MIXING TIMES FOR THE OPEN ASEP AT THE TRIPLE POINT

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Abstract. We consider mixing times for the open asymmetric simple exclusion process (ASEP) at the triple point. We show that the mixing time of the open ASEP on a segment of length N for bias parameter q is of order $N^{3/2+\kappa}$ if $1-q \asymp N^{-\kappa}$ for some $\kappa \in [0, \frac{1}{2})$, and the same result with poly-logarithmic corrections for $\kappa = \frac{1}{2}$. Our proof combines a fine analysis of the current of the open ASEP, moderate deviations of second class particles, the censoring inequality, and various couplings and multi-species extensions of the ASEP. Moreover, we establish a comparison between moderate deviations for the current of the open ASEP and the ASEP on the integers, as well as bounds on mixing times for the open ASEP in the weakly high density phase, which are of independent interest.

1. Introduction

The asymmetric simple exclusion process (ASEP) is a fundamental model in the study of interacting particle systems. Over recent decades, it has been analyzed from various perspectives, including statistical mechanics, probability theory, and combinatorics; see [19, 61, 81] for relevant surveys. In recent years, ASEP has gained further attention as one of the best studied and rigorously established examples belonging to the Kardar–Parisi–Zhang (KPZ) universality class; see [33, 34] for an overview by Corwin.

In this work, we focus on the asymmetric simple exclusion process with open boundaries, also known as the open ASEP. This model can be intuitively described as follows: Consider a segment of length N, where each site is either occupied by a particle or left empty. Each site has two independent Poisson clocks with rates 1 and q, where $q = q(N) \in [0, 1)$. When the rate 1 clock rings at an occupied site, the particle moves to the right if the neighboring site is empty. Similarly, when the rate q clock rings at an occupied site, the particle moves to the left, provided the destination site is empty. At the left boundary, particles can enter with rate α or exit with rate γ, while at the right boundary, particles exit with rate β or enter with rate δ; see Figure 1. Here, $\alpha, \beta, \gamma, \delta \geq 0$ are parameters which may also depend on N. Depending on the choice of these boundary parameters, the open ASEP exhibits three distinct phases, illustrated in Figure 2. Our analysis focuses on a particular case called the triple point, where the three phases meet, and which is closely linked to constructing solutions to the open KPZ equation [15, 25, 26, 34, 35, 36, 52].

A topic of particular interest for exclusion processes is the analysis of mixing times. Mixing times provide a standard way for Markov chains of quantifying the speed at which a process converges to its stationary distribution; see [59] for an introduction. In the context of the open ASEP, questions about mixing times are closely linked to properties of the ASEP on the integers. It is well-known that the fluctuations of the particle current,

Date: 7. January 2025.
2010 *Mathematics Subject Classification.* Primary: 60K35; Secondary: 60K37, 60J27.
Key words and phrases. exclusion process, KPZ universality class, mixing times, triple point.

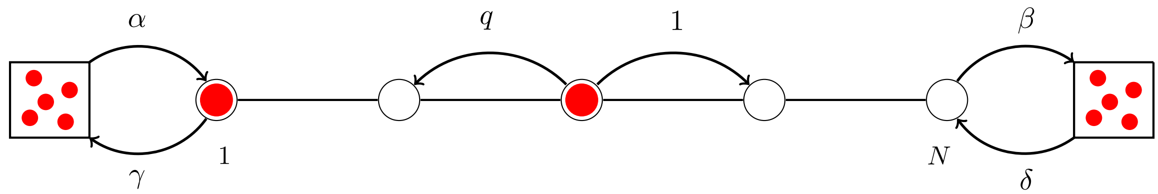

FIGURE 1. Visualization of the open ASEP with respect to parameters $q, \alpha, \beta, \gamma, \delta$.

i.e., the number of particles passing through a given site in the ASEP, can be used to express the speed of perturbation in the system, usually described in terms of second class particles; see [10]. Intuitively, mixing times reflect the time required for any initially placed perturbation, respectively second class particle to disappear. We will elaborate more on this connection after introducing the main model and presenting the main results.

1.1. **Model and results.** We define the **open ASEP** as a continuous-time Markov chain $(\eta_t)_{t\geq 0}$ with state space $\Omega_N = \{0,1\}^N$ for some $N \in \mathbb{N}$, and with generator

$$
\begin{aligned}
\mathcal{L}f(\eta) = &\sum_{x=1}^{N-1} \big(\eta(x)(1-\eta(x+1)) + q\eta(x+1)(1-\eta(x))\big)\left[f(\eta^{x,x+1}) - f(\eta)\right] \\
&+ \alpha(1-\eta(1))\left[f(\eta^1) - f(\eta)\right] + \beta\eta(N)\left[f(\eta^N) - f(\eta)\right] \\
&+ \gamma\eta(1)\left[f(\eta^1) - f(\eta)\right] + \delta(1-\eta(N))\left[f(\eta^N) - f(\eta)\right]
\end{aligned} \tag{1.1}
$$

for all measurable functions $f\colon \Omega_N \to \mathbb{R}$. Here, we use the standard notations

$$
\eta^{x,y}(z) = \begin{cases} \eta(z) & \text{for } z \neq x,y \\ \eta(x) & \text{for } z = y \\ \eta(y) & \text{for } z = x \end{cases} \qquad \text{and} \qquad \eta^a(z) = \begin{cases} \eta(z) & \text{for } z \neq a \\ 1-\eta(z) & \text{for } z = a \end{cases}
$$

to denote swapping of values in a configuration $\eta \in \Omega_N$ at sites $x, y \in [\![N]\!] := \{1, 2, \dots, N\}$, and flipping at $a \in [\![N]\!]$, respectively. We say that site x is **occupied** if $\eta(x) = 1$, and **vacant** otherwise. A visualization of this process is given in Figure 1. In order to study mixing times, it turns out that it is useful to consider certain functions A, C of the boundary rates $\alpha, \beta, \gamma, \delta$ and the bias parameter q. Here and in the following, we usually drop the dependence on N for ease of notation. When $\beta > 0$, set

$$
A = A(\beta,\delta,q) := \frac{1}{2\beta}\left(1 - q - \beta + \delta + \sqrt{(1-q-\beta+\delta)^2 + 4\beta\delta}\right) \tag{1.2}
$$

and similarly, for $\alpha > 0$,

$$
C = C(\alpha,\gamma,q) := \frac{1}{2\alpha}\left(1 - q - \alpha + \gamma + \sqrt{(1-q-\alpha+\gamma)^2 + 4\alpha\gamma}\right) . \tag{1.3}
$$

We will assume that $\alpha, \beta > 0$ in the following, ensuring that the open ASEP has a unique stationary distribution μ_N. As we will see in Section 2, the parameters A and C play a crucial role as **effective reservoir densities**

$$
\rho_{\mathsf{L}} := \frac{1}{1+C} \quad \text{and} \quad \rho_{\mathsf{R}} := \frac{A}{1+A}, \tag{1.4}
$$

which are closely related to basic properties of the stationary distribution μ_N of the open ASEP. Throughout this article, we will impose the assumption that

$$q = \exp(-\psi N^{-\kappa}) = 1 - \frac{\psi}{N^{\kappa}} + \mathcal{O}\left(N^{-2\kappa}\right) \tag{1.5}$$

for constants $\kappa \in [0, \frac{1}{2}]$ and $\psi > 0$ – see Section 1.5 for the respective notation – and

$$\alpha + q\gamma = 1 + \mathcal{O}(N^{-1/2}) \quad \text{and} \quad \beta + q\delta = 1 + \mathcal{O}(N^{-1/2}). \tag{1.6}$$

The special case when the error term $\mathcal{O}(N^{-1/2})$ vanishes is known as **Liggett's condition**; see [60]. We investigate the speed of convergence to the stationary distribution under the **total variation distance**, i.e., for two probability measure ν, ν' on Ω_N, we set

$$\|\nu - \nu'\|_{\mathrm{TV}} := \frac{1}{2} \sum_{x \in \Omega_N} |\nu(x) - \nu'(x)| = \max_{A \subseteq \Omega_N} \left\{\nu(A) - \nu'(A)\right\}. \tag{1.7}$$

We study the **ε-mixing time** of $(\eta_t)_{t \geq 0}$, which is defined as

$$t^N_{\mathrm{mix}}(\varepsilon) := \inf\left\{t \geq 0 \,:\, \max_{\eta' \in \Omega_N} \|\mathbb{P}\left(\eta_t \in \cdot \mid \eta_0 = \eta'\right) - \mu_N\|_{\mathrm{TV}} < \varepsilon\right\} \tag{1.8}$$

for all $\varepsilon \in (0,1)$. We have the following result on the mixing time of the open ASEP when the effective densities are within order $N^{-1/2}$ of the **triple point** $A = C = 1$.

Theorem 1.1. *Let q satisfy (1.5) for some $\kappa \in [0, \frac{1}{2})$ and $\psi > 0$. Let $\alpha, \beta, \gamma, \delta$ satisfy condition (1.6). Assume that there exist some constants $\tilde{A}, \tilde{C} \in \mathbb{R}$ such that A, C satisfy*

$$A = \exp(-\tilde{A}N^{-1/2}) \quad \text{and} \quad C = \exp(-\tilde{C}N^{-1/2}) \tag{1.9}$$

for all N. Then for all $\varepsilon \in (0,1)$ fixed, we get that

$$t^N_{\mathrm{mix}}(\varepsilon) \asymp N^{3/2+\kappa}. \tag{1.10}$$

This confirms Conjecture 1.9 by Gantert et. al in [48] for the open ASEP at the triple point, where $\tilde{A} = \tilde{C} = 0$ and $\kappa = 0$. To our best knowledge, this is the first time a sub-diffusive bound on the mixing time for a partially asymmetric system is established; see [75] for the statement of Theorem 1.1 when $q = 0$. We conjecture that Theorem 1.1 remains valid in the **maximal current phase** when we allow for general $A, C \leq 1$. When $\kappa = \frac{1}{2}$, we obtain the following bound on the mixing time.

Theorem 1.2. *Let q satisfy (1.5) for $\kappa = \frac{1}{2}$ and some $\psi > 0$. Let $\alpha, \beta, \gamma, \delta$ satisfy Liggett's condition. Assume that there exist some constants $\tilde{A}, \tilde{C} \in \mathbb{R}$ such that (1.9) holds for all $N \in \mathbb{N}$. Then for all $\varepsilon \in (0,1)$ fixed, we get that*

$$N^2 \log^{-1}(N) \lesssim t^N_{\mathrm{mix}}(\varepsilon) \lesssim N^2 \log^3(N). \tag{1.11}$$

We conjecture that the correct order of the mixing time in Theorem 1.2 is N^2. This is supported by results of Corwin and Shen, as well as Parekh, who establish in [36] and [65] convergence under the scaling of Theorem 1.2 to the open KPZ equation, and by Knizel and Matetski as well as Parekh, who establish a one force one solution principle for the open KPZ equation [53, 66]. On the way of establishing the main theorems, we will also prove bounds on moderate deviations and the variance of the current of the open ASEP in Proposition 4.15 and Corollary 4.16, as well as upper bounds on the mixing time in the weakly high density phase in Propositions 5.1 and 5.4, which are of independent interest.

1.2. **Related work.** Exclusion processes can be studied from a variety of different perspectives. Classical studies on exclusion processes focus on characterizing the invariant measure; see [62] for an introduction. While the set of invariant measures for ASEP on the integers has a simple explicit description – see Section 3 in Part VIII of [62] – the stationary distribution of the open ASEP is still not fully understood. In a classical result, Liggett establishes in [60] the phase diagram from Figure 2, showing that the overall density in the open ASEP depends on the effective densities at the boundaries. Since the 90s, an important tool to study the invariant measures of the open ASEP is the matrix product ansatz, introduced by Derrida, Evans, Hakim and Pasquier in [38] when particles can move only in one direction; and which was later extended to the full set of parameters $\alpha, \beta, \gamma, \delta, q$; see [20, 79] as well as [19] for an introductory survey. Formally, the matrix product ansatz assigns to every configuration a weight, which can be represented as a product of matrices and vectors. The matrices and vectors must satisfy certain relations, usually called the DEHP algebra named after [38]. Finding suitable weights in the matrix product ansatz is a question that gained lots of recent attention. It also led to numerous combinatorial descriptions such as weighted Catalan paths and staircase tableaux; see for example [22, 32].

A powerful matrix product ansatz representation was established by Uchiyama, Sasamoto and Wadati in [79] using Askey–Wilson polynomials. This led to a series of works by Bryc, Kuznetsov, Wang, Wesołowski, and many others, where the stationary distribution of the open ASEP in the fan region $AC \leq 1$ can be expressed using Askey–Wilson processes. As an application, this representation allows to study for example the limiting density and large deviations for the particle density of the open ASEP, as well as the convergence to a solution to the open KPZ equation, among various other properties [25, 27, 28, 29, 51]. Very recently, Wang, Wesołowski and Yang introduced signed Askey–Wilson measures in [80] to study properties of the open ASEP in the shock region $AC > 1$. Another very recent line of research concerns the two–layer Gibbs representation of the stationary distribution of the open ASEP established in [14, 23, 24]. Intuitively, the stationary distribution can be written as the top curve a coupled pair of suitably reweighted simple random walk trajectories. Moreover, the approximation of the stationary distribution in total-variation distance by product measures was studied in [64, 83]. More generally, similar approaches involving different kinds of orthogonal polynomials or queuing interpretations of the stationary measures also allow to study the open ASEP with second class particles or to multi-species exclusion processes; see for example [5, 6, 31, 46, 63, 77]. Finally, let us mention that for special choices of the boundary parameters in the shock phase of the open ASEP, Schütz describes the stationary measure and the evolution of the open ASEP by a reversible exclusion process with finitely many particles, introducing the concept of reverse duality [78].

Outside of the stationary distribution, the behavior of the open ASEP is closely linked to properties of the current and second class particles for the asymmetric simple exclusion process on the integer lattice. In celebrated work [12], Balázs and Seppäläinen establish that under a Bernoulli-$\frac{1}{2}$-product measure, the fluctuations of the current of the ASEP on the integers at time t are of order $t^{1/3}$. This was refined at the level of limiting distribution by Aggarwal in [2], the first proof in models without determinantal structures of the Baik-Rains limiting distribution [9]; see also [42, 43, 44, 69] for a selection of classical results on the fluctuations of currents and second class particles. Let us mention that the results on the current fluctuations in [12] are established by proving order $t^{2/3}$ fluctuations for a second class particle in a Bernoulli-$\frac{1}{2}$-product measure, and using a scaling relation from [11]; see identity (2.1) therein. This generalizes results from [68], relating the two-point function to the probability density of second class particle, which in turn can be written in terms of height functions; see Proposition 4.1 of [68]. This was used to prove the asymptotic of the law of second class particle for TASEP, see [8, 10, 47] for more details between current fluctuations and second class particles for a more general class of growth models.

In order to establish the fluctuations for second class particles, [12] crucially relies on a multi-species asymmetric simple exclusion processes on the integers, using ideas from [45]. Similar instances for the use of multi-species ASEPs include the ASEP speed process relying on Rezakhanlou's coupling [72]; see also [39] for recent extension to the six vertex model. We will in this paper require similar ideas for the study of multi-species ASEPs, relying instead of Rezakhanlou's coupling on the censoring inequality by Peres and Winkler from [67], which was used in the context of exclusion process for example in [48, 56, 74].

Recently, Landon and Sosoe established in [57] moderate deviations for the current and second class particles for the ASEP on the integers. Their arguments rely on interpreting the ASEP as a limit of the stochastic six vertex model, as well as the so-called Rains–EJS identity, an exact formula developed by Emrah, Janjigian, and Seppäläinen [41] for the moment generation function of stationary exponential last passage percolation, recovering unpublished work by Rains [71]. In a broader context, these scaling relations for the current and second class particle fluctuations manifest the role of the ASEP as a central model in the KPZ universality class; see also [3, 4, 70] for recent results on the existence of the ASEP speed process and convergence to directed landscape, as well as [82] for KPZ scaling of a generalization of the ASEP with long range interactions.

For exclusion processes on a finite state space, mixing time are a key tool to quantify the speed of convergence to the stationary distribution; see [59] for a general introduction to mixing times. Over the last decade, significant progress was achieved in the study of mixing times of the asymmetric exclusion processes on a closed segment. In [54], Labbé and Lacoin verify that the mixing times is linear in the size of the segment and the occurrence of the cutoff phenomenon, a sharp transition in the convergence to equilibrium, where the system goes from being far from equilibrium to being close to equilibrium within a small time window. These results were generalized in [55] to the weakly asymmetric simple exclusion process where $q = q(N) \to 1$ as the system size growths. Recently, the results were further sharpened. Bufetov and Nejjar in [30] establish a Tracy-Widom limit profile using a delicate color position symmetry for the ASEP on the integers from [21], where they interpret the ASEP as a random walk on Hecke algebra; see also [84] for the limit profile for a multi-species ASEP by Zhang. A similar result was established in [50] for the ASEP with one open boundary, using current fluctuations for the ASEP on $\mathbb{N}$ by He [49].

All these sharp results have in common that the respective exclusion processes are reversible, and that the stationary distribution is given by a variant or projection of Mallows measure. This is in contrast to the ASEP with two open boundaries, which is in general not reversible, and where the available results are much more sparse. For the high and low density phase, i.e., when $A > \max(1, C)$ or $C > \max(1, A)$ holds, Gantert et al. establish in [48] a mixing time of order N for the open ASEP on a segment of length N. We will elaborate on their approach in more detail in Section 1.3 when we discuss how to use their strategy in order to study mixing times of the open ASEP at the triple point. In the special case of the open TASEP, i.e., where $\gamma = \delta = q = 0$, mixing times are much better understood. For the maximal current phase $AC \leq 1$, a mixing time of order $N^{3/2}$ and the absence of cutoff are shown in [75, 76]. Similar results are also available for the TASEP on the circle where the mixing time is shown in [75] to be of order $N^2 \min(k, N-k)^{-1/2}$ for a system with k particles. In the high and low density phase, [40] verifies that the cutoff phenomenon occurs while in the co-existence line $A = C > 1$, the mixing time is of order N^2 and we see no cutoff. The reason for this drastically increasing precision in the results is an alternative representation of the open TASEP as a last passage percolation model. In [76], it is shown that mixing time can be expressed by coalescence times of geodesics in exponential last passage percolation. In return, precise bounds on the coalescence of geodesics are due to exact formulas for various quantities in last passage percolation, which are themselves achieved by a connection via RSK correspondence and (Pfaffian) Schur processes to random matrix theory [7, 16, 17, 58, 73]. Let us stress that a similarly powerful set of tools is currently not available for the asymmetric simple exclusion process with $q > 0$, thus requiring different ideas to establish mixing times.

1.3. **Overview of the proof.** As a reader's guide, let us in the following give an outline of the main ideas and the strategy for the proof.

In order to show mixing times at the triple point, we will follow the overall strategy from Gantert et al. in [48] for mixing times for the open ASEP in the high density phase for constant boundary and bias parameters, which we will now briefly summarize. Consider two open ASEPs started from the extremal configurations, where all sites are either fully occupied by particles or are left empty, respectively. Then under the basic coupling, the two open ASEPs can be interpreted as one open ASEP $(\xi_t)_{t\geq 0}$ with second class particles. We obtain an upper bound on the mixing by providing estimates on the time it takes for all initially N many second class particles to exit the segment. In order to study this exit time, Gantert et al. couple the open ASEP with second class particles to a system of two stationary open ASEPs with different boundary parameters – one of them with the same boundary parameters as $(\xi_t)_{t\geq 0}$ and one where α and β are decreased. They show that if the currents of the two stationary systems until some time T differ by at least order N, then this implies an upper bound on the mixing time of order T.

While this approach is sufficient for effective bounds on the mixing time for the high density phase and constant boundary parameters, it comes with three main challenges for the triple point. First, we need effective bounds on the expected current of the open ASEP in the maximal current phase, and when the parameters $\alpha, \beta, \gamma, \delta, q$ are allowed to depend on the system size N. Second, as the most delicate part of the argument, we require bounds on the fluctuations of the current of the open ASEP in order to guarantee that the currents differ at the order of their expected differences with positive probability. Last, even with an optimal bound on the difference and fluctuations of the current, the arguments in [48] a priori only yield an upper bound of order $N^{2+\kappa}$ on the mixing time of the open ASEP at the triple point. Let us now describe how we address all three challenges.

In order to obtain effective bounds on the expected current of the open ASEP in the maximal current phase when the parameters $\alpha, \beta, \gamma, \delta, q$ are allowed to depend on the system size N, we rely on an exact representation of the stationary current by Uchiyama, Sasamoto, Wadati from [79] as a certain contour integral. This formula uses that the invariant measure of the open ASEP can be studied via the matrix product ansatz, and a particular solution related to Askey–Wilson polynomials. We provide a sharp asymptotic analysis of this exact expression for the expected current using various identities of Gamma functions and q-calculus. In particular, we utilize recent fine asymptotic results on q-Pochhammer symbols by Corwin and Knizel from [35].

In order to study the fluctuations of the current of the open ASEP, we introduce a framework to compare moderate deviations for the current of the open ASEP to moderate deviations of the current of the ASEP on the integers. Precise moderate deviations for the ASEP on the integers were recently established by Landon and Sosoe [57]. In order to compare the currents, we use the basic coupling between the open ASEP and the ASEP on the integers. We establish moderate deviations for the speed of second class particles, which are inserted at sites 1 and N, using different hierarchies of particles and the censoring inequality to study the motion among them. As the arguments on moderate deviations take most of the body of this article, we will provide a more detailed outline of the strategy of the proof in Section 4.1.

In order to control the exit time of second class particle, we provide an iterative scheme, using a generalization of the partially ordered multi-species exclusion process introduced by Gantert et al. in [48]. More precisely, by induction over n with $2^n \leq \sqrt{N}$, after a time of order $N^{1+\kappa}2^n$, the law of the open ASEP is with probability at least $1 - \exp(\min(-2^{-n}\sqrt{N}, N^{\kappa'}))$ for some constant $\kappa' > 0$ stochastically dominated from above by a Bernoulli-$(\frac{1}{2} + \frac{1}{2^n})$ product measure, and from below by a Bernoulli-$(\frac{1}{2} - \frac{1}{2^n})$ product measure. This ensures that after time of order $N^{3/2+\kappa}$ there are, with positive probability, at most order $\sqrt{N}$ many second class particles in the open ASEP. This allows us to apply the arguments of Gantert et al. from [48] to obtain an upper bound of order $N^{3/2+\kappa}$ on the mixing time of the open ASEP at the triple point.

1.4. **Outline of the paper.** This paper is structured as follows. In Section 2, we state preliminary results on the open ASEP and the asymmetric simple exclusion process on the integers and the half-line. This includes the basic coupling for multi-species exclusion processes, a characterization of invariant measures for one-dimensional exclusion processes, the phase diagram for the open ASEP, the censoring inequality, and recent results on moderate deviations for the current of the ASEP on the integers. In Section 3, we study the stationary current of the open ASEP and provide sharp estimates in the maximal current phase. In Section 4, we introduce an extended disagreement process, which allows us to compare the motion of second class particles in the ASEP on the integers, the ASEP on the half-line and the open ASEP. We then use this process to convert moderate deviations for the current of the ASEP on the integers to moderate deviations for the current of the open ASEP. Section 5 uses the previous results to achieve an upper bound on the mixing time of the open ASEP in the weakly high and weakly low density phase via an iterative scheme. The respective bounds at the triple point are established in Section 6. We conclude by providing the corresponding lower bounds on the mixing times in Section 7.

1.5. **Notation.** We use standard asymptotic notation throughout this paper. For functions $f, g\colon \mathbb{N} \to \infty$, we will write $f = o(g)$ if $f(N)g(N)^{-1} \to 0$ and $f \sim g$ if $f(N)g(N)^{-1} \to 1$ for $N \to \infty$. Similarly, we write $f = \mathcal{O}(g)$ if $f(N)g(N)^{-1} \leq C_0$ as well as $f = \Theta(g)$ if $c_0 \leq f(N)g(N)^{-1} \leq C_0$ for constants $c_0, C_0 > 0$ and all N large enough. We will sometimes write $\asymp$ instead of Θ, and $\lesssim$ instead of $\mathcal{O}$, as well as $\ll$ instead of o. We allow the parameters $\alpha, \beta, \gamma, \delta, q$, and hence the corresponding functions A, B, C, D, to depend on the system size N, while the constants $c_0, C_0, c_1, C_1, \dots$ do not depend on N and may change from line to line. A special case are the constants κ, κ', ψ as well as $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$, which capture the scaling relations for the open ASEP, and hence do not depend on N.

