Limit distributions for KPZ growth models with spatially homogeneous random initial conditions

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(joint work with S. Chhita and H. Spohn)

For stochastic growth models in the Kardar-Parisi-Zhang (KPZ) universality class over a one-dimensional substrate the height fluctuations "always" broaden as $t^{1/3}$. On the other hand the full probability density function depends on the choice of the initial data. As well known, for a flat initial surface, $h(x, t = 0) = 0$, the large t fluctuations of $h(0, t)$ are distributed according to GOE Tracy-Widom distribution [5, 18, 22]. In contrast, if the height profile is macroscopically curved, then GOE has to be replaced by GUE [2, 3, 7, 13, 16, 20, 21].

If as a surface growth model we consider the one-dimensional KPZ equation,

$$\partial_t h = \tfrac{1}{2}(\partial_x h)^2 + \tfrac{1}{2}\partial_x^2 h + \xi$$

with $\xi(x, t)$ normalized space-time white noise, then the time-stationary initial data are

$$h(x, 0) = B(x)$$

with $B(x)$ a two-sided Brownian motion. As shown in [6] (for other KPZ models, see [1, 4, 12, 15]),

$$h(0, t) \simeq -\tfrac{1}{24}t + (t/2)^{1/3}\xi_{\rm BR}$$

for large t and the random amplitude $\xi_{\rm BR}$ is Baik-Rains distributed [4]. Recently, Quastel and Remenik [19] identified a large domain of attraction for GOE Tracy-Widom distribution. Roughly speaking, for a macroscopically flat initial profile, if it satisfies $|h(x, 0) - h(0, 0)| \simeq |x|^{1/2}$ for large $|x|$ is the borderline below which the height fluctuations are GOE Tracy-Widom distributed.

We consider translation invariant random initial data, for which height differences typically grow as $|x|^{1/2}$. More precisely, for the totally asymmetric simple exclusion process, TASEP, with initial slopes $\eta_j(t = 0) = \eta_j \in \{0, 1\}$, we allow initial conditions such that $\{\eta_j | j \in \mathbb{Z}\}$ is a stationary stochastic process satisfying the functional central limit theorem

$$\lim_{\ell\to\infty} \frac{1}{\sqrt{\ell}} \sum_{j=0}^{[\gamma x \ell]} \big(\eta_j - \langle\eta_0\rangle\big) = \sigma B(x)$$

for some $\sigma \geq 0$. Here γ is a scaling constant set by the fact that $\sigma = 1$ corresponds to stationary initial condition. We show that for each σ there is a distinct distribution function $F^{(\sigma)}(s)$.

Denote by ρ the expected density of particles and j the expected (infinitesimal) current of particles. Then if $\mathsf{j}'(\rho) = 0$ holds, the time correlations are relevant around the origin and the height fluctuations, as obtained from $\eta_j(t)$, are governed by $F^{(\sigma)}(s)$ in the large t limit. If $\mathsf{j}'(\rho) \neq 0$, then correlations spread at a non-zero velocity and $F^{(\sigma)}(s)$ will be observed after properly centering (see e.g. [12] in the $\sigma = 1$ case).

For the TASEP we prove that the limiting distribution is determined through a variational formula,

$$F^{(\sigma)}(s) = \mathbb{P}\Big(\sup_{x\in\mathbb{R}}\{\sqrt{2}\sigma B(x) + \mathcal{A}_2(x) - x^2\} \leq s\Big),$$

where $\mathcal{A}_2(x)$ is the Airy process and is independent of the two-sided Brownian motion $B(x)$. The proof employs several non-trivial results obtained only recently, the most important ones being tightness [9] for the point-to-point process with ending points on horizontal lines, and the one-point slow-decorrelation [10]. Finally one also needs to know the convergence of the finite-dimensional distributions [8]. These ingredients can be used to obtain a functional slow-decorrelation result, see [11] for the discrete time counterpart. Interestingly, this latter result then implies tightness of the point-to-point process along generic lines, which is a result not covered by the elegant and soft arguments of [9].

As already proved in [17], $F^{(0)}(s) = F_{\rm GOE}(2^{2/3}s)$, with $F_{\rm GOE}$ the GOE Tracy-Widom distribution. Our result indirectly implies that $F^{(1)}(s)$ equals the Baik-Rains distribution. The only other explicit solution corresponds formally to the limit $\sigma \to \infty$, which reads (after scaling s with $\sigma^{4/3}$)

$$\mathbb{P}\Big(\sup_{x\in\mathbb{R}}\{B(x) - x^2\} \leq s\Big).$$

An explicit representation is provided in [14]. Its probability density vanishes for $s < 0$ and decays as a stretched exponential with power $\frac{3}{2}$ for $s \to \infty$.

For all other values of σ we have to rely on Monte Carlo simulations, see Figure 1 for a plot of the densities of $F^{(\sigma)}$ for some values of σ.

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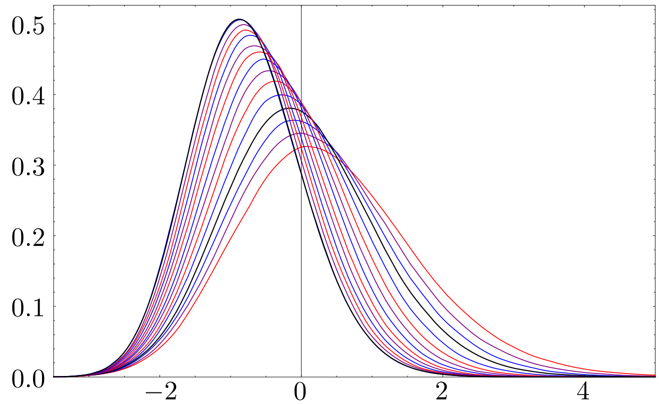

FIGURE 1. Probability densities of $F^{(\sigma)}(s)$ with $\sigma = \sqrt{\alpha/(1-\alpha)}$ from TASEP simulation until time $t_{\max} = 10^3$ and 10^6 runs. The different plots corresponds to the values $\alpha = 0, 0.05, 0.1, 0.15, 0.20, 0.25, 0.3, 0.35, 0.4, 0.45, 0.5, 0.54, 0.58, 0.62$. The left-most black line is the exact rescaled GOE distribution ($\sigma = 0$), which overlaps with $\alpha = 0$ from the simulations. The black line in the middle is the stationary case ($\sigma = 1$).

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