The Airy$_1$ and Airy$_2$ processes in the TASEP

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We consider a stochastic interacting particle system, the totally asymmetric simple exclusion process (TASEP) on $\mathbb{Z}$ in continuous time. At any given time t, every site $j \in \mathbb{Z}$ can be occupied at most by one particle. Thus a configuration of the TASEP can be described by $\eta = \{\eta_j, j \in \mathbb{Z} | \eta_j \in \{0,1\}\}$. η_j is called the *occupation variable* of site j, which is defined by $\eta_j = 1$ if site j is occupied and $\eta_j = 0$ if site j is empty.

The dynamics of the TASEP is defined as follows. Particles jumps on the neighboring right site with rate 1 provided that the site is empty. This means that jumps are independent of each other and occur after an exponential waiting time with mean 1, which is counted from the time instant when the right neighbor site is empty.

On a macroscopic scale the density of particles $u(x,t)$ evolves deterministically according to the Burger's equation $\partial_t u + \partial_x(u(1-u)) = 0$ [15]. Therefore it is natural to focus on fluctuations properties and large deviations, which have some interesting and unexpected features. The observables analyzed in our recent works [2, 3] are the positions of given particles, which are closely related to integrated particle currents. It turns out that the observables fluctuation depends on the initial condition. Thus the natural question is to analyze which kind of initial conditions leads to a common limit distribution and limit process.

The first result in this direction has been obtained with *step initial conditions.* To be precise, let us denote by $x_k(t)$ the position at time t of the particle with label k. Then step initial condition means $x_k(0) = -k$, $k \in \mathbb{N}$, which is studied by Johansson [8, 9] in terms of a corner growth model. The positions of particles fluctuate on a $t^{1/3}$-scale while two particles are (in this scale) non-trivially correlated if they are at a distance of order $t^{2/3}$. For example,

$$\lim_{t\to\infty} \frac{x_{[t/4+u(t/2)^{2/3}]}(t) - (-2u(t/2)^{2/3} + u^2(t/2)^{1/3})}{-(t/2)^{1/3}} = \mathcal{A}_2(u) \tag{1}$$

where $\mathcal{A}_2$ is the Airy$_2$ process (usually simply called Airy process), first discovered in the polynuclear growth (PNG) model under droplet growth [13]. The 1/3 and 2/3 exponents are the one of the KPZ universality class [10]. The Airy$_2$ process is the marginal of the determinantal point process with extended Airy kernel. $\mathcal{A}_2$ appears also in Dyson's Brownian Motion [4], where the motion of the largest eigenvalue properly rescaled converges to the Airy$_2$ process [9]. In particular, the one-point distribution of $\mathcal{A}_2$ is the GUE Tracy-Widom distribution [19]. The same result holds if one focuses around $k \sim \alpha t$, $\alpha \in (0,1)$, but with different numerical factors.

Besides the step-initial condition explained above, two other situations are of particular interest. One is the *stationary* initial condition, where the one-point distribution has been obtained in [7]. The second are *deterministic* initial conditions leading to a macroscopically uniform density profile, thus called *flat initial conditions*. The simplest realization is obtained by setting $x_k(0) = -2k$, $k \in \mathbb{Z}$.

In [16] an important new result has been discovered, allowing the analysis of such initial conditions. First of all, as expected by universality, the fluctuations of the position of a particle is governed by the GOE Tracy-Widom distribution, F_1 [20]. This result is a combination of [6, 16], that is,

$$\lim_{t\to\infty} \mathbb{P}(x_{[t/4]}(t) \leq -st^{1/3}) = F_1(2s). \tag{2}$$

More importantly, for flat initial condition, the analogue of the Airy$_2$ process has been determined, which we denote by $\mathcal{A}_1$ and call Airy$_1$ process. It is the marginal of the determinantal point measure with the extended kernel K_{F_1} given as follows. Let $B_0(x, y) = \mathrm{Ai}(x + y)$ and Δ the one-dimensional Laplacian, then

$$K_{\mathrm{F}_1}(u_1, s_1; u_2, s_2) = -(e^{(u_2-u_1)\Delta})(s_1, s_2)\mathbb{1}(u_2 > u_1) + (e^{-u_1\Delta}B_0 e^{u_2\Delta})(s_1, s_2). \tag{3}$$

The process $\mathcal{A}_1$ has m-point joint distributions at $u_1 < u_2 < \ldots < u_m$ given by a Fredholm determinant (regarded as its Fredholm series)

$$\mathbb{P}\Big(\bigcap_{k=1}^{m}\{\mathcal{A}_1(u_k) \leq s_k\}\Big) = \det(\mathbb{1} - \chi_s K_{\mathrm{F}_1}\chi_s)_{L^2(\{u_1,\ldots,u_m\}\times\mathbb{R})} \tag{4}$$

where $\chi_s(u_i, x) = \mathbb{1}(x > s_i)$.

In [3] we analyze the continuous-time TASEP with $x_k(0) = -2k$, $k \in \mathbb{Z}$, and show that the joint distributions of particle positions are given by a Fredholm determinant of a kernel. Then in the appropriate scaling limit we obtain point-wise convergence of the kernel to K_{F_1}. The analysis starts from a determinantal formula of the joint distributions of particle position obtained by Schütz [17]. In [2] we consider the discrete-time TASEP with sequential update for which the corresponding of Schütz formula has been determined in [14]. The analogue formula for parallel update has been obtained in a recent work [11], but whether a similar approach as in [2, 3] can be applied has still to be investigated. There are other update rules introduced in the literature, but we will not discuss them. For a review, see [18].

Instead of restricting to density 1/2 (the $d = 2$ case) we consider a more general set of initial conditions: for any integer $d \geq 2$, we take $x_k(0) = -dk$, $k \in \mathbb{Z}$. By universality it is expected that the limit process is independent of d (if $d \geq 2$). This is proven in [2], where we show convergence of Fredholm determinants too, thus convergence in the sense of finite-dimensional distributions to the Airy$_1$ process. The final result, rewritten for continuous-time TASEP, is

$$\lim_{t\to\infty} \frac{x_{\lfloor \alpha t + \mu t^{2/3}\rfloor}(t) + d\mu u t^{2/3}}{-\kappa t^{1/3}} = \mathcal{A}_1(u) \tag{5}$$

with $\kappa = \frac{2^{1/3}(d(d-1))^{2/3}}{d}$, $\alpha = \frac{d-1}{d^2}$, and $\mu = \frac{2^{5/3}(d(d-1))^{1/3}}{d^2}$.

As briefly discussed in [3], the TASEP can also be reinterpreted as a stochastic growth model, a directed last passage percolation, and a directed polymer model. Step initial conditions corresponds to point-to-point directed polymers [8,9] and corner growth [13]. There the Airy$_2$ process appears. Flat initial condition translates into growth on a flat substrate [1, 5, 12] and point-to-line directed polymers. In particular, $d \geq 3$ is growth on a *flat but tilted surface*, and to our knowledge, the analysis of the limit distribution and/or limit process has not been carried out before for models in the KPZ class.

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