2. Preliminaries

In the following, we collect preliminary results on the asymmetric simple exclusion process, which we will use throughout this paper.

2.1. **The asymmetric simple exclusion process.** Recall the asymmetric simple exclusion process with open boundaries (open ASEP), which we defined in (1.1). We will now define two related exclusion processes, which can both be interpreted as a limit as $N \to \infty$ for the open ASEP; the asymmetric simple exclusion process on the half-line $\mathbb{N}$ and on the integers $\mathbb{Z}$. For parameters $\alpha, \gamma \geq 0$ and $q \in [0, 1)$, we define the **ASEP on the half-line** as the Markov process $(\eta^{\mathbb{N}}_t)_{t \geq 0}$ with state space $\{0,1\}^{\mathbb{N}}$ and generator

$$\begin{aligned} \mathcal{L}_{\mathbb{N}} f(\eta) = \sum_{x=1}^{\infty} \big(\eta(x)(1-\eta(x+1)) + q\eta(x+1)(1-\eta(x))\big) \left[f(\eta^{x,x+1}) - f(\eta)\right] \\ + \alpha(1-\eta(1)) \left[f(\eta^1) - f(\eta)\right] + \gamma\eta(1) \left[f(\eta^1) - f(\eta)\right] \end{aligned} \tag{2.1}$$

with respect to all cylinder functions $f\colon \{0,1\}^{\mathbb{N}} \to \mathbb{R}$. Similar, we let $(\eta^{\mathbb{Z}}_t)_{t \geq 0}$ denote the **ASEP on the integers** with state space $\{0,1\}^{\mathbb{Z}}$ and generator

$$\mathcal{L}_{\mathbb{Z}} f(\eta) = \sum_{x \in \mathbb{Z}} \big(\eta(x)(1-\eta(x+1)) + q\eta(x+1)(1-\eta(x))\big) \left[f(\eta^{x,x+1}) - f(\eta)\right]. \tag{2.2}$$

for all cylinder functions $f\colon \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$. We refer to Liggett [62] for an introduction. Moreover, on the integer lattice, we define the **multi-species ASEP** $(\zeta_t)_{t \geq 0}$. This is a Markov process taking values in $(\mathbb{N} \cup \{\infty\})^{\mathbb{Z}}$, i.e., we assign types in $\mathbb{N} \cup \{\infty\}$ to all vertices in $\mathbb{Z}$. Along each edge $\{x, x+1\}$ with $x \in \mathbb{Z}$, we place two independent rate 1 and q Poisson clocks. Whenever the rate 1 clock rings, we sort the endpoints in decreasing order, whenever the rate q clock rings, we sort the endpoints in increasing order. Note that for any $k \in \mathbb{N}$, we obtain from $(\zeta_t)_{t \geq 0}$ an ASEP on the integers by identifying types $1, 2, \dots, k$ with particles and types $k+1, k+2, \dots$ (including $k = \infty$) with empty sites.

2.2. **The basic coupling.** Next, we introduce the basic coupling for simple exclusion processes. In contrast to [18] or [48], we do neither assume that the underlying configurations are componentwise ordered, nor that the exclusion processes are defined with respect to the same underlying graphs. Let $G = (V, E)$ and $G' = (V', E')$ be either the interval $[\![N]\!]$, the half-line $\mathbb{N}$, or the integer lattice $\mathbb{Z}$. Without loss of generality assume that $E \subseteq E'$.

Definition **2.1** (Basic coupling)**.** *Let $q \in [0,1)$. We define the basic coupling* $\mathbf{P}$ *between the asymmetric simple exclusion processes on G and G' as follows. We place independent rate $1+q$ Poisson clocks on all edges $e \in E$ and use the same clocks in both processes. Whenever the clock of an edge $e = \{x, x+1\}$ rings, and the respective exclusion processes are in states η and η', respectively, we sample an independent Uniform-$[0,1]$-distributed random variable U and distinguish two cases:*

- *If $U \leq (1+q)^{-1}$, and $\eta(x) = 1 - \eta(x+1) = 1$ holds, we move the particle at site x to site $x+1$ in configuration η.*
- *If $U > (1+q)^{-1}$, and $\eta(x) = 1 - \eta(x+1) = 0$ holds, we move the particle at site $x+1$ to site x in configuration η.*

The configuration η' is updated in the same way, using the same random variable U. In addition, when G is either the interval or the half-line, we place a rate α Poisson clock (respectively a rate γ Poisson clock) on site 1. Whenever the clock rings, we place a particle (respectively an empty site) at site 1, irrespective of the current value of $\eta(1)$. Similarly, when G is the interval, we place a rate β Poisson clock (respectively a rate δ Poisson clock) on site N. Whenever the clock rings, we place an empty site (respectively a particle) at site N, irrespective of the current value of $\eta(N)$. As before, we use the same Poisson clocks to update η'.

The basic coupling allows us to naturally define a process $(\xi_t)_{t\geq 0}$ taking values in $(\{0,1\} \times \{0,1\})^{\mathbb{Z}}$, which has the laws of $(\eta_t)_{t\geq 0}$ on V and $(\eta'_t)_{t\geq 0}$ on V' as marginals. We say that $(\xi_t)_{t\geq 0}$ is occupied by a **first class particle** at site x if $\xi_t(x) = (1,1)$. Similar, we say that x is occupied by a **second class particle of type** A if $\xi_t(x) = (1,0)$, by a **second class particle of type** B if $\xi_t(x) = (0,1)$, and by an **empty site** if $\xi_t(x) = (0,0)$. We refer to $(\xi_t)_{t\geq 0}$ as the **disagreement process** between $(\eta_t)_{t\geq 0}$ and $(\eta'_t)_{t\geq 0}$.

Note that the basic coupling respects the partial order, where first class particles have a higher priority than second class particles, and second class particles have a higher priority than empty sites. On the other hand, whenever two adjacent second class particles of different types are updated, they are replaced by a pair of first class particles and empty sites. Hence, $(\xi_t)_{t\geq 0}$ can also be interpreted as a continuous-time Markov chain on the state space

$$\Omega_N^2 := \{(0,0),(1,0),(0,1),(1,1)\}^N.$$

Remark **2.2.** *Note that whenever the disagreement process contains only second class particles of one type (either type A or type B), we can identify the disagreement process as a multi-species exclusion process of three types $1, 2, \infty$.*

2.3. **The stationary distribution of asymmetric simple exclusion processes.** In the following, we collect results on the stationary distribution of the open ASEP as well as the ASEP on $\mathbb{N}$ and $\mathbb{Z}$. Note that whenever $\alpha, \beta > 0$, the open ASEP has a unique stationary distribution μ_N, which was intensively studied over the past decades. The next result is due to Liggett [60] when

$$\gamma = q(1-\alpha) \quad \text{and} \quad \delta = q(1-\beta); \tag{2.3}$$

see also Proposition A.2 and Remark A.3 in [64] for general parameters $\alpha, \beta, \gamma, \delta, q$. We write $\mathrm{Ber}_n(\rho)$ for the Bernoulli-ρ-product measure on $\{0,1\}^n$ for some $n \in \mathbb{N}$ and $\rho \in [0,1]$.

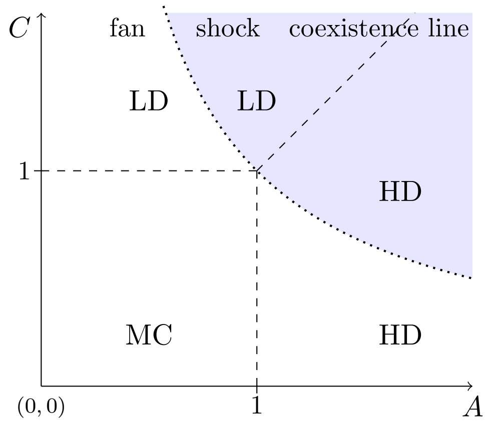

FIGURE 2. Phase diagrams for the open ASEP stationary measures. The terms LD, HD, and MC, respectively, correspond to the low density, high density and maximal current phase. Moreover, we distinguish between the fan and the shock region for the open ASEP.

THEOREM 2.3. *Let $q \in [0,1)$. Assume that the parameters $\alpha, \beta, \gamma, \delta \geq 0$ and q do not depend on N. Let $M > 0$ be finite and consider intervals $I = I_N = [\![a_N, a_N + M - 1]\!]$ with*

$$\lim_{N \to \infty} \min(a_N, N - a_N) = \infty. \tag{2.4}$$

Let μ_N^I denote the projection of μ_N onto the sites I. Then

$$\mu_N^I \to \begin{cases} \mathrm{Ber}_M(\rho_{\mathsf{L}}) & \text{if } C > \max(A,1) \\ \mathrm{Ber}_M(\rho_{\mathsf{R}}) & \text{if } A > \max(C,1) \\ \mathrm{Ber}_M\left(\frac{1}{2}\right) & \text{if } \max(A,C) \leq 1 \end{cases} \tag{2.5}$$

in the sense of weak convergence, where we recall ρ_{L} and ρ_{R} from (1.4).

Theorem 2.3 motivates the following phase decomposition for the open ASEP. We say that the open ASEP is in the **maximal current phase** when $AC \leq 1$, it is in the **high density phase** when $A > \max(1, C)$, and in the **low density phase** when $C > \max(1, A)$. We refer to the remaining case $A = C > 1$ as the **coexistence line**. A visualization of the different phases is given in Figure 2.

Let us mention that the above results on the stationary distribution can be strengthened; see [29, 80] for scaling limits of the particle density under μ_N, and [64, 83] for an approximation of the stationary measure as $M = M(N) \to \infty$ with $N \to \infty$. Let us stress that we will in the following allow for $A = A_N$ and $C = C_N$ to depend on N. Whenever $A_N \to 1$ while $A_N > \max(C_N, 1)$, we say that the open ASEP belongs to the **weakly high density phase**. Similarly, when $C_N \to 1$ while $C_N > \max(A_N, 1)$, we refer to the **weakly low density phase**. The next lemma states that in the special case $AC = 1$, the stationary measure of the open ASEP has a simple product form.

LEMMA 2.4. *Recall $\rho_{\mathsf{L}}, \rho_{\mathsf{R}}$ from (1.4). Whenever $AC = 1$, we have that*

$$\mu_N = \mathrm{Ber}_N(\rho_{\mathsf{L}}) = \mathrm{Ber}_N(\rho_{\mathsf{R}}). \tag{2.6}$$

Moreover, for all $\alpha,\beta,\gamma,\delta>0$ and $q=q_N\in(0,1)$, we have that

$$\mathrm{Ber}_N\big(\max\left(\rho_{\mathsf{L}},\rho_{\mathsf{R}}\right)\big)\succeq\mu_N\succeq\mathrm{Ber}_N\big(\min\left(\rho_{\mathsf{L}},\rho_{\mathsf{R}}\right)\big),\tag{2.7}$$

where we denote by $\succeq$ stochastic domination between probability measures on $\{0,1\}^N$.

Proof. The first statement is a simple computation; see Proposition 2 in [22]. The second statement can be found for example as Lemma 2.10 in [48]. □

The condition $AC=1$ provides another partition of the parameter space for the open ASEP. We refer to $AC<1$ as the **fan region** and to $AC>1$ as the **shock region** of the open ASEP. We now investigate the stationary measure of the half-line ASEP as well as the ASEP on the integers. The next result is standard and follows from a straightforward computation; see Theorem 2.1 in Section VIII of [62].

Lemma **2.5.** *For all $\rho\in[0,1]$ and all $q\in[0,1]$, the Bernoulli-ρ-product measures on $\{0,1\}^{\mathbb{Z}}$ are invariant measure for the ASEP on the integers. Recall $A=A(\alpha,\gamma,q)$ from (1.2). Then the Bernoulli-ρ-product measure with $\rho=(A+1)^{-1}$ is an invariant measure for the ASEP on $\mathbb{N}$ with respect to parameters $\alpha,\gamma>0$ and $q\in[0,1)$.*

We refer to [83] for a more detailed discussion by Yang on the existence and structure of further invariant measures of the half-line ASEP when $A\geq 1$. For the ASEP on the integers, we consider the family of invariant measures called **blocking measures** $\nu^{(n)}$ on

$$\mathcal{A}_n:=\Big\{\eta\in\{0,1\}^{\mathbb{Z}}:\ \sum_{i>n}(1-\eta(i))=\sum_{i\leq n}\eta(i)<\infty\Big\}.\tag{2.8}$$

Here, for $q>0$, we denote by $\tilde{\nu}$ the Bernoulli product measure on $\{0,1\}^{\mathbb{Z}}$ with marginals

$$\tilde{\nu}(\eta(x)=1)=\frac{q^{-x}}{1+q^{-x}}\tag{2.9}$$

for all $x\in\mathbb{Z}$, and $\nu^{(n)}:=\tilde{\nu}(\,\cdot\,|\mathcal{A}_n)$ for all $n\in\mathbb{Z}$, noting that $\tilde{\nu}(\mathcal{A}_n)>0$ for all $n\in\mathbb{N}$ and $q>0$. We make the following observation on properties of the blocking measures, which we will frequently use throughout this article.

Lemma **2.6.** *Let $q\in(\varepsilon,1)$ for some $\varepsilon>0$. For all $n\in\mathbb{Z}$, the ASEP on the integers restricted to $\mathcal{A}_n$ is reversible with respect to $\nu^{(n)}$. Moreover, there exist some constants $c_0,C_1,C_0>0$, depending only on $\varepsilon>0$, such that for all $x>C_1(1-q)^{-1}$,*

$$\begin{aligned}\nu^{(n)}\Big(\eta(y)=1\text{ for some }y<n-\frac{x}{1-q}\Big)&\leq C_0\exp(-c_0x)\\ \nu^{(n)}\Big(\eta(y')=0\text{ for some }y'>n+\frac{x}{1-q}\Big)&\leq C_0\exp(-c_0x).\end{aligned}\tag{2.10}$$

Proof. Reversibility of the measure can be checked directly by a simple computation, see Theorem 2.1 in Section VIII of [62]. For the second statement, assume without loss of generality that $n=0$. Note that (2.10) holds under the measure $\tilde{\nu}$ as

$$\begin{aligned}\tilde{\nu}\left(\eta(z)=1\text{ for some }z\in\Big(-\infty,\frac{-y}{1-q}\Big]\right)&\leq\frac{1}{1-q}\exp(-c_2y)\\ \tilde{\nu}\left(\eta(z')=0\text{ for some }z'\in\Big[\frac{y}{1-q},\infty\Big)\right)&\leq\frac{1}{1-q}\exp(-c_2y)\end{aligned}$$

for some constant $c_2 > 0$ and all $q \in (\varepsilon, 1]$, and all $y \geq 1$. Since

$$\tilde{\nu}(\mathcal{A}_0) \geq c_3(1-q)$$

for some constant $c_3 > 0$ and all $q \in (\varepsilon, 1)$, this allows us to conclude. $\square$

2.4. **The censoring inequality.** The censoring inequality by Peres and Winkler in [67] intuitively states for monotone spin systems that leaving out transitions only increases the distance from equilibrium. This was applied by Lacoin in [56] to the symmetric simple exclusion processes on a closed segment. We will in the following apply the censoring inequality with respect to the ASEP $(\eta_t^{\mathbb{Z}})_{t\geq 0}$ on the integers, restricted to $\mathcal{A}_n$ from (2.8) for some $n \in \mathbb{N}$.

Definition 2.7 (Censoring). *Let $(\eta_t^{\mathbb{Z}})_{t\geq 0}$ be an ASEP on $\mathcal{A}_n$ for some $n \in \mathbb{Z}$ and let $\mathcal{P}(E)$ denotes the power set of $E(\mathbb{Z})$. We call a random càdlàg function*

$$\mathcal{C} \colon \mathbb{R}_0^+ \to \mathcal{P}(E) \tag{2.11}$$

*a **censoring scheme** for the exclusion process $(\eta_t^{\mathbb{Z}})_{t\geq 0}$ when $\mathcal{C}$ is independent of the process $(\eta_t^{\mathbb{Z}})_{t\geq 0}$. We define the process $(\eta_t^{\mathcal{C}})_{t\geq 0}$ as the **censored exclusion process**, where an update along an edge e at time t is performed if and only if $e \notin \mathcal{C}(t)$.*

Define the partial order $\succeq_{\mathrm{h}}$ on $\bigcup_{n\in\mathbb{Z}} \mathcal{A}_n$ by the relation

$$\eta \succeq_{\mathrm{h}} \eta' \quad \Leftrightarrow \quad \sum_{y\geq x} \eta(y) \geq \sum_{y\geq x} \eta'(y) \text{ for all } x \in \mathbb{Z}. \tag{2.12}$$

Similarly, we write $\nu \succeq_{\mathrm{h}} \nu'$ for laws ν, ν' if there exists a coupling such that (2.12) holds almost surely. The following result is Proposition 2.12 and Remark 2.13 in [48].

Lemma 2.8. *Let $\mathcal{C}$ be a censoring scheme. Let $P_\eta(\eta_t \in \cdot)$ and $P_\eta(\eta_t^{\mathcal{C}} \in \cdot)$ denote the law of $(\eta_t^{\mathbb{Z}})_{t\geq 0}$ and its censored dynamics $(\eta_t^{\mathcal{C}})_{t\geq 0}$ at time $t \geq 0$, respectively, starting from some $\eta \in \mathcal{A}_n$. For all $n \in \mathbb{Z}$, we have that*

$$P_{\vartheta_n}(\eta_t^{\mathcal{C}} \in \cdot) \preceq_{\mathrm{h}} P_{\vartheta_n}(\eta_t \in \cdot) \quad \textit{for all } t \geq 0, \tag{2.13}$$

where we set ϑ_n with $\vartheta_n(x) := \mathbb{1}_{\{x>n\}}$ for all $x \in \mathbb{Z}$.

We record the following consequence of Lemma 2.6 and Lemma 2.8.

Lemma 2.9. *Let q satisfy (1.5) for some $\kappa \in [0,1]$ and $\psi > 0$. Let $(\eta_t^{\mathcal{C}})_{t\geq 0}$ denote a censored exclusion process on $\mathcal{A}_n$ for some $n \in \mathbb{Z}$. Let $(L_t)_{t\geq 0}$ and $(R_t)_{t\geq 0}$ denote the position of the leftmost particle and rightmost empty site in $(\eta_t^{\mathcal{C}})_{t\geq 0}$, respectively. Then there exist constants $c_0, C_0, C_1 > 0$ such that for any censoring scheme $\mathcal{C}$, and all $x \geq C_1 \log(N)$,*

$$\mathbb{P}\left(R_t, L_t \in \left[n - xN^\kappa, n + xN^\kappa\right] \textit{ for all } 0 \leq t \leq N^3\right) \geq 1 - C_0 \exp(-c_0 x). \tag{2.14}$$

Proof. Note that by Lemma 2.8 with $t \to \infty$, the bounds from Lemma 2.6 continue to hold for the censored exclusion process $(\eta_t^{\mathcal{C}})_{t\geq 0}$ on $\mathcal{A}_n$ started from $\nu^{(n)}$. Thus, we get that for some constants $c_2, C_2, C_3 > 0$

$$\mathbb{P}\left(R_m, L_m \in \left[n - xN^\kappa, n + xN^\kappa\right] \text{ for all } m \in [\![N^3]\!]\right) \geq 1 - C_2 \exp(-c_2 x) \tag{2.15}$$

for all $x \geq C_3 \log(N)$ and N large enough. For all $m \in \mathbb{N}$ and all $\tilde{\eta} \in \mathcal{A}_0$, we see that for some constant $c_3 > 0$ and all x large enough

$$\mathbb{P}\Big(\sup_{t \in [m,m+1]} (|R_t - R_m| + |L_t - L_m|) \geq x \,\Big|\, \eta_m = \tilde{\eta} \Big) \leq \exp(-c_3 x) \tag{2.16}$$

as particles can move at most at speed 2. This allows us to conclude by (2.15) and a union bound on $m \in [\![N^3]\!]$ in (2.16). □

We will primarily apply Lemma 2.9 with respect to censoring schemes, which result from a projection of a multi-species exclusion process, e.g., where we can interpret the dynamics of second class particles with respect to empty sites as an exclusion process with censoring by erasing all first class particles (and the corresponding sites), merging the respective edges, and preventing all updates along such newly created edges; see also Lemma 4.8.

Remark **2.10.** *Note that Lemma 2.8 and Lemma 2.9 continue to hold for a stationary ASEP on a closed interval I by censoring all moves outside of I at all times; see also Proposition 5.1 in* [74] *for a precise statement.*

2.5. **Current fluctuations for the ASEP on the integers.** A key observable for the simple exclusion process is the number of particles passing through a given site over time. For the ASEP $(\eta_t^{\mathbb{Z}})$ on the integers, we define the **current** $(\mathcal{J}_t^{\mathbb{Z}}(x))_{t \geq 0}$ through a site x as the net number of particles which have passed through site x until time t, i.e., set $\mathcal{J}_0^{\mathbb{Z}}(x) = 0$ and increase (respectively decrease) the current by one every time a particle jumps from $x-1$ to x (respectively from x to $x-1$). Note that $\mathcal{J}_t^{\mathbb{Z}}(x)$ is almost surely finite for all $t \geq 0$ and $x \in \mathbb{Z}$. Similarly, we denote by $(\mathcal{J}_t^{\mathbb{N}}(x))_{t \geq 0}$ the current of the ASEP on $\mathbb{N}$ through x for all $x \in \mathbb{N}$, and by $(\mathcal{J}_t(x))_{t \geq 0}$ the current of the open ASEP for all $x \in [\![N]\!]$. Note that the ergodic theorem Markov chain ensures that

$$J_N := \lim_{t \to \infty} \frac{\mathcal{J}_t(1)}{t} \qquad \text{almost surely,} \tag{2.17}$$

where J_N is the **stationary current** of the open ASEP given by

$$J_N := \mu_N(\eta(1) = 1, \eta(2) = 0) - q\mu_N(\eta(1) = 0, \eta(2) = 1). \tag{2.18}$$

Similarly, suppose that the ASEP on $\mathbb{N}$ or $\mathbb{Z}$ is stationary, started from a Bernoulli-ρ-product measure with some $\rho \in [0, 1]$. We denote by $J_{\mathbb{Z}}$ and $J_{\mathbb{N}}$ with

$$J_{\mathbb{Z}} = J_{\mathbb{N}} = (1-q)\rho(1-\rho)$$

the respective stationary currents. Note that

$$\mathbb{E}[\mathcal{J}_t^{\mathbb{Z}}(0)] = J_{\mathbb{Z}} t \quad \text{and} \quad \mathbb{E}[\mathcal{J}_t^{\mathbb{N}}(1)] = J_{\mathbb{N}} t$$

for all $t \geq 0$. A simple observation from Lemma 2.4 is that whenever $AC = 1$, the stationary current of the respective open ASEP satisfies

$$J_N = (1-q)\frac{A}{(1+A)^2}. \tag{2.19}$$

We make the following observation on the stationary current of the open ASEP in the fan region of the (weakly) high and low density phase.

Lemma 2.11. *Let $(\eta_t)_{t\geq 0}$ be an open ASEP with parameters $\alpha,\beta,\gamma,\delta\geq 0$ and $q\in[0,1)$ in the fan region of the high density phase, i.e., where $AC\leq 1$ and $A>\max(1,C)$, allowing the parameters also to depend on N. Then the stationary current satisfies for all $N\in\mathbb{N}$*

$$J_N \geq (1-q)\frac{A}{(1+A)^2}. \tag{2.20}$$

Similarly, in the fan region of the low density phase, where $AC\leq 1$ and $C>\max(1,A)$, the stationary current satisfies

$$J_N \geq (1-q)\frac{C}{(1+C)^2}. \tag{2.21}$$

Proof. We will consider only the high density phase as the result for the low density phase follows from the standard particle–hole symmetry. Let $(\eta'_t)_{t\geq 0}$ be an open ASEP with respect to parameters $\alpha',\beta,\gamma,\delta\geq 0$ and $q\in[0,1)$ for some $\alpha'\leq\alpha$ such that $C'=C(\alpha',\gamma,q)$ satisfies $AC'=1$. By Lemma 2.1 in [48], there exists a coupling $\mathbf{P}$ between $(\eta_t)_{t\geq 0}$ and $(\eta'_t)_{t\geq 0}$, started from empty initial conditions, such that

$$\mathbf{P}\left(\eta_t(x)\geq \eta'_t(x) \text{ for all } t\geq 0 \text{ and } x\in[\![N]\!]\right)=1.$$

Hence, the respective currents $(\mathcal{J}_t(0))_{t\geq 0}$ and $(\mathcal{J}'_t(0))_{t\geq 0}$ of $(\eta_t)_{t\geq 0}$ and $(\eta'_t)_{t\geq 0}$ satisfy

$$\mathbf{P}\left(\mathcal{J}_t(0)\geq \mathcal{J}'_t(0) \text{ for all } t\geq 0\right)=1. \tag{2.22}$$

Since both processes satisfy (2.17), we use (2.19) and (2.22) to get the desired result. □

We remark that for constant parameters $\alpha,\beta,\gamma,\delta,q$, the lower bound in (2.20) is asymptotically sharp; see [60]. We expect that a similar result can be achieved in the weakly high and low density phase using Askey–Wilson signed measures introduced in [80]. However, since we only require a lower bound on the stationary current in the (weakly) high and (weakly) low density phase, we leave this task to future work.

