pattern. There are four type of dominos, depending on the direction and on the position with respect to the checkerboard table, called East, West, North and South. It is possible to draw continuous lines in the Aztec diamond by adding in each domino a line pattern depending on its type as shown in Figure 3. Let t be the horizontal coordinate with $t = 0$ in the middle of the diamond and denote by $b_{\ell,N}(t)$, $\ell = 0, \ldots, N-1$ be the position of the ℓ^{th} line at "time" t. Then it is proven [12] that $(b_{0,N}(0) - N/\sqrt{2})/(2^{-5/6}N^{1/3}) \to \zeta_2$ as $N \to \infty$, with ζ_2 a random variable F_2-distributed.

6) The problem of **vicious random walks** [21] and of **non-colliding Brownian particles** [15]. In the vicious random walks, i.e., non-intersecting, problem one considers particles on $\mathbb{Z}$ which starts from $0, -2, -4, \ldots$. At every unit time-step, they move randomly to one of the nearest neighbouring sites. Up to time T there is the constraint that two particles can not occupy the same position at same time. It is shown in [21] that the distribution of the first particle at time τT with $\tau \in (0, 1)$ converges to F_2 and at $\tau = 1$ there is a transition to F_1.

The F_1 distribution shows up also in the PNG model when the nucleations are not constraint to occur above the first island [23], i.e., translation invariant, or above the origin of the half space growth model with sources at the origin [27]. In this last model, if no sources are there, then F_4 arises.

Finally, in [4] it is studied the problem of the longest increasing subsequence with additional constraints and symmetries. They find the distributions F_1, F_2, F_4 and others [4].

Remark: The results of PNG and the 3D Ising corner model uses a space-time extension of the determinantal processes. The results of these papers are stronger than the one discussed here. For example in the Ising problem it is proven the convergence of the 1^{st} line to the so called Airy process introduced in [25]. For the Aztec diamond it is proved in [14] that the first line converges to the Airy process too.

References

[1] M. Abramowitz and I. A. Stegun, *Pocketbook of Mathematical Functions*, Verlag Harri Deutsch, Thun-Frankfurt am Main, 1984.

[2] D. Aldous and P. Diaconis, *Longest increasing subsequences: from patience sorting to the Baik-Deift-Johansson theorem*, Bull. Amer. Math. Soc. **36** (1999), 413–432.

[3] J. Baik, P.A. Deift, and K. Johansson, *On the distribution of the length of the longest increasing subsequence of random permutations*, J. Amer. Math. Soc. **12** (1999), 1119–1178.

[4] J. Baik and E. M. Rains, *Symmetrized random permuationsm* In *" Random Matrix Models and Their Applications"*, vol. 40, Cambridge University Press, 2001.

[5] A. Borodin, *Biorthogonal ensembles*, Nuclear Phys. B **536** (1999), 704–732.

[6] F. J. Dyson, *The three fold way. Algebraic structure of symmetry groups and ensembles in quantum mechanics*, J. Math. Phys. **3** (1962), 1199–1215.

[7] P. L. Ferrari and H. Spohn, *Step fluctations for a faceted crystal*, J. Stat. Phys. **113** (2003), 1–46.

[8] P. J. Forrester, *The spectrum edge of random matrix ensembles*, Nuclear Phys. B **402** (1993), 709–728.

[9] P. J. Forrester, T. Nagao, and G. Honner, *Correlations for the orthogonal-unitary and symplectic-unitary transitions at the hard and soft edges*, Nuclear Phys. B **553** (1999), 601–643.

[10] J. Gravner, C.A. Tracy, and H. Widom, *Limit theorems for height fluctuations in a class of discrete space and time growth models*, J. Stat. Phys. **102** (2001), 1085–1132.

[11] W. Jockush, J. Propp, and P. Shor, *Random domino tilings and the arctic circle theorem*, preprint, arXiv:math.CO/9801068 (1995).

[12] K. Johansson, *Shape fluctuations and random matrices*, Comm. Math. Phys. **34** (2000), 437–476.

[13] K. Johansson, *Discrete polynuclear growth and determinantal processes*, preprint, arXiv:math.PR/0206208 (2002).

[14] K. Johansson, *The arctic circle boundary and the Airy process*, preprint, arXiv:math.PR/0306216, To appear in Comm. Math. Phys. (2003).

[15] M. Katori, T. Nagao, and H. Tanemura, *Infinite systems of non-colliding Brownian particles*, To appear in: Adv. Stud. Pure Math. (2003).

[16] H. Kunz, *Matrices aléatoires en physique*, Presse Polytechniques et Universitaires Romandes, Lausanne, 1998.

[17] A. Lenard, *States of classical statistical mechanical system of infinitely many particles I*, Arch. Rational Mech. Anal. **59** (1975), 219–239.

[18] A. Lenard, *States of classical statistical mechanical system of infinitely many particles II*, Arch. Rational Mech. Anal. **59** (1975), 240–256.

[19] M. L. Mehta, *Random Matrices*, 2nd ed., Academic Press, San Diego, 1991.

[20] G. Moore, *Matrix models of 2D quantum gravity and isomonodromic deformation*, Prog. Theoret. Phys. Suppl. **102** (1990), 255–285.

[21] T. Nagao, M. Katori, and H. Tanemura, *Dynamical correlations among vicious random walkers*, Phys. Lett. A **307** (2003), 29–33.

[22] L. Pastur, *Random matrices as paradigm*, Mathematical Physics 2000 (Singapore), World Scientific, 2000, pp. 216–266.

[23] M. Prähofer and H. Spohn, *Universal distributions for growth processes in* $1+1$ *dimensions and random matrices*, Phys. Rev. Lett. **84** (2000), 4882–4885.

[24] M. Prähofer and H. Spohn, *Exact scaling function for one-dimensional stationary KPZ growth*, http://www-m5.ma.tum.de/KPZ/ (2002).

[25] M. Prähofer and H. Spohn, *Scale invariance of the PNG droplet and the Airy process*, J. Stat. Phys. **108** (2002), 1071–1106.

[26] E. M. Rains, *Correlation functions for symmetrized increasing subsequences*, arXiv:math.CO/0006097 (2000).

[27] T. Sasamoto and T. Imamura, *Fluctuations of a one-dimensional polynuclear growth model in a half space*, preprint, arXiv:cond-mat/0307011, To appear in J. Stat. Phys. (2003).

[28] A. Soshnikov, *Determinantal random point fields*, Russian Math. Surveys **55** (2000), 923–976.

[29] A. Soshnikov, *Janossy densities II. Pfaffian ensembles*, arXiv:math-ph/0301003 (2003).

[30] G. Szegö, *Orthogonal Polynomials*, 3th ed. ed., American Mathematical Society Providence, Rhode Island, 1967.

[31] C. A. Tracy and H. Widom, *Correlation functions, cluster functions, and spacing distributions for random matrices*, J. Stat. Phys. **92** (1998), 809–835.

[32] C. A. Tracy and H. Widom, *Distribution functions for largest eigenvalues and their applications*, Proceedings of ICM 2002, vol. 3, 2002.

[33] S. M. Ulam, *Monte Carlo calculations in problems of mathematical physics*, Modern Mathematics for the Engineer (E.F.Beckenbach ed., McGraw-Hill, New York, Toronto, London), vol. II, 1961, pp. 261–277.