Next, we revisit the current $(\mathcal{J}^{\mathbb{Z}}_t)_{t\geq 0}$ of the ASEP on the integers. We will use a recent moderate deviation result on the current of the ASEP on $\mathbb{Z}$ by Landon and Sosoe [57]. Let us note that for convenience, we adapt at this point already the notation used in Section 4 to establish moderate deviations for the current of the open ASEP.

Theorem 2.12 (c.f. Theorem 2.4 in [57])**.** *Let q satisfy (1.5) for some $\kappa\in[0,\frac{1}{2}]$ and $\psi>0$. Consider the current of the ASEP on the integers started from a Bernoulli-ρ_N-product measure, where $\rho_N=\frac{1}{2}+2^{-n}$ for some $n\in[\![\frac{1}{2}\log_2(N)]\!]$. Set*

$$T=\theta^{-1}2^n N(1-q)^{-1} \tag{2.23}$$

for some $\theta\geq 1$, allowed to depend on N. Then there exist universal constants $c_0,C_0>0$ such that for all $1\leq y\leq 2^n N^{1/2}\theta^{-1}$ and all $m\in\mathbb{Z}$, we get that

$$\mathbb{P}(|\mathcal{J}^{\mathbb{Z}}_T(m)-T\rho_N(1-\rho_N)(1-q)|\geq yN^{1/2})\leq C_0\exp\left(-c_0\min(y^{3/2}\sqrt{\theta},y^2\theta)\right) \tag{2.24}$$

for all N large enough. The same result holds when $\rho_N=\frac{1}{2}$ and $T=N^{3/2}(1-q)^{-1}$.

Proof. This follows from Theorem 2.4 in [57] with a suitable change of notation. More precisely, in their notation $L-R=(1-q)$, $u=\sqrt{N}y$, and $x-x_0=(1-q)T(1-2\rho_N)=\theta^{-1}N$. We need to ensure that

$$((1-q)T)^{1/3}=(2^n\theta^{-1}N)^{1/3}\leq u=y\sqrt{N}\leq(1-q)T=2^nN\theta^{-1},$$

which is the case for all $1\leq y\leq 2^nN^{1/2}\theta^{-1}$, and the above choices for n. □

Remark **2.13.** *Let us emphasize that we require $\rho_N\in(\mathfrak{a},1-\mathfrak{a})$ for some constant $\mathfrak{a}>0$ uniformly in N, so that the constants $C_0,c_0>0$ in Theorem 2.12 only depend on $\mathfrak{a}$ (e.g. with $\mathfrak{a}=\frac{1}{5}$ above). This assumption is stated in Theorem 2.3 of [57] for the stochastic six vertex model and carries over to the ASEP in Theorem 2.4 of [57], when applying the convergence result by Aggarwal from [1]; see Section 7 in [57] for more details.*

3. Current estimates in the maximal current phase

In this section, we consider the stationary current J_N for the open ASEP when the boundary rates $\alpha,\beta,\gamma,\delta$ and the bias q depend on the system size N. We establish second order asymptotics on the stationary current in the maximal current phase relying on a representation by Uchiyama, Sasamoto and Wadati [79]. Recall the functions $A=A_N$ and $C=C_N$ from (1.2) and (1.3), and define the constants $B,D\in[-1,0]$, when $\alpha>0$ as

$$D=D(\alpha,\gamma,q):=\frac{1}{2\alpha}\left(1-q-\alpha+\gamma-\sqrt{(1-q-\alpha+\gamma)^2+4\alpha\gamma}\right),\tag{3.1}$$

and similarly, when $\beta>0$ as

$$B=B(\beta,\delta,q):=\frac{1}{2\beta}\left(1-q-\beta+\delta-\sqrt{(1-q-\beta+\delta)^2+4\beta\delta}\right).\tag{3.2}$$

The expected current of the system of size N is given by (see (2.11) and (6.1) of [79])

$$J_N=\frac{Z_{N-1}}{Z_N}\quad\text{with}\quad Z_N=\frac{1}{(1-q)^N}\frac{1}{4\pi\mathrm{i}}\oint_\Gamma\frac{dz}{z}e^{Nf(z)}g(z;q),\tag{3.3}$$

where we set

$$\begin{aligned}f(z)&=\log(2+z+z^{-1}),\\ g(z;q)&=\frac{(z^2,z^{-2};q)_\infty}{(Az,Az^{-1};q)_\infty(Bz,Bz^{-1};q)_\infty(Cz,Cz^{-1};q)_\infty(Dz,Dz^{-1};q)_\infty}.\end{aligned}\tag{3.4}$$

The (anticlockwise) integration contour Γ in (3.3) encloses the poles $\{a,aq,aq^2,\dots\}$, but not the poles $\{a^{-1},a^{-1}q^{-1},a^{-1}q^{-2},\dots\}$, where $a\in\{A,B,C,D\}$; see also [27, 28] for similar results using Askey–Wilson processes. In particular, when $A,B,C,D<1$, we can take the contour Γ to be the circle of radius 1. In (3.4), we use the notation $(x,y;q)_\infty=(x;q)_\infty(y;q)_\infty$, where

$$(x;q)_\infty:=\prod_{k\geq 0}(1-xq^k)\tag{3.5}$$

is the **q-Pochhammer symbol**. From (3.3), we get that

$$J_N=\frac{1-q}{4}\left[1+\frac{\oint\frac{dz}{4\pi\mathrm{i}z}e^{Nf(z)}g(z;q)\left(\frac{4}{2+z+z^{-1}}-1\right)}{\oint\frac{dz}{4\pi\mathrm{i}z}e^{Nf(z)}g(z;q)}\right].\tag{3.6}$$

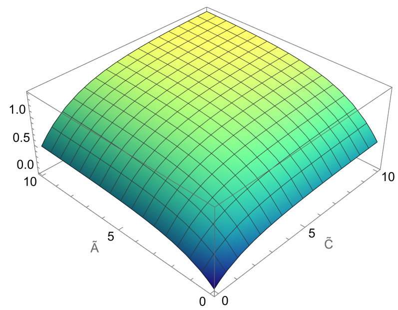

FIGURE 3. Plot of the function $F(\tilde{A}, \tilde{C})$ used in the proof of Proposition 3.1.

We will in the following, under assumption (1.5) for q with some $\kappa \in [0, \frac{1}{2}]$ and $\psi > 0$, assume that the parameters A, B, C, D satisfy

$$\begin{aligned} C &= 1 - \Theta(N^{-1/2}), \quad B = -1 + \Theta(N^{-\kappa}), \\ A &= 1 - \Theta(N^{-1/2}), \quad D = -1 + \Theta(N^{-\kappa}). \end{aligned} \tag{3.7}$$

3.1. **Improved current estimates in the maximal current phase.** In the next two propositions, we get the asymptotics for large N under assumption (1.5) for q with some $\kappa \in [0, \frac{1}{2}]$ and $\psi > 0$. We start with the case where $0 \leq \kappa < \frac{1}{2}$.

PROPOSITION **3.1.** *Consider the scaling $q = e^{-\psi N^{-\kappa}}$ for $\kappa \in [0, \frac{1}{2})$ and some $\psi > 0$. Moreover, we assert that $A = e^{-\tilde{A}N^{-1/2}}$, $C = e^{-\tilde{C}N^{-1/2}}$, with some constants $\tilde{A}, \tilde{C} > 0$. We also set $B = -qe^{-\tilde{B}N^{-1/2}}$ and $D = -qe^{-\tilde{D}N^{-1/2}}$ with some $\tilde{B}, \tilde{D} \in \mathbb{R}$. Then, as $N \to \infty$,*

$$J_N = \frac{1-q}{4}\left[1 + \frac{1}{N}F(\tilde{A}, \tilde{C}) + o(N^{-1})\right], \tag{3.8}$$

where we set

$$F(\tilde{A}, \tilde{C}) := \frac{1}{4}\frac{\int_{\mathbb{R}} dx e^{-x^2/4}\frac{x^4}{(x^2+\tilde{A}^2)(x^2+\tilde{C}^2)}}{\int_{\mathbb{R}} dx e^{-x^2/4}\frac{x^2}{(x^2+\tilde{A}^2)(x^2+\tilde{C}^2)}}. \tag{3.9}$$

A visualization of the function F is given in Figure 3. In the proof of Proposition 3.1, we will provide two separate arguments to cover the (overlapping) regimes $\kappa \in [0, \frac{1}{10})$ and $\kappa \in (0, \frac{1}{2})$ separately. Finally, we also cover the case where $\kappa = 1/2$ as follows.

PROPOSITION **3.2.** *Let $q = e^{-\psi N^{-1/2}}$, $A = e^{-\tilde{A}N^{-1/2}}$, $C = e^{-\tilde{C}N^{-1/2}}$, $B = -qe^{-\tilde{B}N^{-1/2}}$, $D = -qe^{-\tilde{D}N^{-1/2}}$ for constants $\tilde{A}, \tilde{C} > 0$ and $\tilde{B}, \tilde{D} > -\psi$. Then, as $N \to \infty$,*

$$J_N = \frac{1-q}{4}\left[1 + \frac{1}{4N}\frac{\int_{\mathbb{R}} dx e^{-x^2/4}x^2 H(\tilde{A}, \tilde{C}; x)}{\int_{\mathbb{R}} dx e^{-x^2/4} H(\tilde{A}, \tilde{C}; x)} + \mathcal{O}(N^{-9/8})\right], \tag{3.10}$$

where we set

$$H(\tilde{A}, \tilde{C}; x) = \frac{\Gamma((\tilde{A}+\mathrm{i}x)/\psi)\Gamma((\tilde{A}-\mathrm{i}x)/\psi)\Gamma((\tilde{C}+\mathrm{i}x)/\psi)\Gamma((\tilde{C}-\mathrm{i}x)/\psi)}{\Gamma(2\mathrm{i}x/\psi)\Gamma(-2\mathrm{i}x/\psi)}. \tag{3.11}$$

We will discuss strict monotonicity of the current J_N in Section 3.5.

3.2. **Identities and expansions of Gamma functions.** In this short section, we collect some basic properties of Gamma and reciprocal Gamma functions. Below a complex number z is decomposed as $z = x + \mathrm{i}y$, $x, y \in \mathbb{R}$. Note that $\Gamma(z)$ is an analytic function on $\mathbb{C}$ except for the points $z = 0, -1, -2, -3, -4, \dots$ where it has poles with residues $0, -1, 1/2, -1/3!, 1/4!, \dots$. The reciprocal of the Gamma function, $1/\Gamma(z)$, is an entire function with simple zeroes at the points $\dots, -3, -2, -1, 0$. The first identity is (see **6.1.29** in [37])

$$\frac{1}{\Gamma(\mathrm{i}y)\Gamma(-\mathrm{i}y)} = \frac{y\sinh(\pi y)}{\pi}. \tag{3.12}$$

From equation **6.1.15** of [37], $\Gamma(z+1) = z\Gamma(z)$, we have

$$\Gamma(z) = \frac{1}{z}\Gamma(z+1) \tag{3.13}$$

and, since $\Gamma(1) = 1$ and Γ is analytic and bounded in a neighborhood of $z = 1$, for $|z| \to 0$ we get

$$\Gamma(z) = \frac{1}{z}(1 + \mathcal{O}(z)). \tag{3.14}$$

As a consequence, as $\varepsilon \to 0$, it holds that

$$\Gamma((x+\mathrm{i}y)\varepsilon)\Gamma((x-\mathrm{i}y)\varepsilon) = \frac{1}{(x^2+y^2)\varepsilon^2}(1 + \mathcal{O}(\varepsilon)). \tag{3.15}$$

Inequality **6.1.26** of [37] states that

$$|\Gamma(x+\mathrm{i}y)| \le |\Gamma(x)|. \tag{3.16}$$

Finally, for any $x > 0, y \in \mathbb{R}$,

$$\Gamma(x+\mathrm{i}y)\Gamma(x-\mathrm{i}y) > 0. \tag{3.17}$$

This follows from

$$\Gamma(x+\mathrm{i}y)\Gamma(x-\mathrm{i}y) = |\Gamma(x+\mathrm{i}y)|^2 e^{\mathrm{i}[\arg(x+\mathrm{i}y)+\arg(x-\mathrm{i}y)]}$$

as using formula **6.1.27** of [37], we get that $\arg(x+\mathrm{i}y) + \arg(x-\mathrm{i}y) = 0$.

Remark **3.3.** *Note that for $\tilde{A}, \tilde{C} > 0$ and $x \in \mathbb{R}$, we see that H from (3.11) satisfies $H(\tilde{A}, \tilde{C}; x) > 0$ by (3.12) and (3.17). This means that the integrals in the numerator and denominator are of positive functions, i.e., there are no cancellations.*

3.3. **Asymptotics for q-Pochhammer expressions.** In the following, we parameterize the integration path as $z = e^{\mathrm{i}\phi}$. We collect some lemmas to control the contribution for ϕ away from 0 as well as the expansions for ϕ close to 0.

Lemma **3.4.** *The critical point of $f(z)$ is $z = 1$. Moreover,*

$$f(z) = \log(4) + \frac{1}{4}(z-1)^2 + \mathcal{O}((z-1)^3) \tag{3.18}$$

and for all $\phi \in [-\pi, \pi)$,

$$f(z = e^{\mathrm{i}\phi}) \le \log(4) - \frac{1}{4}\phi^2. \tag{3.19}$$

Proof. We have
$$\frac{d}{dz}f(z)=\frac{z-1}{z(z+1)}$$
which is equal to 0 for $z=1$. Moreover,
$$\frac{d^2}{dz^2}f(z)\big|_{z=1}=\frac{1}{2}$$
so (3.19) is obtained using the explicit parametrization $f(e^{\mathrm{i}\phi})=\log[2(1+\cos(\phi))]$. Indeed, setting $g(\phi)=\log(4)-\phi^2/4-\log[2(1+\cos(\phi))]$, we have $g(0)=0$ and, for $\phi\in[0,\pi)$,
$$\frac{dg(\phi)}{d\phi}=\tan(\phi/2)-\phi/2\geq 0,$$
which implies $g(\phi)\geq 0$, and thus (3.19). ☐

The next estimates are useful in the case that $1-A$ goes to 1 much faster than $1-q$ to 0 as we let $N\to\infty$.

Lemma **3.5.** *Let $q\in(0,1)$ and $w\in\mathbb{C}$ such that $|(1-w)/(1-q)|\leq\frac{1}{2}$. Then*
$$(qw;q)_\infty=(q;q)_\infty e^{\mathcal{O}(|1-w|/(1-q)^2)}. \tag{3.20}$$

Proof. We start with
$$\log((qw;q)_\infty)-\log((q;q)_\infty)=\sum_{n\geq 0}\log\left(\frac{1-wq^{n+1}}{1-q^{n+1}}\right)=\sum_{n\geq 0}\log\left(1+\frac{(1-w)q^{n+1}}{1-q^{n+1}}\right). \tag{3.21}$$
Notice that for all $n\geq 0$ and $q\in[0,1)$, we have that
$$0\leq\frac{1}{1-q^{n+1}}\leq\frac{1}{1-q},\quad 0\leq q^{n+1}\leq q. \tag{3.22}$$
Denoting $z=1-w$, our assumptions give $|z/(1-q)|\leq\frac{1}{2}$. Then by (3.22) for all n we also get $|zq^{n+1}/(1-q^{n+1})|\leq\frac{1}{2}$. Therefore, the series of the logarithm in the summands in (3.21) is convergent for all n. This implies
$$(3.21)=\sum_{n\geq 0}\sum_{\ell\geq 1}\frac{(-1)^\ell z^\ell q^{\ell(n+1)}}{(1-q^{n+1})^\ell\ell}. \tag{3.23}$$
Using (3.22), we have the bound
$$\begin{aligned}|(3.23)|&\leq\sum_{\ell\geq 1}\frac{|z|^\ell}{(1-q)^\ell\ell}\sum_{n\geq 1}q^{\ell n}=\sum_{\ell\geq 1}\frac{|z|^\ell}{(1-q)^\ell\ell}\frac{q^\ell}{1-q^\ell}\\&\leq\frac{1}{1-q}\sum_{\ell\geq 1}\frac{|z|^\ell q^\ell}{(1-q)^\ell}=\frac{1}{1-q}\frac{|z|q}{1-q}\frac{1}{1-\frac{|z|q}{1-q}}\leq 2\frac{|z|q}{(1-q)^2},\end{aligned} \tag{3.24}$$
as our assumptions imply $\frac{|z|q}{1-q}\leq\frac{1}{2}$. ☐

Lemma **3.6.** *Let $q\in(0,1)$ and $w\in\mathbb{C}$ be such that $|(1-w)/(1-q)|\leq\frac{1}{2}$. Then*
$$(-qw;q)_\infty=(-q;q)_\infty e^{\mathcal{O}(|1-w|/(1-q)^2)}. \tag{3.25}$$

Proof. We have

(3.26)
$$\log((-qw;q)_\infty)-\log((-q;q)_\infty)=\sum_{n\geq 0}\log\left(\frac{1+wq^{n+1}}{1+q^{n+1}}\right)=\sum_{n\geq 0}\log\left(1-\frac{(1-w)q^{n+1}}{1+q^{n+1}}\right).$$

Notice that for all $n\geq 0$ and $q\in[0,1)$, it holds that

$$\frac{1}{1+q}\leq\frac{1}{1+q^{n+1}}\leq 1,\quad 0\leq q^{n+1}\leq q. \tag{3.27}$$

Denoting $z=1-w$, we see that $|z/(1-q)|\leq\frac{1}{2}$. Then by (3.27) for all n, we get $|zq^{n+1}/(1+q^{n+1})|\leq\frac{1}{2}$. Therefore, the series of the logarithm is convergent for all n, so

$$|(3.26)|\leq\sum_{n\geq 0}\sum_{\ell\geq 1}\frac{|z|^\ell q^{\ell(n+1)}}{(1+q^{n+1})^\ell\ell}\leq\sum_{\ell\geq 1}|z|^\ell\sum_{n\geq 0}q^{\ell(n+1)}=\sum_{\ell\geq 1}\frac{|z|^\ell q^\ell}{1-q^\ell}\leq\frac{2|z|q}{(1-q)^2} \tag{3.28}$$

since our assumptions imply $|z|q\leq 1/2$. □

The following estimate will be used when $0<\kappa\leq\frac{1}{2}$. It is a special case of Corwin–Knizel's Proposition 2.3 in [35] (with $m=1$ and *their* $\varepsilon=\frac{1}{4}$, $b=\frac{1}{2}$, $\alpha=2$).

Lemma **3.7** (Special case of Proposition 2.3 of [35])**.** *Let $q=e^{-\varepsilon}$. Define the functions*

$$\begin{aligned}\mathcal{A}^+(\varepsilon,w)&=-\frac{\pi^2}{6\epsilon}-\left(w-\frac{1}{2}\right)\log(\varepsilon)+\tfrac{1}{2}\log(2\pi)-\log(\Gamma(w)),\\ \mathcal{A}^-(\varepsilon,w)&=\frac{\pi^2}{12\epsilon}-\left(w-\frac{1}{2}\right)\log(2).\end{aligned} \tag{3.29}$$

For all $w\in\mathbb{C}$ with $|\operatorname{Im}(w)|\leq 2/\varepsilon$,

$$\log((q^w;q)_\infty)=\mathcal{A}^+(\varepsilon,w)+E^+(\varepsilon,w), \tag{3.30}$$

$$\log((-q^w;q)_\infty)=\mathcal{A}^-(\varepsilon,w)+E^-(\varepsilon,w), \tag{3.31}$$

with $E^\pm(\varepsilon,w)=\mathcal{O}(\varepsilon(1+|w|)^2+\varepsilon^{1/2}(1+|w|)^{2+1/4})$.

Lemma **3.8.** *Let $q=e^{-\varepsilon}$ with $0<\varepsilon\leq 1$ and $z=e^{\mathrm{i}\phi}$. Then*

$$|(z^2,z^{-2};q)_\infty|\leq 4\sin(\phi)^2(-q;q)_\infty^2\leq c_0e^{\pi^2/(6\varepsilon)} \tag{3.32}$$

for some constant c_0. Furthermore, for $|\phi|\leq 1$,

$$(z^2,z^{-2};q)_\infty=\frac{e^{-\frac{\pi^2}{3\varepsilon}+\log(\varepsilon)+\log(2\pi)}}{\Gamma(-2\mathrm{i}\phi/\varepsilon)\Gamma(2\mathrm{i}\phi/\varepsilon)}e^{\mathcal{O}(\phi^2\varepsilon^{-7/4};\sqrt{\varepsilon})}. \tag{3.33}$$

Proof. Taking out the first terms in the q-Pochhammer products, we get

$$\begin{aligned}(z^2,z^{-2};q)_\infty&=-(z^{-1}-z)^2(qz^2,qz^{-2};q)_\infty\\ &=4\sin^2(\phi)\prod_{k\geq 0}(1-e^{2\mathrm{i}\phi}q^{k+1})(1-e^{-2\mathrm{i}\phi}q^{k+1}).\end{aligned}$$

The last product is bounded by $((-q,q)_\infty)^2$. By Lemma 3.7 (Equation (3.31) with $w=1$), we get that

$$((-q,q)_\infty)^2\leq C_0e^{\pi^2/(6\varepsilon)}$$

for some constant $C_0 > 0$. Next, notice that $z^2 = e^{2\mathrm{i}\phi} = q^w = e^{-\varepsilon w}$ for $w = -2\mathrm{i}\phi/\varepsilon$. As $|\phi| \leq 1$ gives $|\operatorname{Im}(w)| \leq 2/\varepsilon$, we can use the expansion of Lemma 3.7 to see that

$$(z^2; q)_\infty = \frac{e^{-\frac{\pi^2}{6\varepsilon} + (2\mathrm{i}\phi/\varepsilon + 1/2)\log(\varepsilon) + \frac{1}{2}\log(2\pi)}}{\Gamma(-2\mathrm{i}\phi/\varepsilon)} e^{\mathcal{O}(\phi^2\varepsilon^{-7/4}; \sqrt{\varepsilon})},$$

where $\mathcal{O}(\phi^2\varepsilon^{-7/4})$ comes from the case where $|w| \gg 1$ and $\sqrt{\varepsilon}$ from the case where $|w|$ remains bounded. Similarly, by replacing ϕ with $-\phi$, we get the expansion for $(z^{-2}; q)_\infty$. Multiplying them together we get the claimed result. $\square$

The next result gives us the asymptotics for A close to the critical point.

Lemma 3.9. *Let $q = e^{-\varepsilon}$ with $0 < \varepsilon \leq 1$, $A = e^{-\tilde{A}N^{-1/2}}$ for some $\tilde{A} \geq 0$ and $z = e^{\mathrm{i}\phi}$. Then*

$$|(Az, Az^{-1}; q)_\infty| \geq c_0 \tilde{A}^2 N^{-1} e^{-\tilde{c}/\varepsilon} \tag{3.34}$$

for some constants $c_0, \tilde{c} > 0$. Furthermore, for $|\phi| \leq 1$, we see that

$$(Az, Az^{-1}; q)_\infty = \frac{e^{-\frac{\pi^2}{3\varepsilon} + \log(\varepsilon) + \log(2\pi) - \frac{2\tilde{A}\log(\varepsilon)}{N^{1/2}\varepsilon}}}{\Gamma\left(\frac{\tilde{A}}{N^{1/2}\varepsilon} - \mathrm{i}\frac{\phi}{\varepsilon}\right)\Gamma\left(\frac{\tilde{A}}{N^{1/2}\varepsilon} + \mathrm{i}\frac{\phi}{\varepsilon}\right)} e^{\mathcal{O}(\phi^2\varepsilon^{-7/4}; \sqrt{\varepsilon})}. \tag{3.35}$$

Proof. We have that

$$\begin{aligned}
|(Az, Az^{-1}; q)_\infty| &= \left| \prod_{k\geq 0} (1 - Azq^k)(1 - Az^{-1}q^k) \right| \\
&\geq \prod_{k \geq 0} (1 - Aq^k)^2 = ((A, q)_\infty)^2 \geq (1 - A)^2 \left((q; q)_\infty\right)^2,
\end{aligned}$$

where we used $|A| \leq 1$ for the first inequality. Note that $(1 - A)^2 \simeq \tilde{A}^2 N^{-1}$ and $(q; q)_\infty \simeq e^{-\pi^2/(6\varepsilon)}$. For the expansion around $\phi = 0$, we can use Lemma 3.7. Defining w by $Az = q^w$, we have that $w = \tilde{A}/(N^{1/2}\varepsilon) - \mathrm{i}\phi/\varepsilon$. This leads to

$$(Az; q)_\infty = \frac{e^{-\frac{\pi^2}{6\varepsilon} - \left(\frac{\tilde{A}}{N^{1/2}\varepsilon} - \frac{\mathrm{i}\phi}{\varepsilon} - \frac{1}{2}\right)\log(\varepsilon) + \frac{1}{2}\log(2\pi)}}{\Gamma\left(\frac{\tilde{A}}{N^{1/2}\varepsilon} - \mathrm{i}\frac{\phi}{\varepsilon}\right)} e^{\mathcal{O}(\phi^2\varepsilon^{-7/4}; \sqrt{\varepsilon})}.$$

With $\phi \to -\phi$, we get the expression for $(Az^{-1}; q)_\infty$. Multiplying them leads to

$$(Az, Az^{-1}; q)_\infty = \frac{e^{-\frac{\pi^2}{3\varepsilon} + \log(\varepsilon) + \log(2\pi) - \frac{2\tilde{A}\log(\varepsilon)}{N^{1/2}\varepsilon}}}{\Gamma\left(\frac{\tilde{A}}{N^{1/2}\varepsilon} - \mathrm{i}\frac{\phi}{\varepsilon}\right)\Gamma\left(\frac{\tilde{A}}{N^{1/2}\varepsilon} + \mathrm{i}\frac{\phi}{\varepsilon}\right)} e^{\mathcal{O}(\phi^2\varepsilon^{-7/4}; \sqrt{\varepsilon})},$$

which is the claimed result. $\square$

The next result tells us that in the asymptotics, the terms with B (and D) can be replaced by constants close to the critical point.

Lemma 3.10. *Let $q = e^{-\varepsilon}$ with $0 < \varepsilon \leq 1$, $B = -qe^{-\tilde{B}N^{-1/2}}$ for some $\tilde{B} > -\varepsilon N^{1/2}$ and $z = e^{\mathrm{i}\phi}$. Then*

$$|(Bz, Bz^{-1}; q)_\infty| \geq c_0(\tilde{B}N^{-1/2} + \varepsilon)^2 e^{-\tilde{c}/\varepsilon} \tag{3.36}$$

for some constants $c_0, \tilde{c} > 0$. *Furthermore, for* $|\phi| \leq 1$,

$$(Bz, Bz^{-1}; q)_\infty = e^{\frac{\pi^2}{6\varepsilon} - 2\log(2) - 2\frac{\tilde{B}}{N^{1/2}\varepsilon}\log(2)} e^{\mathcal{O}(\phi^2\varepsilon^{-7/4}; \sqrt{\varepsilon})}. \tag{3.37}$$

Proof. The first bound is as in Lemma 3.9, replacing ϕ with $\pi - \phi$. The expansion for small ϕ is obtained using the second formula of Lemma 3.7. Setting w through $Bz = -q^w$ we get $w = -\mathrm{i}\phi/\varepsilon + 1 + \tilde{B}/(N^{1/2}\varepsilon)$, and hence

$$(Bz; q)_\infty = e^{\frac{\pi^2}{12\varepsilon} - \left(1 - \frac{\mathrm{i}\phi}{\varepsilon} + \frac{\tilde{B}}{N^{1/2}\varepsilon}\right)\log(2)} e^{\mathcal{O}(\phi^2\varepsilon^{-7/4}; \sqrt{\varepsilon})},$$

and similarly for $(Bz^{-1}; q)_\infty$. Multiplying the two terms, at first order the dependence on ϕ vanishes, and we get the claimed result. $\square$

Remark **3.11.** *We will use the following estimate to bound an extra factor in the numerator in* (3.9). *For* $z = e^{\mathrm{i}\phi}$, *we have that*

$$\frac{4}{2 + z + z^{-1}} - 1 = (\tan(\phi/2))^2. \tag{3.38}$$

As $\phi \to \pi$, *the right-hand side in* (3.38) *diverges, but will be compensated by the factor of* $e^{Nf(z)}$, *which converges to* 0 *much faster. Furthermore, for* ϕ *small, we have that*

$$\frac{4}{2 + z + z^{-1}} - 1 = \frac{1}{4}\phi^2(1 + \mathcal{O}(\phi^2)). \tag{3.39}$$

3.4. **Proof of the improved current estimates.** We have now all tools to show the improved estimates on the stationary current in the maximal current phase.

Proof of Proposition 3.1 for $\kappa \in [0, \frac{1}{10})$. Consider first the integral in the denominator of (3.6). Notice that for any $a \in [-1, 1)$ and $q \in [0, 1)$, with $z = e^{\mathrm{i}\phi}$,

$$0 < (a; q)_\infty^2 \leq |(az, az^{-1}; q)_\infty| \leq (-1; q)_\infty^2 < \infty. \tag{3.40}$$

Also, we can write

$$(z^2, z^{-2}; q) = (1 - z^2)(1 - z^{-2})(qz^2, qz^{-2}; q)_\infty$$

as well as

$$(Az, Az^{-1}; q) = (1 - Az)(1 - Az^{-1})(qAz, qAz^{-1}; q)_\infty,$$

and similarly for B, C and D. For the choice of A, B, C, D, the rough bounds from (3.40) implies that $|g(z, q)| \leq c_1 e^{c_2 N^{1/10}}$ for some constants $c_1, c_2 > 0$ (if $\kappa = 0$ it is even only polynomial in N, while for $\kappa > 0$ it grows at most as $e^{c_2 N^\kappa}$). This term is dominated by $e^{-N\phi^2/4}$ for $|\phi| > N^{-2/5}$. Therefore, the contribution to $\oint \frac{dz}{4\pi \mathrm{i} z} e^{Nf(z)} g(z; q)$ for $|\phi| > N^{-2/5}$ is of order $\mathcal{O}(e^{-N^{1/5}})$ smaller than the leading term.

Next consider $|\phi| \leq N^{-2/5}$. By using Taylor expansion for the simple factors and Lemma 3.5 applied to the terms with the q-Pochhammer terms we get

$$\begin{aligned}(z^2, z^{-2}; q) &= (1 - z^2)(1 - z^{-2})(qz^2, qz^{-2}; q)_\infty \\ &= 4\phi^2(1 + \mathcal{O}(N^{-4/5}))((q; q)_\infty)^2 e^{\mathcal{O}(N^{-(2/5 - 2\kappa)})},\end{aligned}$$

as well as

$$\begin{aligned}(Az, Az^{-1}; q) &= (1 - Az)(1 - Az^{-1})(qAz, qAz^{-1}; q)_\infty \\ &= (\tilde{A}^2/N + \phi^2)(1 + \mathcal{O}(N^{-2/5}))((q;q)_\infty)^2 e^{\mathcal{O}(N^{-(2/5-2\kappa)})},\end{aligned}$$

and similarly for C. As a consequence, if we denote $\phi = xN^{-1/2}$, we have that

$$\frac{1}{(Az, Az^{-1}; q)} = \frac{N}{\tilde{A}^2 + x^2} \frac{1}{((q;q)_\infty)^2}(1 + \mathcal{O}(N^{-(2/5-2\kappa)})).$$

Using Lemma 3.6, we get that

$$\begin{aligned}(Bz, Bz^{-1}; q) &= (1 - Bz)(1 - Bz^{-1})(qAz, qAz^{-1}; q)_\infty \\ &= ((-q;q)_\infty)^2 e^{\mathcal{O}(N^{-(2/5-2\kappa)})}(1 + \mathcal{O}(N^{-2/5})).\end{aligned}$$

Finally, we have

$$e^{Nf(z)} = e^{N\log(4)} e^{-N\phi^2/4} e^{\mathcal{O}(N\phi^3)} = e^{N\log(4)} e^{-N\phi^2/4} e^{\mathcal{O}(N^{-1/5})}.$$

Collecting all the terms we get, with $\phi = xN^{-1/2}$, that

$$g(z;q) \sim \frac{4Nx^2}{(\tilde{A}^2 + x^2)(\tilde{C}^2 + x^2)((q;q)_\infty)^2((-q;q)_\infty)^4} e^{N\log(4) - x^2/4}$$

up to errors of order $\mathcal{O}(N^{-(2/5-2\kappa)}; N^{-1/5}) = \mathcal{O}(N^{-1/5})$. Removing the error terms for the integral over $|x| \leq N^{1/10}$ leads to an error term of at most $\mathcal{O}(N^{-1/5})$ with respect to the leading term. After this step, extending the integral from $|x| \leq N^{1/10}$ for $x \in \mathbb{R}$ can be made up to an error of order $\mathcal{O}(e^{-N^{1/5}})$. Summing up, the denominator is given by

$$\Xi(N,q)(1 + \mathcal{O}(N^{-1/5})) \int_{\mathbb{R}} dx e^{-x^2/4} \frac{x^2}{(x^2 + \tilde{A}^2)(x^2 + \tilde{C}^2)}$$

with $\Xi(N,q) = e^{N\log(4)} \frac{4N}{((q;q)_\infty)^2((-q;q)_\infty)^4}$. Similarly, one deals with the numerator, which is given by

$$\Xi(N,q)(1 + \mathcal{O}(N^{-1/5})) \frac{1}{N} \int_{\mathbb{R}} dx e^{-x^2/4} \frac{x^4}{(x^2 + \tilde{A}^2)(x^2 + \tilde{C}^2)}.$$

Taking the ratio, we get the claimed result. □

Next, we consider the case where $\kappa \in (0, \frac{1}{2})$.

Proof of Proposition 3.1 for $\kappa \in (0, \frac{1}{2})$. Let us parameterize $z = e^{\mathrm{i}\phi}$ with $\phi \in [-\pi, \pi)$. We decompose the integration into the following subsets:

$$\Gamma_0 = \{e^{\mathrm{i}\phi}, -\delta_1 \leq \phi \leq \delta_1\}, \quad \Gamma_1 = \{e^{\mathrm{i}\phi}, |\phi| \in (\delta_1, 1]\}, \quad \Gamma_2 = \{e^{\mathrm{i}\phi}, |\phi| \in (1, \pi/2]\}. \tag{3.41}$$

Here, we choose $\delta_1 = N^{-(2\kappa+1)/4}$. We also define a small parameter ε by the relation $q = e^{-\varepsilon}$, that is,

$$\varepsilon = \psi N^{-\kappa} \gg N^{-1/2}.$$

We first consider the integral in the denominator. The analysis for the numerator is almost identical, except for the extra term $\frac{4}{2+z+z^{-1}} - 1$; see Remark 3.11. We bound the contributions from Γ_2 and Γ_1, and then determine the asymptotics from the contribution by Γ_0.

(a) Bound for $z \in \Gamma_2$. For $z \in \Gamma_2$, by Lemma 3.4, we have that

$$|e^{Nf(z)}| \leq e^{N\log(4)} e^{-N/4}.$$

By Lemma 3.8, we get that

$$|(z^2, z^{-2}; q)_\infty| \leq c_0 e^{\pi^2 N^\kappa/(6\psi)} \leq c_0 e^{\pi^2 N^{1/2}/6},$$

where in the second inequality, we used that $1/\varepsilon \gg N^{-1/2}$. Similarly, by Lemma 3.9,

$$\frac{1}{|(Az, Az^{-1}; q)_\infty (Cz, Cz^{-1}; q)_\infty|} \leq c_1 \frac{N^2}{\tilde{A}^2 \tilde{C}^2} e^{2\tilde{c}N^{1/2}}$$

for some constants $c_1, \tilde{c} > 0$. Finally, Lemma 3.10 gives that

$$\frac{1}{|(Bz, Bz^{-1}; q)_\infty (Dz, Dz^{-1}; q)_\infty|} \leq c_1 N^2 e^{2\tilde{c}N^{1/2}}$$

for some constant $c_1 > 0$. Therefore,

$$e^{-N\log(4)} \left| \int_{\Gamma_2} \frac{dz}{4\pi \mathrm{i} z} e^{Nf(z)} g(z; q) \right| \leq c_2 e^{c_3 N^{1/2}} e^{-N/4} \tag{3.42}$$

for some constants $c_2, c_3 > 0$.

(b) Bound for $z \in \Gamma_1$. For $z \in \Gamma_1$, by Lemma 3.4 we have that

$$|e^{Nf(z)}| \leq e^{N\log(4)} e^{-N\phi^2/4}.$$

Using that $\kappa \leq \frac{1}{2}$ in the error term, Lemma 3.8 yields

$$(z^2, z^{-2}; q)_\infty = \frac{e^{-\frac{\pi^2}{3\varepsilon} + \log(\varepsilon) + \log(2\pi)}}{\Gamma(-2\mathrm{i}\phi/\varepsilon)\Gamma(2\mathrm{i}\phi/\varepsilon)} e^{\mathcal{O}(\phi^2 N^{7/8}; N^{-\kappa/2})}$$

while Lemma 3.9 leads to

$$\begin{aligned} &\frac{1}{(Az, Az^{-1}; q)_\infty (Cz, Cz^{-1}; q)_\infty} \\ &= \frac{\Gamma\left(\frac{\tilde{A}}{N^{1/2}\varepsilon} - \mathrm{i}\frac{\phi}{\varepsilon}\right)\Gamma\left(\frac{\tilde{A}}{N^{1/2}\varepsilon} + \mathrm{i}\frac{\phi}{\varepsilon}\right)\Gamma\left(\frac{\tilde{C}}{N^{1/2}\varepsilon} - \mathrm{i}\frac{\phi}{\varepsilon}\right)\Gamma\left(\frac{\tilde{C}}{N^{1/2}\varepsilon} + \mathrm{i}\frac{\phi}{\varepsilon}\right)}{e^{-\frac{2\pi^2}{3\varepsilon} + 2\log(\varepsilon) + 2\log(2\pi) - \frac{2(\tilde{A}+\tilde{C})\log(\varepsilon)}{N^{1/2}\varepsilon}}} e^{\mathcal{O}(\phi^2 N^{7/8}; N^{-\kappa/2})}. \end{aligned}$$

Finally, by Lemma 3.10, we get that

$$\frac{1}{(Bz, Bz^{-1}; q)_\infty (Dz, Dz^{-1}; q)_\infty} = e^{-\frac{\pi^2}{3\varepsilon} + 4\log(2) + 2\frac{(\tilde{B}+\tilde{D})}{N^{1/2}\varepsilon}\log(2)} e^{\mathcal{O}(\phi^2 N^{7/8}; N^{-\kappa/2})}.$$

Collecting all the terms which are not dependent on ϕ, we define

$$\Xi_N = e^{-\log(\varepsilon) + 4\log(2) - \log(2\pi)} e^{\frac{2(\tilde{A}+\tilde{C})\log(\varepsilon)}{N^{1/2}\varepsilon}} e^{2\frac{(\tilde{B}+\tilde{D})}{N^{1/2}\varepsilon}\log(2)}. \tag{3.43}$$

The terms involving the Gamma functions are given by

$$G_N(\phi) = \frac{\Gamma\left(\frac{\tilde{A}}{N^{1/2}\varepsilon} - \mathrm{i}\frac{\phi}{\varepsilon}\right)\Gamma\left(\frac{\tilde{A}}{N^{1/2}\varepsilon} + \mathrm{i}\frac{\phi}{\varepsilon}\right)\Gamma\left(\frac{\tilde{C}}{N^{1/2}\varepsilon} - \mathrm{i}\frac{\phi}{\varepsilon}\right)\Gamma\left(\frac{\tilde{C}}{N^{1/2}\varepsilon} + \mathrm{i}\frac{\phi}{\varepsilon}\right)}{\Gamma(-2\mathrm{i}\phi/\varepsilon)\Gamma(2\mathrm{i}\phi/\varepsilon)}.$$

In these terms $\frac{\tilde{A}}{N^{1/2}\varepsilon} \ll \frac{\phi}{\varepsilon}$ since $1 \geq \phi \geq \delta_1$. By the identity (3.12) on Gamma functions, we have that $\frac{1}{\Gamma(-2\mathrm{i}\phi/\varepsilon)\Gamma(2\mathrm{i}\phi/\varepsilon)} \sim e^{2\pi\phi/\varepsilon}$, which is much smaller than $e^{-N\phi^2/4}$. Thus,

$$e^{-N\phi^2/4}e^{2\pi\phi/\varepsilon} = e^{-N\phi^2\left(\frac{1}{4}-2\pi\frac{N^{\kappa-1}}{\phi}\right)}$$

and, for $1 \geq \phi \geq \delta_1$,

$$2\pi\frac{N^{\kappa-1}}{\phi} \leq 2\pi N^{\frac{3}{4}(2\kappa-1)} \to 0 \text{ as } N \to \infty. \tag{3.44}$$

By (3.15) and (3.16), the terms with the Gamma functions in the numerator grow at most with a power law in N, and thus are much smaller than the contribution of the two Gamma functions in the denominator (these are very rough bounds, but good enough for ϕ in Γ_1). Thus, for all N large enough,

$$G_N(\phi) \leq e^{N\phi^2/16}.$$

Furthermore, all the error terms $e^{\mathcal{O}(\phi^2 N^{7/8};N^{-\kappa/2})}$ together, are also bounded by $e^{N\phi^2/16}$ for all N large enough. As a consequence, for all N large enough, all the ϕ-dependent terms are bounded by $e^{-N\phi^2/8}$. Since we have $|\phi| \geq \delta_1 = N^{-(2\kappa+1)/4}$, this yields a contribution of order $e^{-c_4 N^{1/2-\kappa}}$. Thus, altogether we have

$$e^{-N\log(4)}\left|\int_{\Gamma_1}\frac{dz}{4\pi\mathrm{i}z}e^{Nf(z)}g(z;q)\right| \leq \Xi_N e^{-c_5 N^{(1-2\kappa)/2}} \tag{3.45}$$

for some constant $c_5 > 0$. We remark that the bound in (3.42) is subleading with respect to (3.45).

(c) Expansion for $z \in \Gamma_0$. Consider $|\phi| \leq N^{-(2\kappa+1)/4}$. All the expansions we have collected for case (b) still holds true for $z \in \Gamma_0$, i.e.,

$$e^{-N\log(4)}\int_{\Gamma_0}\frac{dz}{4\pi\mathrm{i}z}e^{Nf(z)}g(z;q) = \Xi_N\int_{-N^{-(2\kappa+1)/4}}^{N^{-(2\kappa+1)/4}}\frac{d\phi}{4\pi}e^{-N\phi^2/4}G_N(\phi)e^{\mathcal{O}(N^{7/8}\phi^2;N^{-\kappa/2})}, \tag{3.46}$$

where we recall Ξ_N from (3.43). By the change of variable $\phi = N^{-1/2}x$, we obtain

$$(3.46) = \Xi_N N^{-1/2}\int_{-N^{(1-2\kappa)/4}}^{N^{(1-2\kappa)/4}}\frac{dx}{4\pi}e^{-x^2/4}G_N(xN^{-1/2})e^{\mathcal{O}(x^2N^{-1/8};N^{-\kappa/2})}. \tag{3.47}$$

Letting $\tilde{\varepsilon} = N^{\kappa-1/2}/\psi$, the terms involving Gamma functions are given by

$$G_N(xN^{-1/2}) = \frac{\Gamma\big((\tilde{A}-\mathrm{i}x)\tilde{\varepsilon}\big)\Gamma\big((\tilde{A}+\mathrm{i}x)\tilde{\varepsilon}\big)\Gamma\big((\tilde{C}-\mathrm{i}x)\tilde{\varepsilon}\big)\Gamma\big((\tilde{C}+\mathrm{i}x)\tilde{\varepsilon}\big)}{\Gamma(-2\mathrm{i}x\tilde{\varepsilon})\Gamma(2\mathrm{i}x\tilde{\varepsilon})}.$$

The error term in (3.47) can be removed up to an error $\mathcal{O}(N^{-1/8};N^{-\kappa/2})$ with respect to the leading term (using the usual inequality $|e^y-1| \leq |y|e^{|y|}$ with y replaced by the error term). Notice that the integration is now over $|x| \leq N^{(1-2\kappa)/4}$, which means that

$|\tilde{\varepsilon}x| \leq \frac{1}{\psi} N^{(2\kappa-1)/4} \to 0$ as $N \to \infty$. Therefore, all entries in the Gamma functions are very small. More precisely, using the expansion (3.15), we get that

$$G_N(xN^{-1/2}) = \frac{4x^2}{(\tilde{A}^2 + x^2)(\tilde{C}^2 + x^2)\tilde{\varepsilon}^2}(1 + \mathcal{O}(x\tilde{\varepsilon})). \tag{3.48}$$

At this point all the error terms can be estimated by $\mathcal{O}(N^{(2\kappa-1)/4}; N^{-1/8}; N^{-\kappa/2}) = o(1)$, and are thus smaller than the leading term. Removing the error terms and then extending the integration over $x \in \mathbb{R}$, the error term is not larger than the ones we already have. Consequently, for the denominator, we get that

$$\int_{|z|=1} \frac{dz}{4\pi \mathrm{i} z} e^{Nf(z)} g(z;q) = e^{N\log(4)} \Xi_N N^{-1/2} \tilde{\varepsilon}^{-2}(1 + o(1)) \int_{\mathbb{R}} \frac{dx}{4\pi} e^{-x^2/4} \frac{4x^2}{(\tilde{A}^2 + x^2)(\tilde{C}^2 + x^2)}$$

with $o(1) = \mathcal{O}(N^{(2\kappa-1)/4}; N^{-1/8}; N^{-\kappa/2})$. The prefactors will cancels exactly with the ones of the numerator. The computations for the denominators are essentially the same. The only difference is that we have the additional factor $\frac{4}{2+z+z^{-1}} - 1$, which by Remark 3.11, under the change of variables $z = e^{\mathrm{i}xN^{-1/2}}$, is given by

$$\frac{4}{2 + z + z^{-1}} - 1 = \frac{x^2}{4N}(1 + \mathcal{O}(x^2/N)).$$

Putting everything together we get the claimed result with an error term given by $o(N^{-1}) = N^{-1}\mathcal{O}(N^{(2\kappa-1)/4}; N^{-1/8}; N^{-\kappa/2})$. □

Given the proof of Proposition 3.1 for $\kappa \in (0, \frac{1}{2})$, we now describe the necessary adjustments in order deduce Proposition 3.2 for $\kappa = \frac{1}{2}$.

Sketch of proof of Proposition 3.2 . The proof in this case is almost identical to the one of Proposition 3.1. Since the Gamma functions remain in the final expression, it is even easier. This time we can simply take $\delta_1 = N^{-1/4}$. The arguments apply mutatis mutandis, except that we do not have to expand the Gamma functions, i.e., we do not need the expansion (3.48). It is in that step that we used the fact that we cn take $\delta_1 = N^{-(2\kappa+1)/4}$ to ensure that $|\tilde{\varepsilon}x| \to 0$. Everything else works also for $\delta_1 = N^{-1/4}$ and collecting the error terms with the new choice of δ_1, we get an error term of order $\mathcal{O}(N^{-1/8})$ smaller than the leading term. □

3.5. **Strict mononicity of the second order current estimates.** For the proof of Theorem 1.1, we will require that the function $F(\tilde{A}, \tilde{C})$ from (3.9) is strict monotone. This is the content of the following lemma.

Lemma **3.12.** *The function $F(\tilde{A}, \tilde{C})$ is positive, symmetric in $\tilde{A}, \tilde{C}$, and for fixed $\tilde{C} \geq 0$, it is strictly increasing in $\tilde{A}$. Moreover, we have that* $\lim_{\tilde{A},\tilde{C}\to\infty} F(\tilde{A}, \tilde{C}) = \frac{3}{2}$.

Proof. The only non-trivial property to be verified is the strict monotonicity. Let us define $\rho(x) = \frac{x^2}{x^2+\tilde{C}^2} e^{-x^2/4}$. Then

$$F(\tilde{A}, \tilde{C}) = \frac{1}{4} \frac{\int dx \rho(x) \frac{x^2}{x^2+\tilde{A}^2}}{\int dx \rho(x) \frac{1}{x^2+\tilde{A}^2}}, \tag{3.49}$$

where here and below all the integrals are over $\mathbb{R}$. Thus

$$\frac{dF(\tilde{A},\tilde{C})}{d\tilde{A}} = \frac{\int dx\rho(x)\frac{x^2(-2\tilde{A})}{(x^2+\tilde{A}^2)^2}\int dy\rho(y)\frac{1}{y^2+\tilde{A}^2} - \int dx\rho(x)\frac{(-2\tilde{A})}{(x^2+\tilde{A}^2)^2}\int dy\rho(y)\frac{y^2}{y^2+\tilde{A}^2}}{\left(\int dx\rho(x)\frac{1}{x^2+\tilde{A}^2}\right)^2}.$$

Denoting $f(x) = \rho(x)/(x^2+\tilde{A}^2)^2$, we can rewrite the derivative as

$$\begin{aligned}\frac{dF(\tilde{A},\tilde{C})}{d\tilde{A}} &= \frac{\int dxf(x)x^2(-2\tilde{A})\int dyf(y)(y^2+\tilde{A}^2) - \int dxf(x)(-2\tilde{A})\int dyf(y)y^2(y^2+\tilde{A}^2)}{\left(\int dxf(x)(x^2+\tilde{A}^2)\right)^2} \\ &= 2\tilde{A}\frac{\int dxf(x)\int dyf(y)y^2(y^2+\tilde{A}^2) - \int dxf(x)x^2\int dyf(y)(y^2+\tilde{A}^2)}{\left(\int dxf(x)(x^2+\tilde{A}^2)\right)^2}.\end{aligned}$$

Thus, we need to prove that

$$\int dxf(x)\int dyf(y)y^2(y^2+\tilde{A}^2) > \int dxf(x)x^2\int dyf(y)(y^2+\tilde{A}^2),$$

which is equivalent to showing that

$$\int dxf(x)\int dyf(y)y^4 > \int dxf(x)x^2\int dyf(y)y^2.$$

If we denote by $d\mu(x) = f(x)dx$, this rewrites as

$$\left(\langle 1, x^2\rangle_{L^2(d\mu)}\right)^2 < \langle 1,1\rangle_{L^2(d\mu)}\langle x^2,x^2\rangle_{L^2(d\mu)} = \|1\|^2_{L^2(d\mu)}\|x^2\|_{L^2(d\mu)}.$$

Notice that this, with $\leq$ is nothing else than the Cauchy-Schwarz inequality. The strict inequality holds true since the functions 1 and x^2 are not collinear. Finally, the limit as $\tilde{A},\tilde{C}\to\infty$ is given by

$$\frac{1}{4}\frac{\int dxe^{-x^2/4}x^4}{\int dxe^{-x^2/4}x^2} = \frac{3}{2},$$

which allows us to conclude. □

REMARK **3.13.** *Note that the limit* $\lim_{\tilde{A},\tilde{C}\to\infty} F(\tilde{A},\tilde{C}) = \frac{3}{2}$ *in Lemma 3.12 agrees with the fact that when* $\alpha=\beta=1$ *and* $q=0$*, the stationary current satisfies* $J_N = Z_{N-1}/Z_N = \mathsf{C}_N/\mathsf{C}_{N+1} = \frac{1}{4}+\frac{3}{8N}+o(N^{-1})$*, where* C_N *denotes the* N^{th} *Catalan number; see for example Section 2.3 in* [64].

Similarly, we require for the proof of Theorem 1.2 strict monotonicity for all sufficiently large $\tilde{A},\tilde{C}>0$ for the function $\tilde{F}(\tilde{A},\tilde{C})$ defined in the following.

LEMMA **3.14.** *Let us define*

$$\tilde{F}(\tilde{A},\tilde{C}) = \frac{1}{4}\frac{\int_{\mathbb{R}} dxe^{-x^2/4}x^2 H(\tilde{A},\tilde{C};x)}{\int_{\mathbb{R}} dxe^{-x^2/4}H(\tilde{A},\tilde{C};x)}. \tag{3.50}$$

For any given $\tilde{C}$*, for* $\tilde{A}\gg 1$*,*

$$\frac{d\tilde{F}(\tilde{A},\tilde{C})}{d\tilde{A}} > 0. \tag{3.51}$$

Proof. Let us rewrite $\tilde{F}(\tilde{A},\tilde{C})$ as

$$\tilde{F}(\tilde{A},\tilde{C}) = \frac{1}{4}\frac{\int_{\mathbb{R}} dx x^2 \rho(x) g(\tilde{A},x)}{\int_{\mathbb{R}} dx \rho(x) g(\tilde{A},x)}$$

with

$$\rho(x) = \frac{e^{-x^2/4}\Gamma((\tilde{C}+\mathrm{i}x)/\psi)\Gamma((\tilde{C}-\mathrm{i}x)/\psi)}{\Gamma(2\mathrm{i}x/\psi)\Gamma(-2\mathrm{i}x/\psi)},$$
$$g(\tilde{A},x) = \Gamma((\tilde{A}+\mathrm{i}x)/\psi)\Gamma((\tilde{A}-\mathrm{i}x)/\psi).$$

We have that

$$\frac{d\tilde{F}(\tilde{A},\tilde{C})}{d\tilde{A}} = \frac{1}{4}\frac{\int dx x^2 \rho(x)\frac{dg(\tilde{A},x)}{d\tilde{A}}\int d\tilde{x}\rho(\tilde{x})g(\tilde{A},\tilde{x}) - \int dx x^2\rho(x)g(\tilde{A},x)\int d\tilde{x}\rho(\tilde{x})\frac{dg(\tilde{A},\tilde{x})}{d\tilde{A}}}{\left(\int dx \rho(x) g(\tilde{A},x)\right)^2}$$

and

$$\frac{dg(\tilde{A},x)}{d\tilde{A}} = g(\tilde{A},x)h(\tilde{A},x),$$

where $h(\tilde{A},x) = c^{-1}[\Gamma(0,(\tilde{A}+\mathrm{i}x)/\psi) + \Gamma(0,(\tilde{A}-\mathrm{i}x)/\psi)]$. Here, $\Gamma(0,z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is the 0-polygamma function. Denoting $f(x) = \rho(x)g(\tilde{A},x)$, we get that

$$\frac{d\tilde{F}(\tilde{A},\tilde{C})}{d\tilde{A}} = \frac{1}{4}\frac{\int dx x^2 f(x) h(\tilde{A},x)\int d\tilde{x} f(\tilde{x}) - \int dx x^2 f(x)\int d\tilde{x} f(\tilde{x}) h(\tilde{A},\tilde{x})}{\left(\int dx f(x)\right)^2}. \tag{3.52}$$

Note that by (3.12) and (3.15), we have that for all x large enough (independent of $\tilde{A}$), $f(x) \le e^{-x^2/8}$. Thus, the contribution to the integrals in (3.52) for $|x|, |\tilde{x}| \ge \tilde{A}^{1/4}$ is at least $e^{-\tilde{A}^{1/2}/16}$ smaller than the leading term. For this reason, we can restrict the integrals in (3.52) to $|x|, |\tilde{x}| \le \tilde{A}^{1/4}$. Next, we use the asymptotic expansion of the 0-polygamma function for large z, which is given in **6.3.18** of [37] as

$$\Gamma(0,z) = \log(z) - \frac{1}{2z} - \frac{1}{12z^2} + \mathcal{O}(z^{-4}), \qquad \text{for } z \to \infty.$$

This leads to

$$h(\tilde{A},x) = \left(\left[\frac{2}{\psi}\log(\tilde{A}/\psi) - \frac{1}{\tilde{A}} - \frac{\psi}{6\tilde{A}^2}\right] + \frac{x^2}{\psi\tilde{A}^2}\right)(1+\mathcal{O}(\tilde{A}^{-1/2}))$$

for $|x| \le \tilde{A}^{1/4}$. Inserting this expansion into (3.52), the x-independent term cancels exactly. The remaining error terms as well as the terms in x^2 yield

$$(3.52) = \frac{1}{4c\tilde{A}^2}\frac{\int dx x^4 f(x)\int d\tilde{x} f(\tilde{x}) - \int dx x^2 f(x)\int d\tilde{x}\tilde{x}^2 f(x)}{\left(\int dx f(x)\right)^2}(1+\mathcal{O}(\tilde{A}^{-5/2})).$$

By the Cauchy-Schwarz argument as in the proof of Lemma 3.12, we get that the numerator is strictly positive for all $\tilde{A}$ large enough. $\square$

4. Moderate deviations for the current of the open ASEP

In this section, we establish moderate deviation estimates for the current $(\mathcal{J}_t)_{t\geq 0}$ of the open ASEP. Our goal is to compare the current of the ASEP on the integers and the open ASEP. To simplify notation, we establish the current estimates only for $\rho = \rho_N$ with either

$$\rho_N = \frac{1}{2} + 2^{-n} \tag{4.1}$$

with some $n \in [\![\frac{1}{2}\log(N)]\!]$ or $\rho_N = \frac{1}{2}$. Intuitively, under assumption (4.1), the fluctuations of a second class particle in the ASEP on $\mathbb{Z}$, initially placed at the origin in a Bernoulli-ρ_N-product measure, are until time $T = 2^n N(1-q)^{-1}$ of order at most N. This suggests that under the basic coupling, the ASEP on $\mathbb{Z}$ should differ from the open ASEP until time T in at most order N many sites, and hence their currents at order at most $\sqrt{N}$. This intuition will be formalized in Propositions 4.12 and 4.15, respectively.

4.1. **Strategy for the proof.** We start by comparing the ASEP on the integers to an ASEP on $\mathbb{N}$ under the basic coupling. Suppose that both processes are started from the same configuration on $\mathbb{N}$ chosen according to a Bernoulli-ρ-product measure for some $\rho \in (0,1)$. Recall Definition 2.1 and note that under the basic coupling, second class particles of types A and B can only enter at site 1 in the disagreement process. In order to follow the second class particles, we define in Section 4.2 an extended disagreement process, where instead of annihilation, an update of a type A/B second class particle pair results in a change to types A' and B', respectively. This construction has the advantage that type B and B' second class particles can be seen as a coloring of empty sites in an ASEP on the integers within a Bernoulli-ρ-product measure.

In Section 4.3, we establish moderate deviations on the maximal displacement for a collection of second class particles in the ASEP on the integers. This is achieved by combining moderate deviations for a single second class particle and the censoring inequality. We get in Section 4.4 moderate deviations on the rightmost type B or type B' second class particle in the extended disagreement process until time $T \asymp 2^n N(1-q)^{-1}$ when $\rho_N = \frac{1}{2} + 2^{-n}$, and $T \asymp N^{3/2}(1-q)^{-1}$ for $\rho = \frac{1}{2}$. Similarly, whenever the disagreement process contains only type A and A' second class particles beyond a certain site, we can see these type A and type A' second class particles as a coloring of first class particles in an ASEP on $\mathbb{Z}$ in a Bernoulli-ρ-product measure. With a similar argument as for the type B and B' second class particles, we obtain moderate deviations for the location of the rightmost type A and A' second class particles, respectively.

In Section 4.5, we convert these moderate deviations on the location of the rightmost second class particle in the extended disagreement process into moderate deviations of the respective currents, expressing the difference in the currents as the difference in the number of type A and type B second class particles in the extended disagreement process at a given time. Controlling the location of the rightmost second class particle in a disagreement process between the ASEP on the half-line and the open ASEP in a similar way, this allows us to achieve moderate deviations for the current of the open ASEP in Section 4.6. Together with the results from Section 2 and 3 on the stationary current of the open ASEP, we obtain in Section 4.7 lower bounds on the probability that the currents of two open ASEPs with different boundary conditions deviate by the order of their stationary currents.

4.2. **The extended basic coupling.** Recall the basic coupling defined in Section 2.2 for the ASEP on $\mathbb{N}$ and $\mathbb{Z}$. In the following, we introduce two additional types of second class particles: type A' and type B' to which we refer as a **marked second class particles** for type A and type B, respectively. We consider the ordering $\succeq$ given by

$$\mathbf{1} \succeq \mathsf{A}' \succeq \mathsf{A} \succeq \mathsf{B} \succeq \mathsf{B}' \succeq \mathbf{0}, \tag{4.2}$$

where we identify $\mathbf{1}$ with first class particles and $\mathbf{0}$ with empty sites in the disagreement process. Note that the ordering $\mathsf{A} \succeq \mathsf{B}$ is not consistent with the updates of the underlying exclusion process, as A and B are incomparable. We remedy this issue by an extension to the basic coupling, where every time a pair of unmarked second class particles of different types is updated, they receive a mark, that is, we turn a type A and B pair into A' and B'.

Definition 4.1 (Extended disagreement process). *We define the extended disagreement process $(\xi_t^{\mathrm{mod}})_{t\geq 0}$ as a Markov process on the state space*

$$\{\mathbf{1}, \mathsf{A}', \mathsf{A}, \mathsf{B}, \mathsf{B}', \mathbf{0}\}^{\mathbb{Z}}. \tag{4.3}$$

For all edges $e = \{x, x+1\}$, assign a rate $1+q$ clock. Whenever a clock rings at time t, we sample a Uniform-$[0,1]$-random variable U independently of all previous samples. For $x \neq 0$, we distinguish four cases:

- *If $U \leq (1+q)^{-1}$ and $\{\xi_{t_-}^{\mathrm{mod}}(x), \xi_{t_-}^{\mathrm{mod}}(x+1)\} \neq \{\mathsf{A}, \mathsf{B}\}$, then sort the endpoints in ξ_t^{mod} with respect to (4.2) in increasing order,*
- *if $U > (1+q)^{-1}$ and $\{\xi_{t_-}^{\mathrm{mod}}(x), \xi_{t_-}^{\mathrm{mod}}(x+1)\} \neq \{\mathsf{A}, \mathsf{B}\}$, then sort the endpoints in ξ_t^{mod} with respect to (4.2) in decreasing order,*
- *if $U \leq (1+q)^{-1}$ and $\{\xi_{t_-}^{\mathrm{mod}}(x), \xi_{t_-}^{\mathrm{mod}}(x+1)\} = \{\mathsf{A}, \mathsf{B}\}$, then set $\xi_t^{\mathrm{mod}}(x+1) = \mathsf{A}'$ and $\xi_t^{\mathrm{mod}}(x) = \mathsf{B}'$,*
- *if $U > (1+q)^{-1}$ and $\{\xi_{t_-}^{\mathrm{mod}}(x), \xi_{t_-}^{\mathrm{mod}}(x+1)\} = \{\mathsf{A}, \mathsf{B}\}$, then set $\xi_t^{\mathrm{mod}}(x) = \mathsf{A}'$ and $\xi_t^{\mathrm{mod}}(x+1) = \mathsf{B}'$.*

The rules for updates at site 1 and along the edge $\{0,1\}$ remain as under the basic coupling, treating A' as a particle and B' as an empty site, i.e.,

- *if $U \leq (1+q)^{-1}$ and $\xi_{t_-}^{\mathrm{mod}}(0) = \mathsf{A}$ as well as $\xi_{t_-}^{\mathrm{mod}}(1) = \mathbf{0}$, then set $\xi_t^{\mathrm{mod}}(1) = \mathsf{A}$ and $\xi_t^{\mathrm{mod}}(0) = \mathbf{0}$,*
- *if $U \leq (1+q)^{-1}$ and $\xi_{t_-}^{\mathrm{mod}}(0) = \mathsf{A}$ as well as $\xi_{t_-}^{\mathrm{mod}}(1) \in \{\mathsf{B}, \mathsf{B}'\}$, then set $\xi_t^{\mathrm{mod}}(1) = \mathbf{1}$ and $\xi_t^{\mathrm{mod}}(0) = \mathbf{0}$,*
- *if $U > (1+q)^{-1}$ and $\xi_{t_-}^{\mathrm{mod}}(0) = \mathbf{0}$ as well as $\xi_{t_-}^{\mathrm{mod}}(1) \in \{\mathsf{A}', 1\}$, then set $\xi_t^{\mathrm{mod}}(1) = \mathsf{B}$ and $\xi_t^{\mathrm{mod}}(0) = \mathsf{A}$,*
- *if $U > (1+q)^{-1}$ and $\xi_{t_-}^{\mathrm{mod}}(0) = \mathbf{0}$ as well as $\xi_{t_-}^{\mathrm{mod}}(1) = \mathsf{A}$, then set $\xi_t^{\mathrm{mod}}(1) = \mathbf{0}$ and $\xi_t^{\mathrm{mod}}(0) = \mathsf{A}$.*

Moreover, at rate α, we update site 1 as follows:

- *If $\xi_{t_-}^{\mathrm{mod}}(1) \in \{\mathbf{0}, \mathsf{B}'\}$, we set $\xi_t^{\mathrm{mod}}(1) = \mathsf{B}$,*
- *if $\xi_{t_-}^{\mathrm{mod}}(1) \in \{\mathsf{A}, \mathsf{A}'\}$, we set $\xi_t^{\mathrm{mod}}(1) = \mathbf{1}$.*

At rate γ, we perform the following update:

- *If $\xi_{t_-}^{\mathrm{mod}}(1) \in \{\mathsf{B}', \mathsf{B}\}$, we set $\xi_t^{\mathrm{mod}}(1) = \mathbf{0}$,*
- *if $\xi_{t_-}^{\mathrm{mod}}(1) \in \{\mathsf{A}', \mathbf{1}\}$, we set $\xi_t^{\mathrm{mod}}(1) = \mathsf{A}$.*

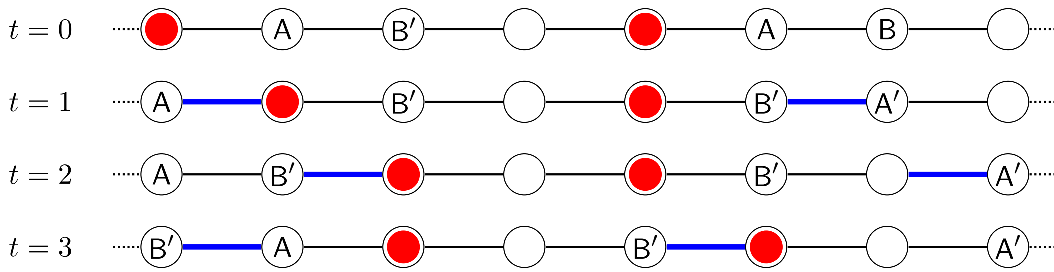

FIGURE 4. Visualization of a possible evolution of the extended disagreement process for $q = 0$. Edges which received an update since the previous step are marked in blue. First class particles are red, second class particles are given by their types. Note that the type of second class particles changes if and only if a type A/B pair receives an update (as from $t = 0$ to $t = 1$).

A visualization of this process is given in Figure 4. In words, the process $(\xi_t^{\mathrm{mod}})_{t\geq 0}$ has the same transition rules as a disagreement process under the basic coupling (identifying $\mathsf{A}\widehat{=}(1,0)$ and $\mathsf{B}\widehat{=}(0,1)$ as well as $\mathbf{1}\widehat{=}(1,1)$ and $\mathbf{0}\widehat{=}(0,0)$), but every time a pair of unmarked second class particles of different types receives an update, we mark them. Let $(\eta_t^{\mathsf{A}})_{t\geq 0}$ onto $\{0,1\}^{\mathbb{Z}}$ and $(\eta_t^{\mathsf{B}})_{t\geq 0}$ onto $\{0,1\}^{\mathbb{N}}$ be two projections of the extended disagreement process $(\xi_t^{\mathrm{mod}})_{t\geq 0}$ given by

$$\eta_t^{\mathsf{A}}(x) := \mathbb{1}_{\xi_t^{\mathrm{mod}}(x)\in\{\mathsf{A},\mathsf{A}',\mathbf{1}\}}$$
$$\eta_t^{\mathsf{B}}(y) := \mathbb{1}_{\xi_t^{\mathrm{mod}}(y)\in\{\mathsf{B},\mathsf{A}',\mathbf{1}\}}$$

for all $x \in \mathbb{Z}$ and $y \in \mathbb{N}$. Moreover, we define the process $(\xi_t^{\mathsf{AB}})_{t\geq 0}$ as the projection of $(\xi_t^{\mathrm{mod}})_{t\geq 0}$ onto $\{(1,1),(1,0),(0,1),(0,0)\}^{\mathbb{Z}}$, which is given for all $x \in \mathbb{Z}$ by

$$\xi_t^{\mathsf{AB}}(x) := \begin{cases} (0,1) & \text{if } \xi_t^{\mathrm{mod}}(x) = \mathsf{A} \\ (1,0) & \text{if } \xi_t^{\mathrm{mod}}(x) = \mathsf{B} \\ (1,1) & \text{if } \xi_t^{\mathrm{mod}}(x) \in \{\mathsf{A}',\mathbf{1}\} \\ (0,0) & \text{if } \xi_t^{\mathrm{mod}}(x) \in \{\mathsf{B}',\mathbf{0}\}. \end{cases} \tag{4.4}$$

The next lemma states that the above three projections have a simple interpretation as exclusion and disagreement processes.

LEMMA **4.2.** *The process $(\eta_t^{\mathsf{A}})_{t\geq 0}$ has the law of an ASEP on the integers with respect to drift q. The process $(\eta_t^{\mathsf{B}})_{t\geq 0}$ has the law of an ASEP on the half-line $\mathbb{N}$ with respect to drift q and entry and exit rates α and γ, respectively. The process $(\xi_t^{\mathsf{AB}})_{t\geq 0}$ has the law of a disagreement process between $(\eta_t^{\mathsf{A}})_{t\geq 0}$ and $(\eta_t^{\mathsf{B}})_{t\geq 0}$, with the convention that sites on $(-\infty, 0]$ in the ASEP on the integers are always either A or 0 (since $(\eta_t^{\mathsf{B}})_{t\geq 0}$ is only defined on $\{0,1\}^{\mathbb{N}}$.)*

Proof. This is immediate from verifying the transition rates in the construction of the extended disagreement process. □

Furthermore, we define the two projections $(\xi^{\mathsf{A}}_t)_{t\geq 0}$ and $(\xi^{\mathsf{B}}_t)_{t\geq 0}$ as follows. We obtain $(\xi^{\mathsf{A}}_t)_{t\geq 0}$ by setting

$$\xi^{\mathsf{A}}_t(x) = \begin{cases} \mathbf{1} & \text{if } \xi^{\text{mod}}_t(x) = \mathbf{1} \\ \mathsf{A} & \text{if } \xi^{\text{mod}}_t(x) \in \{\mathsf{A}', \mathsf{A}\} \\ \mathbf{0} & \text{if } \xi^{\text{mod}}_t(x) \in \{\mathbf{0}, \mathsf{B}', \mathsf{B}\} \end{cases} \tag{4.5}$$

for all $t \geq 0$ and $x \in \mathbb{Z}$. Intuitively, $(\xi^{\mathsf{A}}_t)_{t\geq 0}$ acts on sites ≥ 2 like a multi-species ASEP, but where some of the first class particles in $(\eta^{\mathsf{A}}_t)_{t\geq 0}$ are turned into second class particles. Similarly, let

$$\xi^{\mathsf{B}}_t(x) = \begin{cases} \mathbf{1} & \text{if } \xi^{\text{mod}}_t(x) \in \{\mathbf{1}, \mathsf{A}', \mathsf{A}\} \\ \mathsf{B} & \text{if } \xi^{\text{mod}}_t(x) \in \{\mathsf{B}', \mathsf{B}\} \\ \mathbf{0} & \text{if } \xi^{\text{mod}}_t(x) = \mathbf{0} \end{cases} \tag{4.6}$$

for all $t \geq 0$ and $x \in \mathbb{Z}$, and note that $(\xi^{\mathsf{B}}_t)_{t\geq 0}$ has the same law as $(\eta^{\mathsf{A}}_t)_{t\geq 0}$, but where some of the empty sites in $(\eta^{\mathsf{A}}_t)_{t\geq 0}$ are turned into second class particles. Let us stress that in all of the above processes, second class particles can enter and exit only at site 1.

Remark **4.3.** *Note that instead of* (4.2)*, we can also define the ordering* $\succeq_{\text{r}}$ *given by*

$$\mathbf{1} \succeq_{\text{r}} \mathsf{B}' \succeq_{\text{r}} \mathsf{B} \succeq_{\text{r}} \mathsf{A} \succeq_{\text{r}} \mathsf{A}' \succeq_{\text{r}} \mathbf{0}. \tag{4.7}$$

where we reverse the priority of the second class particles of types $\mathsf{A}', \mathsf{A}, \mathsf{B}$ *and* B'*. The extended disagreement process is defined accordingly, that is, we have the same transition rules as in a disagreement process between an ASEP on the integers and an ASEP on the half-line. However, every adjacent pair of* A *and* B *is replaced by* $(\mathsf{B}', \mathsf{A}')$ *at rate* 1 *and* $(\mathsf{A}', \mathsf{B}')$ *at rate* q*.*

As we will see in Sections 4.4 to 4.6, this interpretation of the extended disagreement process as a multi-type exclusion process (with a marking procedure and special rules along the edge $\{0, 1\}$ and at site 1) allows us to study the position of type A and B second class particles in the vein of [12] for the fluctuations of a second class particle in the ASEP on the integers.

4.3. **Moderate deviations for second class particles.** In this section, we establish moderate deviation results for second class particles in the ASEP on the integers. Recall the multi-species $(\zeta_t)_{t\geq 0}$ defined in Section 2.1. We will restrict ourselves to three particle types, $1, 2$ and ∞, to which we refer as first class particles, second class particles, and empty sites, respectively. Consider the $(\zeta_t)_{t\geq 0}$ with initial configuration ζ such that ζ contains a single second class particle at the origin and is chosen according to a Bernoulli-ρ-product measure for some $\rho \in (0, 1)$ on all other sites. We denote by $(Z_t)_{t\geq 0}$ the location of the unique second class particle in $(\zeta_t)_{t\geq 0}$. We set in the following

$$\mathcal{I}_{q,\rho}(t, y, z) := [z + (1-q)(1-2\rho)t - y - 1, z + (1-q)(1-2\rho)t + y + 1]\,, \tag{4.8}$$

where $(1-q)(1-2\rho)$ is the stationary speed of a second class particle, i.e.,

$$\mathbb{E}[Z_t] = t(1-q)(1-2\rho)$$

for all $t \geq 0$; see Theorem 2.8 in Part III of [61]. The following moderate deviation result for $(Z_t)_{t\geq 0}$ is due to Landon and Sosoe [57].

Theorem 4.4 (c.f. Theorem 2.5 in [57]). *Suppose that there exists some $\mathfrak{a} \in (0,1)$ such that $\rho \in [\mathfrak{a}, 1-\mathfrak{a}]$. Then there exist constants c_0, C_0, depending only on $\mathfrak{a}$, such that for all $q \in (0,1)$, and all $1 \leq w \leq (1-q)s^{1/3}$*

$$\mathbb{P}\big(Z_s \notin \mathcal{I}_{q,\rho}(s, ws^{2/3}, 0)\big) \leq C_0 \exp(-c_0 w^3). \tag{4.9}$$

Remark 4.5. *As in Remark 2.13, let us stress that the dependence of the constants c_0, C_0 on $\mathfrak{a}$ can be found as Proposition 5.3 of [57] for the stochastic six vertex model, which transfers to the ASEP using the results by Aggarwal from [1]; see also Section 7 in [57].*

In the following, we prove a moderate deviation result for the maximal displacement of a second class particle in the ASEP on the integers. This will be further extended to a bound on the maximal displacement of multiple second class particles. The next lemma is similar to Theorem 4.7 in [77], which states moderate deviations for the maximal displacement of a single second class particles when $w \geq \log(s)$ above. However, we require refined bounds, which we obtain by a multi-scale argument similar to the chaining argument of Theorem 11.1 in [17] for the transversal fluctuations of geodesics in last passage percolation.

Lemma 4.6. *For $N \in \mathbb{N}$, and $n = n(N) \in [\![\frac{1}{2}\log_2(N)]\!]$, we let*

$$T = \theta^{-1} 2^n N (1-q)^{-1} \tag{4.10}$$

for some $\theta \geq 1$, allowed to depend on N. Set $\rho_N = \frac{1}{2} + 2^{-n}$. Let $(Z_t)_{t\geq 0}$ denote the position of a second class particle started from the origin in an ASEP on the integers $(\zeta_t)_{t\geq 0}$ in a Bernoulli-ρ_N-product measure. Then there exist constants $c_0, C_0, c_1 > 0$, such that for all q from (1.5) with $\kappa \in [0, \frac{1}{2}]$ and some $\psi > 0$, and all y with

$$1 \leq y \leq c_1 \theta^{-1/3} N^{(1-2\kappa)/3} \log^{-1}(N) 2^{n/3}, \tag{4.11}$$

we get that for all N large enough

$$\mathbb{P}\big(\exists s \in [0,T] \colon Z_s \notin \mathcal{I}_{q,\rho_N}(s, y(\theta^{-1} N 2^n)^{2/3}, 0)\big) \leq C_0 \exp(-c_0 y^3). \tag{4.12}$$

Moreover, when $\rho_N = \frac{1}{2}$, we obtain with $T = \theta^{-1} N^{3/2} (1-q)^{-1}$ that

$$\mathbb{P}\big(\exists s \in [0,T] \colon Z_s \notin \mathcal{I}_{q,\rho_N}(s, y\theta^{-2/3} N, 0)\big) \leq C_0 \exp(-c_0 y^3) \tag{4.13}$$

for all $1 \leq y \leq c_1 \theta^{-1/3} N^{1/2 - 2\kappa/3} \log^{-1}(N)$, and all N large enough.

Proof. To simplify notation, we will in the following write

$$\mathcal{I}(y,k) := \mathcal{I}_{q,\rho_N}(k 2^{-i}, y(\theta^{-1} N 2^n)^{2/3}, 0), \tag{4.14}$$

and consider only the case (4.12) as the same arguments as for $n = \lfloor \frac{1}{2}\log_2(N) \rfloor$ will give (4.13). We define $(h_i)_{i\in\mathbb{N}}$ as

$$h_i := \frac{1}{2}\Big(\prod_{m=1}^{\infty}(1+2^{-m/4})\Big)^{-1} \prod_{j=1}^{i}(1+2^{-j/4}),$$

where we note that $h_1 > 0$ as well as $h_i < \frac{1}{2}$ for all $i \in \mathbb{N}$. For all $i \in \mathbb{N}$, we define the events

$$\mathcal{A}_i := \left\{ Z_{k2^{-i}T} \in \mathcal{I}(h_i y, k) \text{ for all } k \in [\![2^i]\!] \right\}.$$

We will in the following argue that there exist some constants $c_2, C_2 > 0$ such that for all $i = \mathcal{O}(\log_2 \log(T))$, we get that

$$\mathbb{P}(\mathcal{A}_{i+1} \,|\, \mathcal{A}_i) \geq 1 - C_2 \exp(-c_2 y^3 2^{i/4}). \tag{4.15}$$

Since we get from Theorem 4.4 that for some constants $c_3, C_3 > 0$,

$$\mathbb{P}(\mathcal{A}_1) \geq 1 - C_3 \exp(-c_3 y^3),$$

this yields that by choosing $c_1 > 0$ in (4.11) small enough,

$$\mathbb{P}(\mathcal{A}_i \text{ for all } i \leq \log_2 \log(T)) \geq 1 - C_4 \exp(-c_4 y^3) \tag{4.16}$$

for some $c_4, C_4 > 0$ and all y from (4.11). In order to show (4.15), we define the sets

$$\mathcal{I}^+_{k,i} := \mathcal{I}_{q,\rho_N}\left(k2^{-i}T, h_i y(\theta^{-1}N2^n)^{2/3}, \tfrac{1}{4}y^3 2^{i/5}\right) \setminus \mathcal{I}_{q,\rho_N}(k2^{-i}T, h_i y(\theta^{-1}N2^n)^{2/3}, 0)$$
$$\mathcal{I}^-_{k,i} := \mathcal{I}_{q,\rho_N}\left(k2^{-i}T, h_i y(\theta^{-1}N2^n)^{2/3}, -\tfrac{1}{4}y^3 2^{i/5}\right) \setminus \mathcal{I}_{q,\rho_N}(k2^{-i}T, h_i y(\theta^{-1}N2^n)^{2/3}, 0),$$

and consider the events

$$\mathcal{B}^{k,+}_i := \left\{ \eta_{k2^{-i}T}(v^1_+) = 1 \text{ and } \eta_{k2^{-i}T}(v^0_+) = 0 \text{ for some } v^1_+, v^0_+ \in \mathcal{I}^+_{k,i} \right\}$$
$$\mathcal{B}^{k,-}_i := \left\{ \eta_{k2^{-i}T}(v^1_-) = 1 \text{ and } \eta_{k2^{-i}T}(v^0_-) = 0 \text{ for some } v^1_-, v^0_- \in \mathcal{I}^-_{k,i} \right\}.$$

We set in the following

$$\mathcal{B}_i := \bigcap_{k=1}^{2^i - 1} \left(\mathcal{B}^{k,+}_i \cap \mathcal{B}^{k,-}_i \right)$$

and note that for some constants $c_5, C_5 > 0$,

$$\mathbb{P}(\mathcal{B}_i) \geq 1 - C_5 \exp(-c_5 y^3 2^{i/5}). \tag{4.17}$$

This follows from the observation that the ASEP at time $k2^{-i}T$ has a Bernoulli-ρ_N-product law on $\mathcal{I}^+_{k,i}$ and $\mathcal{I}^-_{k,i}$, together with a union bound over $2^i - 1$ many events. Fix $i \in \mathbb{N}$ and $k \in [\![2^i - 1]\!]$. Then on the event $\mathcal{B}^{k,-}_i \cap \mathcal{B}^{k,+}_i \cap \mathcal{A}_i$, we consider four ASEPs on the integers, started from $\eta_{k2^{-i}T}$, where we place second class particles at sites $v^1_+, v^0_+, v^1_-, v^0_-$, respectively, and where we replace $\eta_{k2^{-i}T}(Z_{k2^{-i}T})$ by an independent Bernoulli-ρ_N-distributed random variable (the same for all four processes). Let $Z^{1,+}_t, Z^{0,+}_t, Z^{1,-}_t, Z^{0,-}_t$ denote the positions of the second class particles at time t in the respective exclusion processes. Observe that under the basic coupling for all five processes, we get that

$$\mathbf{P}\Big(\min(Z^{1,-}_t, Z^{0,-}_t) \leq Z_{k2^{-i}T+t} \leq \max(Z^{1,+}_t, Z^{0,+}_t) \,\Big|\, \mathcal{A}_i \Big) = 1. \tag{4.18}$$

To see this, distinguish whether the second class particle in $\eta_{k2^{-i}T}$ is replaced by a first class particle or an empty site. In the first case, observe that $(Z_{t+k2^{-i}T})_{t\geq 0}$ is dominated by $(Z^{1,+}_t)_{t\geq 0}$ and $(Z^{1,-}_t)_{t\geq 0}$, in the second case it is dominated by $(Z^{0,+}_t)_{t\geq 0}$ and $(Z^{0,-}_t)_{t\geq 0}$, respectively. For i and $k \in [\![2^i - 1]\!]$, we define the events $\mathcal{C}^k_i$ as

$$\mathcal{C}^k_i := \left\{ Z^{1,+}_{2^{-i}T}, Z^{0,+}_{2^{-i}T}, Z^{1,-}_{2^{-i}T}, Z^{0,-}_{2^{-i}T} \in \mathcal{I}(h_{i+1} y, k) \right\}. \tag{4.19}$$

We only show that for some constants $c_6, C_6 > 0$

$$\mathbb{P}\Big(Z^{1,+}_{2^{-i}T} \in \mathcal{I}(h_{i+1}y,k) \,\Big|\, \mathcal{A}_i, \mathcal{B}_i \Big) \geq 1 - C_6 \exp(-c_6 y^3 2^{i/5}). \tag{4.20}$$

A similar argument applies by symmetry for the other three processes in the events $\mathcal{C}^k_i$. Note that the law of $\eta_{k2^{-i}T}$ around the particle $Z^{1,+}_0$ differs by construction from a Bernoulli-ρ_N-product measure in at most $\frac{1}{4}y^3 2^{i/5}$ many sites (corresponding to the sites inspected to the right of $Z^{1,+}_0$). Since $\rho_N \in [\mathfrak{a}, 1-\mathfrak{a}]$, by a change of measure to an ASEP on the integers in a Bernoulli-ρ_N-product measure with a single second class particle initially at a fixed site, we see that for some constants $C_7, c_8, C_8 > 0$, depending only on $\mathfrak{a}$, and all y from

$$\begin{aligned}
&\sup_{x \in \mathcal{I}^+_{k,i}} \mathbb{P}\Big(Z^{1,+}_{k2^{-i}T} \notin \mathcal{I}(h_{i+1}y,k) \,\Big|\, Z^{1,+}_0 = x \Big) \\
&\leq \exp(C_7 y^3 2^{i/5}) \sup_{x \in \mathcal{I}^+_{k,i}} \mathbb{P}\Big(Z_{k2^{-i}T} \notin \mathcal{I}(h_{i+1}y,k) \,\Big|\, Z_0 = x \Big) \\
&\leq C_8 \exp(-c_8 y^3 2^{i/4}).
\end{aligned}$$

Here, we applied Theorem 4.4 with $s = 2^{-i}T$ and $w = y2^{i/12}$ for the last step, using that $h_{i+1} - h_i = 2^{-i/4}$ for all $i \in \mathbb{N}$. This gives (4.20) for all four processes in $\mathcal{C}^k_i$, which by the virtue of (4.17) and (4.18) implies (4.15). Next, we define the event

$$\tilde{\mathcal{A}} := \Big\{ Z_s \notin \mathcal{I}_{q,\rho_N}(s, y(\theta^{-1}N2^n)^{2/3}, 0)\big) \text{ for all } s \in [0,T] \Big\}.$$

Condition on the events $\mathcal{A}_i$ for all $i < j = \lceil \log_2(\log(T)) \rceil$ and $\mathcal{B}_j$. Then there exist some constants $C_9, c_{10}, C_{10}, c_{11}, C_{11} > 0$, such that by a change of measure and a union bound for the first inequality, and Theorem 4.7 in [77] for the second inequality (which states Lemma 4.6 for $y \geq \log(N)$),

$$\begin{aligned}
\mathbb{P}\big(\tilde{\mathcal{A}} \mid \cap_{i=1}^{j-1} \mathcal{A}_j, \mathcal{B}_{j+1}\big) &\geq 1 - 2^{j-1}\exp(C_9 y^3 2^{j/5}) \mathbb{P}\big(\exists s \in [0, 2^{-j}T] \colon Z_s \notin \mathcal{I}(y/2, 0) \,\big|\, Z_0 = 0\big) \\
&\geq 1 - C_{10} \exp(C_9 y^3 2^{j/5}) 2^j \exp(-c_{10} y^3 2^{3j}) \geq 1 - C_{11} \exp(-c_{11} y^3),
\end{aligned}$$

for all y from (4.11) and N large enough. Together with (4.16), this finishes the proof. ☐

Next, we recall the microscopic concavity coupling of two ASEPs with second class particles, introduced by Balázs and Seppäläinen in [11], which allows us to compare the location of second class particles for different initial conditions.

Theorem 4.7 (c.f. Theorem 3.1 in [11]). *Let $(\zeta^1_t)_{t\geq 0}$ and $(\zeta^2_t)_{t \geq 0}$ be two ASEPs on the integers with initial configurations ζ^1 and ζ^2 such that*

$$\zeta^1(x) \geq \zeta^2(x) \tag{4.21}$$

with $\zeta^1(x), \zeta^2(x) \in \{0,1\}$ for all $x \neq 0$, and $\zeta^1(0) = \zeta^2(0) = 2$. Let $(Z^1_t)_{t\geq 0}$ and $(Z^2_t)_{t \geq 0}$ be the positions of the respective second class particles. There exists a coupling $\bar{\mathbf{P}}$ such that

$$\bar{\mathbf{P}}\left(Z^2_t \geq Z^1_t \text{ for all } t \geq 0 \,\middle|\, \zeta^1_0 = \zeta^1 \text{ and } \zeta^2_0 = \zeta^2 \right) = 1. \tag{4.22}$$

Our goal is to convert moderate deviations on the maximal displacement of a single second class particle to the maximal displacement of order N^κ many second class particles. To achieve this, the second class particles have to be initially place sufficiently distant from each other. To do so, recall q from (1.5) with some $\kappa \in [0, \frac{1}{2}]$, and let $\rho_N = \frac{1}{2} + 2^{-n}$ for some $n \in [\![\frac{1}{2}\log_2(N)]\!]$ or $\rho_N = \frac{1}{2}$. In the following, we set

$$\bar{\kappa} = \frac{1}{4}(1 - 2\kappa) \geq 0 \quad \text{and} \quad \kappa' = \frac{\bar{\kappa}}{3} \geq 0. \tag{4.23}$$

Note that $\kappa + \bar{\kappa} \leq \frac{1}{2}$. We define a multi-species ASEP on the integers $(\tilde{\zeta}_t)_{t\geq 0}$, where the initial condition is chosen according to the product measure $\tilde{\pi}_N$ with marginals

$$\tilde{\pi}_N(x) = \begin{cases} 1 & \text{with probability } \rho_N - N^{-1/2} \\ 2 & \text{with probability } N^{-1/2} \\ \infty & \text{with probability } 1 - \rho_N \end{cases} \tag{4.24}$$

for all $x \in [\![N^{\kappa+\bar{\kappa}+1/2}]\!]$, and according to

$$\tilde{\pi}_N(x) = \begin{cases} 1 & \text{with probability } \rho_N \\ \infty & \text{with probability } 1 - \rho_N \end{cases} \tag{4.25}$$

for all $x \notin [\![N^{\kappa+\bar{\kappa}+1/2}]\!]$. Let $(L_t)_{t\geq 0}$ and $(R_t)_{t\geq 0}$ denote the position of the left-most and right-most second class particle in the multi-species ASEP $(\tilde{\zeta}_t)_{t\geq 0}$ started from $\tilde{\pi}_N$. We have the following result on the location of second class particles in $(\tilde{\zeta}_t)_{t\geq 0}$ when $\kappa < \frac{1}{2}$.

Lemma 4.8. *Let $\kappa < \frac{1}{2}$, $T = \theta^{-1}N2^n(1-q)^{-1}$ and $\rho_N = \frac{1}{2} + 2^{-n}$ for some $n \in [\![\frac{1}{2}\log_2(N)]\!]$ and $\theta \geq 1$. There exist $c_0, C_0 > 0$ such that for all $1 \leq y, \theta \leq N^{\kappa'}$ with κ' from (4.23)*

$$\mathbb{P}_{\tilde{\pi}_N}\Big(L_s, R_s \in \mathcal{I}_{q,\rho_N}(s, y(\theta^{-1}N2^n)^{2/3} + N^{1-\kappa'}, 0)\, \forall s \in [0,T]\Big) \geq 1 - C_0\exp(-c_0 y^3) \tag{4.26}$$

for all N sufficiently large. Similarly, for $\rho_N = \frac{1}{2}$ and $T = \theta^{-1}N^{3/2}(1-q)^{-1}$, we get that

$$\mathbb{P}_{\tilde{\pi}_N}\Big(L_s, R_s \in \mathcal{I}_{q,\rho_N}(s, y\theta^{-2/3}N + N^{1-\kappa'}, 0)\, \forall s \in [0,T]\Big) \geq 1 - C_0\exp(-c_0 y^3). \tag{4.27}$$

Let us stress that κ' is not optimal, but sufficient for our purposes. Intuitively, Lemma 4.8 ensures that all second class particles in $(\tilde{\zeta}_t)_{t\geq 0}$ started according to $\tilde{\pi}_N$ satisfy similar moderate deviation bounds as a single second class particle. Before giving the proof of Lemma 4.8, we provide a corresponding statement when $\kappa = \frac{1}{2}$. We distinguish two cases, depending on the value of n. For all n such that $2^n \leq N^{1/4}$, we let the initial distribution $\tilde{\pi}_N$ of $(\tilde{\zeta}_t)_{t\geq 0}$ be defined as the measure $\tilde{\pi}_N$ in (4.24) and (4.25), but with respect to the interval

$$\mathcal{I}_{N,n} := [\![\log^2(N)\sqrt{N}2^n]\!].$$

For all n with $2^n \geq N^{1/4}$, and some $a > 0$ fixed (specified later on), we let $\tilde{\pi}_N = \tilde{\pi}_{N,n,a}$ be

$$\tilde{\pi}_N(x) = \begin{cases} 1 & \text{with probability } \rho_N - 2^{-n}\log(N)a^{-2} \\ 2 & \text{with probability } 2^{-n}\log(N)a^{-2} \\ \infty & \text{with probability } 1 - \rho_N \end{cases} \tag{4.28}$$

for all $x \in [\![aN]\!]$ as well as

$$\tilde{\pi}_N(x) = \begin{cases} 1 & \text{with probability } \rho_N \\ \infty & \text{with probability } 1 - \rho_N \end{cases} \tag{4.29}$$

for all $x \notin [\![aN]\!]$. Again, for $\rho_N = \frac{1}{2}$, we assign the same initial distribution $\tilde{\pi}_N = \tilde{\pi}_{N,n,a}$ as in the case $n = \lceil \frac{1}{2}\log_2(N) \rceil$. The next lemma is the analogue of Lemma 4.8 when $\kappa = \frac{1}{2}$.

Lemma 4.9. *Let $\kappa = \frac{1}{2}$. Let $T' = \theta^{-1} N^{3/2} 2^n$ and $\rho_N = \frac{1}{2} + 2^{-n}$ for some $n \in [\![\frac{1}{2}\log_2(N)]\!]$ and $\theta \geq 1$ and assume that $c_1^{-1}\log(N) \leq 2^n \leq N^{1/4}$ for the constant $c_1 > 0$ from Lemma 4.6. Then for every $\phi > 0$, there exists some constant $\theta = \theta(\phi)$ such that*

$$\mathbb{P}\big(L_s, R_s \in [-\phi N, 2\phi N] \text{ for all } s \in [0,T']\big) \geq 1 - N^{-20} \tag{4.30}$$

holds for all N large enough. Assume that $2^n > N^{1/4}$. Then for all $\phi > 0$ and $a > 0$ in the definition of $\tilde{\pi}_N$, we find some $c_0 > 0$ such that (4.30) holds for

$$T' = \frac{c_0}{\log(N)} N^{3/2} 2^n. \tag{4.31}$$

Similarly, when $\rho_N = \frac{1}{2}$, we find some $c_0' > 0$ such that (4.30) holds with respect to

$$T' = \frac{c_0'}{\log(N)} N^2. \tag{4.32}$$

We start by proving Lemma 4.8 on the maximal displacement of the second class particle with $\kappa < \frac{1}{2}$, starting according to $\tilde{\pi}_N$. The proof of Lemma 4.9 will be similar, and we will only provide the required adjustments in the proof of Lemma 4.8 instead of full details.

Proof of Lemma 4.8. For (4.26), we will only prove that there exist $c_0, C_0 > 0$ such that

$$\mathbb{P}(L_s > (1-q)(1-2\rho_N)T - yN^{1-\kappa'} \text{ for all } s \in [0,T]) \geq 1 - C_0 \exp(-c_0 y^3) \tag{4.33}$$

for all $1 \leq y, \theta \leq N^{\kappa'}$. The upper bound on $(R_t)_{t\geq 0}$ follows analogously. For $\tilde{\zeta}_0 \sim \tilde{\pi}_N$, we define a multi-species configuration $\bar{\zeta}$ by flipping every empty site in $\tilde{\zeta}_0$ within the interval $[-N^{\kappa+\bar{\kappa}+1/2}, 0]$ with probability $N^{-1/2}$ into a type 3 particle. Moreover, we replace the rightmost particle, which is located to the left of $-N^{\kappa+\bar{\kappa}+1/2}$, by a type 4 particle. Let $(\bar{\zeta}_t)_{t\geq 0}$ denote the corresponding multi-species ASEP. Note that every site on $[-N^{\kappa+\bar{\kappa}+1/2}, 0]$ is with probability at least $N^{1/2}/4$ occupied by a type 3 particle. We enumerate the locations of the type 2 particles in $\bar{\zeta}$ from left to right as $(X_t^1)_{t\geq 0}, (X_t^2)_{t\geq 0}, \ldots$, and the locations of the type 3 particles from left to right as $(Y_t^1)_{t\geq 0}, (Y_t^2)_{t\geq 0}, \ldots$. The location of the type 4 particle is denoted by $(Z_t)_{t\geq 0}$. A visualization is given in Step 0 of Figure 5. We define

$$\begin{aligned} \mathcal{A}_1 &:= \Big\{ \sum_{x\in\mathbb{Z}} \mathbb{1}_{\{\bar{\zeta}_0(x)=3\}} \geq \frac{1}{5} N^{\kappa+\bar{\kappa}} \Big\} \\ \mathcal{A}_2 &:= \Big\{ Y_t^{\frac{1}{10}N^{\kappa+\bar{\kappa}}} < X_t^1 \text{ for all } 0 \leq t \leq N^3 \Big\} \\ \mathcal{A}_3 &:= \Big\{ Z_t < Y_t^{\frac{1}{10}N^{\kappa+\bar{\kappa}}} \text{ for all } 0 \leq t \leq N^3 \Big\} \end{aligned} \tag{4.34}$$

on the number and location of the particles of types 2, 3 and 4 in the multi-species ASEP $(\bar{\zeta}_t)_{t\geq 0}$ on the integers. We will now argue that for our choice of $\bar{\kappa}$ and κ' in (4.23),

$$\mathbb{P}(\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3) \geq 1 - C_3 \exp(-c_3 N^{3\kappa'})$$

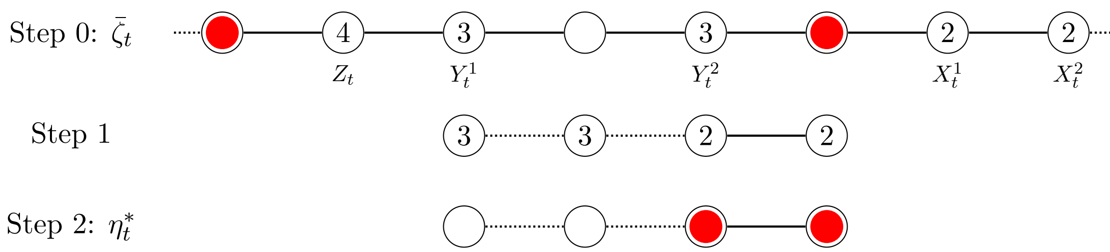

FIGURE 5. Construction of a configuration η_t^* from $\bar{\zeta}_t$ for some $t \geq 0$ and $M_2 = M_3 = 2$. First class particles are depicted in red, while second class particles are drawn with their types. In Step 1, we erase all sites apart from those which contain second class particles of types 2 and 3. In Step 2, we convert the second class particles of types 2 and 3 to first class particles and empty sites, respectively. Censored edges are drawn dashed.

for some constants $c_3, C_3 > 0$ and all N large enough. This follows by showing that there exist some constants $c_4, C_4 > 0$ such that

$$\mathbb{P}(\mathcal{A}_i) \geq 1 - C_4 \exp(-c_4 N^{3\kappa'}). \tag{4.35}$$

for all $i \in [\![3]\!]$, and all N large enough. For $i = 1$, this follows by construction and a standard tail bound for the sum of independent Bernoulli random variables. For the lower bound on the events $\mathcal{A}_2$ and $\mathcal{A}_3$, we will in the following only consider $\mathcal{A}_2$ as the argument for $\mathcal{A}_3$ is similar (projecting to type 3 and type 4 particles in the following). Let M_2 and M_3 denote the number of type 2 and type 3 particles in $(\bar{\zeta}_t)_{t\geq 0}$, respectively. Then for every $t \geq 0$, we define a configuration $\eta_t^* \in \Omega_{M_2+M_3,M_2}$ with

$$\Omega_{m,k} := \Big\{ \eta \in \{0,1\}^m : \sum_{i=1}^{m} \eta(i) = k \Big\} \tag{4.36}$$

for all $m \in \mathbb{N}$ and $k \in [\![m]\!]$, by first deleting all particles and empty sites in $\bar{\zeta}_t$ (as well as removing their sites and merging the resulting edges), which are not of type 2 or 3. We then map all type 3 particles to empty sites and type 2 particles to first particles to obtain a configuration $\eta_t^* \in \Omega_{M_2+M_3,M_2}$. Observe that the resulting process $(\eta_t^*)_{t\geq 0}$ can be interpreted as an asymmetric simple exclusion process on $\Omega_{M_2+M_3,M_2}$ with censoring, i.e., an edge $\{x, x+1\}$ is contained in the censoring scheme at time t if and only if in the construction of $\bar{\zeta}_t$ from η_t^*, we erased a particles (and its site) between the two sites which get mapped to x and $x+1$, respectively. A visualization of this projection is provided in Figure 5. The bound on $\mathcal{A}_2$ in (4.35) is now immediate from Lemma 2.9 and Remark 2.10 applied to the process $(\eta_t^*)_{t\geq 0}$. Observe that on the event $\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3$,

$$\mathbb{P}\big(Z_s \leq L_s \text{ for all } s \in [0, N^3] \,\big|\, \mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3\big) = 1.$$

By Theorem 4.7, $(Z_t)_{t\geq 0}$ dominates the location of a second class particle $(Z_t')_{t\geq 0}$ started from Z_0 under a Bernoulli-$(\rho_N - N^{-1/2})$-product measure. Hence, the desired lower bound (4.33) on the maximal displacement of the left-most second class particle in $(\tilde{\zeta}_t)_{t\geq 0}$ follows from Lemma 4.6 for $(Z_t')_{t\geq 0}$ and all $1 \leq y, \theta \leq N^{\kappa'}$. Note that the same arguments also apply for the case $\rho_N = \frac{1}{2}$, which yields (4.27). $\square$

Next, we describe the necessary changes in the proof of Lemma 4.8 in order to obtain Lemma 4.9 on the maximal displacement of second class particles when $\kappa = \frac{1}{2}$.

Proof of Lemma 4.9. As for Lemma 4.8, we only argue that there exists some θ such that

$$\mathbb{P}(L_s > -\phi N \text{ for all } s \in [0, T']) \geq 1 - N^{-20}/2 \tag{4.37}$$

for all N large enough and $T' = T'(N, \theta)$ as the corresponding bound on $(R_t)_{t\geq 0}$ is similar. We start with the case where $n \leq \frac{1}{4}\log_2(N)$ and $c_1^{-1}\log(N) \leq 2^n$. We define a multi-species ASEP on the integers with initial configuration $\bar{\zeta}$, which we obtain from $\tilde{\zeta} \sim \tilde{\pi}_N$ by converting every particle on the interval $[-\log^2(N)\sqrt{N}2^n, 0]$ independently with probability $N^{-1/2}$ into a type 3 particle, and replacing the rightmost particle located to the left of $-\log^2(N)\sqrt{N}2^n$ by a type 4 particle. Enumerating the particles as in Lemma 4.8, let

$$\begin{aligned}
\mathcal{A}_1' &:= \Big\{ \sum_{x\in\mathbb{Z}} \mathbb{1}_{\{\bar{\zeta}_0(x)=3\}} \geq \frac{1}{5} N^{1/2}\log^2(N) \Big\} \\
\mathcal{A}_2' &:= \Big\{ Y_t^{\frac{1}{10}N^{1/2}\log^2(N)} < X_t^1 \text{ for all } t \in [0, N^3] \Big\} \\
\mathcal{A}_3' &:= \Big\{ Z_t < Y_t^{\frac{1}{10}N^{1/2}\log^2(N)} \text{ for all } t \in [0, N^3] \Big\}.
\end{aligned} \tag{4.38}$$

The remainder follows from the same arguments as Lemma 4.8, showing that the event $\mathcal{A}_1' \cap \mathcal{A}_2' \cap \mathcal{A}_3'$ holds with probability at least $1 - N^{-20}/3$ for all N large enough by virtue of Lemma 2.9 and Remark 2.10 for a suitable projection of type 2 and type 3 particles for the event $\mathcal{A}_2'$, as well as of type 3 and type 4 particles for the event $\mathcal{A}_3'$ to an ASEP on an interval, respectively. On the event $\mathcal{A}_1' \cap \mathcal{A}_2' \cap \mathcal{A}_3'$, we see that

$$\mathbb{P}\big(Z_s \leq L_s \text{ for all } s \in [0, N^3] \,\big|\, \mathcal{A}_1' \cap \mathcal{A}_2' \cap \mathcal{A}_3'\big) = 1,$$

allowing us to conclude using the microscopic concavity coupling from Theorem 4.7 and Lemma 4.6 for moderate deviations on the maximal displacement of a single second class particle. For the case $n \geq \frac{1}{4}\log_2(N)$, we apply the same arguments, placing type 3 particles on the particles in $[-aN/2, 0]$ independently with probability $2^{-n}\log(N)a^{-2}$, and replacing $N^{1/2}\log^2(N)$ above by $a^{-2}N^{1/2}\log(N)$ for a in the definition of $\tilde{\pi}_N$ chosen small enough so that $\mathcal{A}_1' \cap \mathcal{A}_2' \cap \mathcal{A}_3'$ holds with probability at least $1 - N^{-20}/3$ for all N large enough. The remaining case $\rho_N = \frac{1}{2}$ is treated as the case $n = \lfloor \frac{1}{2}\log_2(N) \rfloor$. □

4.4. **Moderate deviations for the extended disagreement process.** Next, our goal is to compare the number of type A and B second class particles in the disagreement process $(\xi_t)_{t\geq 0}$ between an ASEP on the integers $(\eta_t)_{t\geq 0}$ and on the half-line $(\eta_t^{\mathbb{N}})_{t\geq 0}$. Recall that ρ_{L} denotes the effective density of the reservoir of the half-line ASEP. We assume in the following that both processes are started from the same initial configuration on sites $\mathbb{N}$, i.e.,

$$\eta_0(x) = \eta_0^{\mathbb{N}}(x) \tag{4.39}$$

for all $x \in \mathbb{N}$ according to a Bernoulli-ρ_N-product measure for $\rho = \rho_N \in (0,1)$, and coupled according to the extended disagreement process $(\xi_t^{\text{mod}})_{t\geq 0}$ from Definition 4.1. Let $(Z_t^{\mathsf{A}})_{t\geq 0}$ and $(Z_t^{\mathsf{B}})_{t\geq 0}$ denote the position of the rightmost particle of type A or A', respectively of type B or B' in $(\xi_t^{\text{mod}})_{t\geq 0}$. The following proposition covers the speed of propagation of second class particles in the extended disagreement process.

Proposition 4.10. *Let $N \in \mathbb{N}$ and consider q from (1.5) with $\kappa < \frac{1}{2}$. Let $\rho_N = \frac{1}{2} + 2^{-n}$ for some $n \in [\![\frac{1}{2}\log_2(N)]\!]$, and recall $\kappa' > 0$ from (4.23). Then there exists some constants $c_0, C_0 > 0$ such that for all $1 \leq y, \theta \leq N^{\kappa'}$, and all N large enough*

$$\mathbb{P}\left(\sup_{t \in [0, \theta^{-1} N^{1+\kappa} 2^n]} \max(Z_t^{\mathsf{A}}, Z_t^{\mathsf{B}}) \geq \theta^{-1} y N \right) \leq C_0 \exp(-c_0 y^3). \tag{4.40}$$

Similarly, when $\rho_N = \frac{1}{2}$, we get that for all N large enough

$$\mathbb{P}\left(\sup_{t \in [0, \theta^{-1} N^{3/2+\kappa}]} \max(Z_t^{\mathsf{A}}, Z_t^{\mathsf{B}}) \geq \theta^{-1} y N \right) \leq C_0 \exp(-c_0 y^3). \tag{4.41}$$

Let $\kappa = \frac{1}{2}$, and $\rho_N = \frac{1}{2} + 2^{-n}$ for some $n \in [\![\log_2(c_1^{-1}\log(N)), \frac{1}{2}\log_2(N)]\!]$ for $c_1 > 0$ from Lemma 4.6. For every $\phi > 0$, we find some $z > 0$ such that for all N large enough

$$\mathbb{P}\left(\sup_{t \in [0, z N^{1+\kappa} 2^n \log^{-1}(N)]} \max(Z_t^{\mathsf{A}}, Z_t^{\mathsf{B}}) \geq \phi N \right) \leq \frac{1}{N^{12}}. \tag{4.42}$$

Similarly, when $\kappa = \frac{1}{2}$ and $\rho_N = \frac{1}{2}$, we get that for some $z > 0$, and N large enough

$$\mathbb{P}\left(\sup_{t \in [0, z N^{3/2+\kappa} \log^{-1}(N)]} \max(Z_t^{\mathsf{A}}, Z_t^{\mathsf{B}}) \geq \phi N \right) \leq \frac{1}{N^{12}}. \tag{4.43}$$

In words, Proposition 4.10 states that the speed of propagation of a perturbation introduced at the boundary in the extended disagreement process has at most the speed of a second class particle on the integers (with a logarithmic correction for the case $\kappa = \frac{1}{2}$).

In order to show Proposition 4.10, we require some setup. Recall the projection $(\xi_t^{\mathsf{B}})_{t \geq 0}$ from (4.6) for the extended disagreement process, and let $(U_t^{\mathsf{B}})_{t \geq 0}$ denote the position of the rightmost type B second class particle in $(\xi_t^{\mathsf{B}})_{t \geq 0}$ (corresponding to the rightmost type B or type B' second class particle in $(\xi_t^{\mathrm{mod}})_{t \geq 0}$), with the convention that $U_t^{\mathsf{B}} = -\infty$ if ξ_t^{B} contains no type B second class particles at time t. Note that $Z_t^{\mathsf{B}} = U_t^{\mathsf{B}}$ almost surely for all $t \geq 0$. Recall that the process $(\xi_t^{\mathsf{B}})_{t \geq 0}$ has the law of a multi-species ASEP on $\mathbb{Z}$ (with special rules for the updates at site 0 and 1) and the law of a standard ASEP on the integers with a Bernoulli-ρ_N-product law when projecting type B second class particles to empty sites. In order to study $(U_t^{\mathsf{B}})_{t \geq 0}$, it will be convenient to add a new type $*$ to the hierarchy as

$$\mathbf{1} \succeq \mathsf{A}' \succeq \mathsf{A} \succeq * \succeq \mathsf{B} \succeq \mathsf{B}' \succeq \mathbf{0}. \tag{4.44}$$

Recall the measure $\tilde{\pi}_N$ from (4.24), (4.25), (4.28) and (4.29), and recall $\bar{\kappa}$ and $\kappa' > 0$ from (4.23). Using ξ_0^{B}, we define a configuration $\xi_0^* \in \{0, 1, *\}^{\mathbb{Z}}$ as follows. When $\kappa < \frac{1}{2}$ and $1 \leq y, \theta \leq N^{\kappa'}$, we obtain ξ_0^* from $\tilde{\pi}_N$, shifted by $\lfloor \theta^{-1} y N/4 \rfloor$, i.e., let $(\mathcal{U}_x)_{x \in \mathbb{Z}}$ be a family of independent Bernoulli-$N^{-1/2}$-distributed random variables and set

$$\xi_0^*(x) = \begin{cases} * & \text{for } \lfloor \theta^{-1} y N/4 \rfloor \leq x \leq \lfloor \theta^{-1} y N/4 \rfloor + N^{\kappa + \bar{\kappa} + \frac{1}{2}} \text{ and } \mathcal{U}_x = \xi_0^{\mathsf{A}} = 1, \\ \xi_0^{\mathsf{B}}(x) & \text{otherwise.} \end{cases} \tag{4.45}$$

Similarly, we assign type $*$ second class particle according to $\tilde{\pi}_N$ shifted by $\lfloor \phi N/4 \rfloor$ when $\kappa = \frac{1}{2}$. We write $\tilde{\pi}_N^*$ for the law of ξ_0^*, and define $(\xi_t^*)_{t\geq 0}$ as the corresponding multi-species ASEP on $\{0, 1, \mathsf{B}, *\}^{\mathbb{Z}}$ started from $\xi_0^*(x)$, treating the type $*$ particles in $(\xi_t^*)_{t\geq 0}$ as the type B second class particles at site 1. Let τ_* with

$$\tau_* := \inf \{t \geq 0 : \xi_t^*(1) = *\}$$

be the first time at which a type $*$ second class particle reached site 1, and let $(U_t^{\mathsf{B},*})_{t\geq 0}$ denote the location of the rightmost type B second class particle in $(\xi_t^*)_{t\geq 0}$. Our key observations are that until time τ_*, the type $*$ second class particles have the same law as second class particles in an ASEP on the integers started from the measure $\tilde{\pi}_N$ (suitably shifted), and that $\mathbf{P}$-almost surely under the basic coupling

$$U_t^{\mathsf{B},*} = U_t^{\mathsf{B}} \text{ for all } 0 \leq t \leq \tau_*, \tag{4.46}$$

since $*$ and $\mathbf{1}$ have the same priority with respect to B. Let M_* denote the number of type $*$ second class particles in the configuration ξ_0^*. We denote the locations of the type $*$ second class particles in ξ_0^* from left to right by $(Z_t^{*,i})_{t\in[0,\tau_*]}$ for $i \in [\![M_*]\!]$. We have the following control of the rightmost second class particle $(U_t^{\mathsf{B},*})_{t\geq 0}$ of type B in $(\xi_t^*)_{t\geq 0}$.

Lemma 4.11. *Let ρ_N for $\rho_N = \frac{1}{2} + 2^{-n}$ with some $n \in [\![\frac{1}{2}\log_2(N)]\!]$ or $\rho_N = \frac{1}{2}$. Let $(\xi_t^*)_{t\geq 0}$ denote the extended process from (4.45) with respective initial distribution $\tilde{\pi}_N^*$. For $\kappa < \frac{1}{2}$, there exist constants $c_1, C_1 > 0$ such that for all $1 \leq \theta, y \leq N^{\kappa'}$,*

$$\mathbf{P}\left(U_t^{\mathsf{B},*} \leq Z_t^{*,M_*} \text{ for all } t \in [0, \min(\tau_*, \theta^{-1}N^{1+\kappa}2^n)]\right) \geq 1 - C_1 \exp(-c_1 N^{\kappa'}) \tag{4.47}$$

for all N large enough. Similarly, when $\kappa = \frac{1}{2}$, there exists some $z > 0$ such that

$$\mathbf{P}\left(U_t^{\mathsf{B},*} \leq Z_t^{*,M_*} \text{ for all } t \in [0, \min(\tau_*, zN^{1+\kappa}2^n \log^{-1}(N))]\right) \geq 1 - \frac{1}{2}N^{-12}. \tag{4.48}$$

for all N large enough.

Proof. Note that for every configuration ξ_t^* with some $0 \leq t \leq \tau_*$, we can associate a configuration $\tilde{\eta}_t^* \in \mathcal{A}_0$ with $\mathcal{A}_0$ from (2.8) as follows. Let $M_{\mathsf{B}}(t)$ be the number of type B second class particles in $(\xi_t^*)_{t\geq 0}$ at time $t \geq 0$, and denote by $\sigma_t \in \Omega_{M_*+M_{\mathsf{B}}(t),M_*}$ the configuration which we obtain by first deleting all first class particles and empty sites in ξ_t^* (as well as removing their sites and merging the resulting edges), and then mapping all type $*$ particles to first class particles and type B second class particles to empty sites; see also Figure 5. Note that we can extend each configuration in $\Omega_{M_*+M_{\mathsf{B}}(t),M_*}$ uniquely to an element $\tilde{\eta}_t^*$ of $\mathcal{A}_0$ by adding particles to the right and empty sites to the left of σ_t. In particular, we have that $\sigma_0 = \vartheta_0$ for ϑ_0 defined in (2.13). As for Lemma 4.8, observe that we can interpret the process $(\tilde{\eta}_t^*)_{t\geq 0}$ until time τ_* as an asymmetric simple exclusion process on $\mathcal{A}_0$ with censoring. More precisely, an edge $\{x, x+1\}$ is censored at time t if and only if in the construction of $\tilde{\eta}_t^*$ from ξ_t^*, we erased a particles (and its site) between the two sites which get mapped to x and $x+1$, respectively, or we added either x or $x+1$ in the construction of $\tilde{\eta}_t^*$ from σ_t. To see that (4.47) holds, note that there exists some constants $c_0, C_0 > 0$ such that for all N large enough

$$\mathbb{P}\Big(M_* \geq \tfrac{1}{8}N^{\kappa+\kappa'}\Big) \geq 1 - C_0 \exp(-c_0 N^{\kappa'}) \tag{4.49}$$

by the choice of ξ_0^*. The result now follows by Lemma 2.9 for $(\tilde{\eta}_t^*)_{t\geq 0}$, noting that whenever the event in (4.49) holds, we get that $\{U_t^{\mathsf{B},*} \leq Z_t^{*,M_*}\}$ occurs at time $t \geq 0$ if the rightmost empty site in $\tilde{\eta}_t^*$ is to the left of position $\frac{1}{8}N^{\kappa+\kappa'}$. The argument for (4.48) is similar. □

We have now all tools to show Proposition 4.10 on the location of second class particles in the extended disagreement process.

Proof of Proposition 4.10. In the following, we will only give the proof for (4.40) when $\kappa < \frac{1}{2}$ and $\rho_N = \frac{1}{2} + 2^{-n}$ for some $n \in [\![\frac{1}{2}\log_2(N)]\!]$. The remaining three cases follow by the same arguments in virtue of Lemmas 4.8, 4.9, and 4.11. Define the event

$$\mathcal{D}_N^{y,\theta} := \left\{1 < Z_t^{*,1} \leq Z_t^{*,M_*} \leq \lfloor \theta^{-1}yN/2 \rfloor \text{ for all } t \in [0, \min(\tau_*, \theta^{-1}N^{1+\kappa}2^n)]\right\}.$$

Using that the type $*$ second class particles $((Z_t^{*,i})_{t\in[0,\tau_*]})_{i\in[\![M_*]\!]}$ have until time τ_* up to spatial space shift the same law as the second class particles in an ASEP on the integers started from $\tilde{\pi}_N$, we get from Lemma 4.6 that for all $1 \leq y, \theta \leq N^{\kappa'}$ and some $c_1, C_1 > 0$,

$$\mathbb{P}(\mathcal{D}_N^{y,\theta}) \geq C_1 \exp(-c_1 y^3)$$

for all N large enough. Together with Lemma 4.11 and (4.46), this implies for

$$\mathcal{E}_N := \left\{U_t^{\mathsf{B}} \leq \lfloor \theta^{-1}yN/2 \rfloor \text{ for all } t \in [0, \theta^{-1}N^{1+\kappa}2^n)]\right\}$$

that we find constants $c_2, C_2 > 0$ such that for all $1 \leq y, \theta \leq N^{\kappa'}$ and all N large enough

$$\mathbb{P}(\mathcal{E}_N) \geq 1 - C_2 \exp(-c_2 y^3). \tag{4.50}$$

In words, on the event $\mathcal{E}_N$, the rightmost type B second class particle in $(\xi_t^{\mathsf{B}})_{t\geq 0}$, and hence all type B or type B' second class particles in the extended disagreement process $(\xi_t^{\mathrm{mod}})_{t\geq 0}$, will not reach location $\theta^{-1}yN/2$ by time $\theta^{-1}N^{1+\kappa}2^n$.

It remains to bound the location of type A and A' second class particles in the extended disagreement process. Recall the projection $(\xi_t^{\mathsf{A}})_{t\geq 0}$ from (4.5). Observe that on the event $\mathcal{E}_N$, the process $(\xi_t^{\mathsf{A}})_{t\geq 0}$ has on sites $\geq y\theta_N^{-1}N/2$ the same law as the ASEP on the integers in a Bernoulli-ρ_N-product measure, where some of the first class particles may be turned into type A second class particles. Let $(U_t^{\mathsf{A}})_{t\geq 0}$ denote the location of the rightmost second class particle of type A in $(\xi_t^{\mathsf{A}})_{t\geq 0}$. As before, in order to bound $(U_t^{\mathsf{A}})_{t\geq 0}$, we add an additional second class particle type $\star$ and extend the partial order (4.44) to

$$\mathbf{1} \succeq \star \succeq \mathsf{A}' \succeq \mathsf{A} \succeq * \succeq \mathsf{B} \succeq \mathsf{B}' \succeq \mathbf{0}. \tag{4.51}$$

We construct a configuration $\xi_0^\star$ from ξ_0^{A} by

$$\xi_0^\star(x) = \begin{cases} \star & \text{for } \lfloor \frac{3}{4}\theta^{-1}yN \rfloor \leq x \leq \lfloor \frac{3}{4}\theta^{-1}yN \rfloor + N^{\kappa+\bar{\kappa}+\frac{1}{2}} \text{ and } \mathcal{U}_x = \xi_0^{\mathsf{A}} = 1, \\ \xi_0^{\mathsf{A}}(x) & \text{otherwise,} \end{cases}$$

where $(\mathcal{U}_x)_{x\in\mathbb{Z}}$ are independent Bernoulli-$N^{-1/2}$-distributed. Let $(\xi_t^\star)_{t\geq 0}$ denote the corresponding multi-species ASEP on $\{0, 1, \mathsf{A}, \star\}^{\mathbb{Z}}$. We let $(U_t^{\mathsf{A},\star})_{t\geq 0}$ and $(Z_t^\star)_{t\geq 0}$ be the location of the rightmost second class particle of type A and type $\star$, respectively. Let

$$\tau_\star := \inf\left\{t \geq 0 : \xi_t^\star(\lfloor \theta^{-1}yN/2 \rfloor + 1) = \star\right\}.$$

Our key observation is that the event

$$\mathcal{G}_N := \mathcal{E}_N \cap \left\{\tau_\star > \theta^{-1} N^{1+\kappa} 2^n\right\}$$

implies $U_t^{\mathsf{A},\star} = U_t^{\mathsf{A}}$ for all $0 \leq t \leq \tau_\star$. Using Lemma 4.8 and (4.50) for a lower bound on the probability of $\mathcal{G}_N$ as well as an upper bound on the location $(Z_t^\star)_{t\geq 0}$, together with the same arguments as in Lemma 4.11 to dominate $(U_t^{\mathsf{A},\star})_{t\geq 0}$ by $(Z_t^\star)_{t\geq 0}$, this yields (4.40). □

4.5. **From the ASEP on the integers to the ASEP on the half-line.** In the following, we aim to show that under the basic coupling, the current of the ASEP on the integers and on the half-line started from a common initial configuration chosen according to a Bernoulli-$(\frac{1}{2}+2^{-n})$-product measure differ after a time of order $T=(1-q)^{-1}N2^n$ at most by order $\sqrt{N}$. Recall that we denote by $(\mathcal{J}_t^{\mathbb{Z}})_{t\geq 0}$ and $(\mathcal{J}_t^{\mathbb{N}})_{t\geq 0}$ the current of the ASEP on $\mathbb{Z}$ and $\mathbb{N}$ through site 1, respectively.

Proposition **4.12.** *Let q satisfy (1.5) for some $\kappa < \frac{1}{2}$ and $\psi > 0$, and recall $\kappa' > 0$ from (4.23). For all $N \in \mathbb{N}$, suppose that $\rho_N = \frac{1}{2} + 2^{-n}$ for some $n \in [\![\frac{1}{2}\log_2(N)]\!]$, and set*

$$T = 2^n N (1-q)^{-1}. \tag{4.52}$$

Consider the ASEP on the half-line started from a Bernoulli-ρ_N-product measure. Then there exist constants $c_0, C_0 > 0$ such that for all $1 \leq x \leq N^{\kappa'}$, and N large enough

$$\mathbb{P}\left(|\mathcal{J}_T^{\mathbb{N}} - T\rho_N(1-\rho_N)(1-q)| > x\sqrt{N}\right) \leq C_0 \exp(-c_0 x). \tag{4.53}$$

For $\rho_N = \frac{1}{2}$, the statement (4.53) holds with respect to $T = N^{3/2}(1-q)^{-1}$.
When $\kappa = \frac{1}{2}$ and $\rho_N = \frac{1}{2} + 2^{-n}$ with $n \in [\![\log_2(c_1^{-1}\log(N)), \frac{1}{2}\log_2(N)]\!]$ for $c_1 > 0$ from Lemma 4.6, there exists some constant $C_1 > 0$ such that for all N large enough

$$\mathbb{P}\left(|\mathcal{J}_T^{\mathbb{N}} - T\rho_N(1-\rho_N)(1-q)| > C_1\log(N)\sqrt{N}\right) \leq N^{-11}, \tag{4.54}$$

with $T = 2^n N^{3/2}$. For $\rho_N = \frac{1}{2}$, we get that (4.54) holds with respect to $T = N^2$.

Proof. We will in the following only give the arguments for $\kappa < \frac{1}{2}$. The case $\kappa = \frac{1}{2}$ is similar. We aim to show that there exist constants $c_0, C_0 > 0$ such that for all $1 \leq x \leq N^{\kappa'}$

$$\mathbb{P}\left(\left|\mathcal{J}_T^{\mathbb{N}} - T\rho_N(1-\rho_N)(1-q)\right| \leq x\sqrt{N}\right) \geq 1 - C_0\exp(-c_0 x) \tag{4.55}$$

for all N large enough. Note that it suffices for (4.55) to show that for any $y \in [\![N^{\kappa'}]\!]$ and

$$\mathcal{A}_k := \left\{\left|\mathcal{J}_{kT/y}^{\mathbb{N}} - \mathcal{J}_{(k-1)T/y}^{\mathbb{N}} - \rho_N(1-\rho_N)(1-q)T/y\right| \leq \sqrt{N}\right\}, \tag{4.56}$$

we have for all $k \in [\![y]\!]$ that

$$\mathbb{P}(\mathcal{A}_1) = \mathbb{P}(\mathcal{A}_k) \geq 1 - C_2\exp(-c_2 y) \tag{4.57}$$

for some constants $C_2, c_2 > 0$. Since the ASEP on the half-line is stationary, and the event $\mathcal{A}_k$ is measurable with respect to $(\eta_t^{\mathbb{N}})_{t\geq 0}$, this yields the equality in (4.57). In order to show the inequality in (4.57), let

$$\mathcal{B}_k := \left\{|\mathcal{J}_{kT/y}^{\mathbb{Z}} - \mathcal{J}_{(k-1)T/y}^{\mathbb{Z}} - T\rho_N(1-\rho_N)(1-q)/y| \leq \sqrt{N}\right\}.$$

By Theorem 2.12 on moderate deviations for the current of the ASEP on the integers, we get that for some constants $c_3, C_3 > 0$ and all $y \in [\![N^{\kappa'}]\!]$ with κ' from (4.23)

$$\mathbb{P}(\mathcal{B}_k) = \mathbb{P}(\mathcal{B}_1) \geq 1 - C_3 \exp(-c_3 y^{3/2}),$$

using the stationarity of the ASEP on the integers. Note that $\eta^{\mathbb{Z}}_{(k-1)T/y}$ and $\eta^{\mathbb{N}}_{(k-1)T/y}$ have the same law on $\mathbb{N}_0$, and that $\mathcal{J}^{\mathbb{Z}}_{kT/y} - \mathcal{J}^{\mathbb{Z}}_{(k-1)T/y}$ as well as $\mathcal{J}^{\mathbb{N}}_{kT/y} - \mathcal{J}^{\mathbb{N}}_{(k-1)T/y}$ are measurable with respect to $(\eta^{\mathbb{Z}}_t)_{t\in[(k-1)T/y,kT/y)}$ and $(\eta^{\mathbb{N}}_t)_{t\in[(k-1)T/y,kT/y)}$, respectively. Suppose that there exists some coupling $\mathbf{P}_k$ such that the event

$$\mathcal{C}_k := \left\{ \left| \mathcal{J}^{\mathbb{Z}}_{kT/y} - \mathcal{J}^{\mathbb{Z}}_{(k-1)T/y} + \mathcal{J}^{\mathbb{N}}_{kT/y} - \mathcal{J}^{\mathbb{N}}_{(k-1)T/y} \right| \leq \sqrt{N} \right\} \tag{4.58}$$

satisfies for some constants $c_4, C_4 > 0$ and all $y \in [\![N^{\kappa'}]\!]$ and $k \in [\![y]\!]$

$$\mathbf{P}_k\left(\mathcal{C}_k\right) \geq 1 - C_4 \exp(-c_4 y). \tag{4.59}$$

Then we get by a union bound on $k \in [\![y]\!]$

$$\mathbb{P}\left(\left| \mathcal{J}^{\mathbb{N}}_T - \rho_N(1-\rho_N)(1-q)T \right| \leq 2y\sqrt{N} \right) \geq 1 - y\mathbb{P}(\mathcal{B}_1^{\complement}) - \sum_{k=1}^{[\![y]\!]} \mathbf{P}_k(\mathcal{C}_k^{\complement}),$$

implying that (4.57) follows indeed from (4.59) and a change of variables.

It remains to show that (4.59) holds for all $k \in \mathbb{N}$. For the coupling $\mathbf{P}_k$, since $\eta^{\mathbb{Z}}_{(k-1)T/y}$ and $\eta^{\mathbb{N}}_{(k-1)T/y}$ have the same law, assume that $\eta^{\mathbb{Z}}_{(k-1)T/y} = \eta^{\mathbb{N}}_{(k-1)T/y}$, and evolve both processes according to the basic coupling $\mathbf{P}$ between times $(k-1)T/y$ and kT/y. Since the processes $(\eta^{\mathbb{Z}}_t)_{t\geq 0}$ and $(\eta^{\mathbb{N}}_t)_{t\geq 0}$ are stationary, it suffices to show (4.59) for $k = 1$. Recall from the definition of the extended disagreement process that the current of the ASEP on the half-line is at time T/y given by the number of type B second class particles minus type A second class particles plus the current of the ASEP on the integers, i.e.,

$$\mathcal{J}^{\mathbb{N}}_{T/y} = \mathcal{J}^{\mathbb{Z}}_{T/y} + \sum_{x\in\mathbb{N}_0} \mathbb{1}_{\left\{\xi^{\mathrm{mod}}_{T/y}(x)=\mathsf{B}\right\}} - \mathbb{1}_{\left\{\xi^{\mathrm{mod}}_{T/y}(x)=\mathsf{A}\right\}}. \tag{4.60}$$

From Proposition 4.10, we get that for some constants $c_5, C_5 > 0$

$$\mathbf{P}\Bigg(\sum_{v > Ny^{-1/2}} \mathbb{1}_{\left\{\xi^{\mathrm{mod}}_{T/y}(v)=\mathsf{B}\right\}} + \mathbb{1}_{\left\{\xi^{\mathrm{mod}}_{T/y}(v)=\mathsf{A}\right\}} > 0 \Bigg) \leq C_5 \exp(-c_5 y^{3/2})$$

for all $y \in [\![N^{\kappa'}]\!]$, and N large enough. Moreover, there exist constants $c_6, C_6 > 0$ such that

$$\mathbf{P}\Bigg(\Bigg| \sum_{v \leq Ny^{-1/2}} \mathbb{1}_{\left\{\xi^{\mathrm{mod}}_{T/y}(v)=\mathsf{B}\right\}} - \mathbb{1}_{\left\{\xi^{\mathrm{mod}}_{T/y}(v)=\mathsf{A}\right\}} \Bigg| \geq \sqrt{N} \Bigg) \leq C_6 \exp(-c_6 y). \tag{4.61}$$

for all $y \leq N^{\kappa'}$ and N large enough. This follows from the observation that by Lemma 4.2, the marginals $\eta^{\mathsf{A}}_{T/y}$ and $\eta^{\mathsf{B}}_{T/y}$ are Bernoulli-ρ_N-distributed on sites $\leq Ny^{-1/2}$, and a standard moderate deviation bound. Combining now (4.60) and (4.61), we obtain (4.59). $\square$

We get the following immediate consequence on the moderate deviations for the current of the ASEP on $\mathbb{N}$. This complements results by Barraquand et al. in [13] and by He in [49], who obtained limit theorems for the current of the ASEP on $\mathbb{N}$ starting from empty initial conditions, and which are expected to similarly hold for stationary initial data.

Corollary 4.13. *Let $\kappa < \frac{1}{2}$. Then under the same setup as in Proposition 4.12,*

$$\mathrm{Var}[\mathcal{J}_T^{\mathbb{N}}] \leq C_0 N \tag{4.62}$$

for some constant C_0 and all N large enough. Similarly, for $\kappa = \frac{1}{2}$, there exists a constant C_0' such that for all N large enough

$$\mathrm{Var}[\mathcal{J}_T^{\mathbb{N}}] \leq C_0' N \log^4(N). \tag{4.63}$$

Proof. We will only consider the case that $\kappa < \frac{1}{2}$ as the arguments for $\kappa = \frac{1}{2}$ are again similar. In view of (4.53), it suffices for (4.62) to argue that for all N large enough

$$\mathbb{E}\big[(\mathcal{J}_T^{\mathbb{N}})^2 \mathbb{1}_{\mathcal{A}}\big] \leq 1, \tag{4.64}$$

where we define the event

$$\mathcal{A} := \Big\{ |\mathcal{J}_T^{\mathbb{N}} - T(1-\rho_N)(1-q)| \geq N^{1/2}\log^2(N) \Big\}.$$

Applying the Cauchy–Schwarz inequality, we see that

$$\mathbb{E}\big[(\mathcal{J}_T^{\mathbb{N}})^2 \mathbb{1}_{\mathcal{A}}\big] \leq \mathbb{E}\big[(\mathcal{J}_T^{\mathbb{N}})^4\big]\mathbb{P}(\mathcal{A}). \tag{4.65}$$

Using a simple Poisson bound, we get that for some constant $c_1 > 0$,

$$\mathbb{P}(\mathcal{J}_T^{\mathbb{N}} \geq 4yT) \leq \exp(-c_1 yT) \tag{4.66}$$

for all $y \geq 1$ and $N \in \mathbb{N}$, counting the number of times the Poisson clocks at the origin rings until time T. In particular, we have the very rough bound $\mathbb{E}\big[(\mathcal{J}_T^{\mathbb{N}})^4\big] \leq N^9$ for all N large enough. Together with (4.53) for $x = \log^2(N)$, this yields (4.64), and thus (4.62). □

4.6. **From the ASEP on the half-line to the open ASEP.** In the following, we transfer the moderate deviation result for the current of the ASEP on the half-line $(\eta_t^{\mathbb{N}})_{t\geq 0}$ to the current of the open ASEP $(\eta_t)_{t\geq 0}$. Recall that we denote by ρ_{L} and ρ_{R} the effective reservoir densities of the open ASEP. We assume that $(\eta_t^{\mathbb{N}})_{t\geq 0}$ and $(\eta_t)_{t\geq 0}$ share the same boundary parameters $\alpha, \gamma \geq 0$. Recall the basic coupling $\mathbf{P}$ from Definition 2.1 between $(\eta_t^{\mathbb{N}})_{t\geq 0}$ and $(\eta_t)_{t\geq 0}$, and note that second class particles are only created at site N. Suppose that

$$\rho_{\mathsf{L}} = \rho_{\mathsf{R}} = \rho_N \tag{4.67}$$

for some $\rho_N = \frac{1}{2} + 2^{-n}$ with $n \in [\![\frac{1}{2}\log_2(N)]\!]$ or $\rho_N = \frac{1}{2}$. Moreover, assume that

$$\eta_0^{\mathbb{N}}(v) = \eta_0(v)$$

for all $v \in [\![N]\!]$, and that $\eta_0 \sim \mathrm{Ber}_{\mathbb{N}}(\rho_N)$. Let $(Z_t)_{t\geq 0}$ denote the position of the left-most second class particle in the disagreement process $(\xi_t)_{t\geq 0}$ between $(\eta_t^{\mathbb{N}})_{t\geq 0}$ and $(\eta_t)_{t\geq 0}$, with the convention $Z_t = -\infty$ if a second class particle has exited by time t at site 1. We have the following lemma on $(Z_t)_{t\geq 0}$. Since we will apply the same arguments as in Proposition 4.10, we will only describe the necessary changes in the proof.

Lemma 4.14. *Let q from (1.5) satisfy $\kappa < \frac{1}{2}$. Let $n \in [\![\frac{1}{2}\log_2(N)]\!]$. Then there exists constants $c_0, C_0 > 0$ and $\kappa' > 0$ such that for all $1 \leq \theta, y \leq N^{\kappa'}$,*

$$\mathbb{P}(Z_s \geq N - y\theta^{-1}N \text{ for all } s \in [0,T]) \geq 1 - C_0\exp(-c_0 y^3) \tag{4.68}$$

for $T = \theta^{-1}N^{1+\kappa}2^n$ and all N large enough. Similarly, for $\rho_N = \frac{1}{2}$, we get that (4.68) holds with respect to $T = \theta^{-1}N^{3/2+\kappa}$.
Let $\kappa = \frac{1}{2}$. Then for every $\phi > 0$, there exists some $z > 0$ such that for all $n \in [\![\log_2(c_1^{-1}\log(N)), \frac{1}{2}\log_2(N)]\!]$ with $c_1 > 0$ from Lemma 4.6

$$\mathbb{P}(Z_s \geq (1-\phi)N \text{ for all } s \in [0,T]) \geq 1 - \frac{1}{2}N^{-10} \tag{4.69}$$

for $T = zN^{3/2}\log^{-1}(N)2^n$, and all N large enough. Similarly, when $\rho_N = \frac{1}{2}$, then we get that (4.69) holds with respect to $T = zN^2\log^{-1}(N)$.

Sketch of proof. Consider the extended disagreement process $(\xi_t^{\mathrm{mod}})_{t\geq 0}$ between $(\eta_t)_{t\geq 0}$ and $(\eta_t^{\mathbb{N}})_{t\geq 0}$ defined as in Definition 4.1, but where second class particles of types A and B are created at site N. We proceed as in the proof of Proposition 4.10 to bound the location of the leftmost second class particle of types $\mathsf{A}, \mathsf{A}', \mathsf{B}, \mathsf{B}'$ in $(\xi_t^{\mathrm{mod}})_{t\geq 0}$, respectively. However, since second class particles enter from the left in the present setup (corresponding to creating second class particles at site N in Proposition 4.10), we use the reversed partial order from Remark 4.3 and first bound the location of the leftmost type A and type A' second class particles and then the leftmost type B and type B' second class particle. Moreover, we require that the estimates from Lemma 4.8 and Lemma 4.9 also holds for a collection of second class particles placed in the ASEP on the halfline. This is ensured by Proposition 4.10 for the basic coupling between a multi-species ASEP on $\mathbb{Z}$ and on $\mathbb{N}$. $\square$

We are now ready to state the main result on the current of the open ASEP when the invariant measure μ_N is a Bernoulli-ρ_N-product measure, i.e., (4.67) holds. Again, since the arguments are very similar to Proposition 4.12, we will only describe the necessary adjustments required for the proof.

Proposition 4.15. *Let q satisfy (1.5) for some $\kappa < \frac{1}{2}$ and recall κ' from (4.23). For all $N \in \mathbb{N}$, suppose that $\rho_N = \frac{1}{2} + 2^{-n}$ for some $n \in [\![\frac{1}{2}\log_2(N)]\!]$, and set*

$$T = 2^n N(1-q)^{-1}. \tag{4.70}$$

Consider the open ASEP started from a Bernoulli-ρ_N-product measure. Then there exist constants $c_0, C_0 > 0$ such that for all $1 \leq x \leq N^{\kappa'}$ and N large enough

$$\mathbb{P}\left(|\mathcal{J}_T - T\rho_N(1-\rho_N)(1-q)| > x\sqrt{N}\right) \leq C_0\exp(-c_0 x). \tag{4.71}$$

For $\rho_N = \frac{1}{2}$, the statement (4.71) holds with respect to $T = N^{3/2}(1-q)^{-1}$.
When $\kappa = \frac{1}{2}$ and $\rho_N = \frac{1}{2} + 2^{-n}$ for some $n \in [\![\log_2(c_1^{-1}\log(N)), \frac{1}{2}\log_2(N)]\!]$ with $c_1 > 0$ from Lemma 4.6, there exists some constant $C_1 > 0$ such that

$$\mathbb{P}\left(|\mathcal{J}_T - T\rho_N(1-\rho_N)(1-q)| > C_1\log(N)\sqrt{N}\right) \leq N^{-10} \tag{4.72}$$

for $T = 2^n N(1-q)^{-1}$. For $\rho_N = \frac{1}{2}$, we get that (4.72) holds with $T = N^2$.

Sketch of proof. Note that it suffices for (4.55) to show that for $1 \leq y \leq N^{\kappa'}$, the events

$$\begin{aligned} \tilde{\mathcal{B}}_k &:= \Big\{ \Big| \mathcal{J}^{\mathbb{N}}_{kT/y} - \mathcal{J}^{\mathbb{N}}_{(k-1)T/y} - \rho_N(1-\rho_N)(1-q)T/y \Big| \leq \sqrt{N} \Big\} \\ \tilde{\mathcal{C}}_k &:= \Big\{ \Big| \mathcal{J}^{\mathbb{N}}_{kT/y} - \mathcal{J}^{\mathbb{N}}_{(k-1)T/y} + \mathcal{J}_{kT/y} - \mathcal{J}_{(k-1)T/y} \Big| \leq \sqrt{N} \Big\} \end{aligned} \tag{4.73}$$

satisfy for all $k \in [\![y]\!]$ with some constants $c_2, C_2, c_3, C_3 > 0$

$$\mathbb{P}(\tilde{\mathcal{B}}_k) = \mathbb{P}(\tilde{\mathcal{B}}_1) \geq 1 - C_2 \exp\big(-c_2 y^{3/2} \big) \tag{4.74}$$

$$\mathbb{P}(\tilde{\mathcal{C}}_k) = \mathbb{P}(\tilde{\mathcal{C}}_1) \geq 1 - C_3 \exp(-c_3 y). \tag{4.75}$$

The lower bound in (4.74) follows from Proposition 4.12. The lower bound in (4.75) follows from the same arguments as (4.58) in the proof of Proposition 4.12, using Lemma 4.14 instead of Proposition 4.10 to bound the location of the second class particles. $\square$

Using Propositions 4.12 and 4.15, we get the following immediate consequence on the moderate deviations of the current of the open ASEP. The proof is identical to Corollary 4.13 and therefore omitted.

COROLLARY **4.16.** *Consider the open ASEP started from its stationary distribution* $\mu_N = \mathrm{Ber}(\rho_N)$ *for some* $\rho_N = \frac{1}{2} + 2^{-n}$ *with* $n \in [\![\frac{1}{2}\log_2(N)]\!]$ *or* $\rho_N = \frac{1}{2}$. *For* q *from* (1.5) *with* $\kappa < \frac{1}{2}$, *let* $T = (1-q)^{-1}N2^n$ *when* $\rho_N = \frac{1}{2} + 2^{-n}$ *with* $n \in [\![\frac{1}{2}\log_2(N)]\!]$, *and set* $T = (1-q)^{-1}N^{3/2}$ *when* $\rho_N = \frac{1}{2}$. *Then*

$$\mathrm{Var}[\mathcal{J}_T] \leq C_1 N \tag{4.76}$$

for some constant C_1 *and all* N *large enough.*
Let $\kappa = \frac{1}{2}$, *and set* $T = N^{3/2}2^n$ *when* $\rho_N = \frac{1}{2} + 2^{-n}$ *for some* $n \in [\![\frac{1}{2}\log_2(N)]\!]$, *and* $T = N^2$ *when* $\rho_N = \frac{1}{2}$. *Then we get that*

$$\mathrm{Var}[\mathcal{J}_T] \leq C_2 N \log^4(N) \tag{4.77}$$

for some constant C_2 *and all* N *large enough.*

Next, we study moderate deviations for the current of the open ASEP when $\rho_{\mathsf{L}} \neq \rho_{\mathsf{R}}$. We consider two open ASEPs $(\eta^1_t)_{t\geq 0}$ and $(\eta^2_t)_{t\geq 0}$ with currents $(\mathcal{J}^{(1)}_t)_{t\geq 0}$ and $(\mathcal{J}^{(2)}_t)_{t\geq 0}$ and invariant measures μ^1_N, μ^2_N, respectively, where the associated boundary parameters satisfy

$$\max\big(|\alpha^2 - \alpha^1|, |\beta^2 - \beta^1|, |\gamma^2 - \gamma^1|, |\delta^2 - \delta^1| \big) = \mathcal{O}(N^{-1/2}), \tag{4.78}$$

and the invariant measure μ^1_N of $(\eta^1_t)_{t\geq 0}$ is a Bernoulli-$\frac{1}{2}$-product measure on $\{0,1\}^N$.

LEMMA **4.17.** *Let* q *from* (1.5) *satisfy* $\kappa < \frac{1}{2}$ *and* $\psi > 0$, *and assume that* (4.78) *holds. Then*

$$\mathrm{Var}[\mathcal{J}^{(2)}_{N^{3/2}(1-q)^{-1}}] \leq C_1 N \tag{4.79}$$

holds for some constant $C_1 > 0$, *and all* N *large enough. For* $\kappa = \frac{1}{2}$,

$$\mathrm{Var}[\mathcal{J}^{(2)}_{N^2}] \leq C_2 N \log^4(N) \tag{4.80}$$

holds for some constant $C_2 > 0$, *and all* N *large enough.*

Proof. We will only consider the case $\kappa < \frac{1}{2}$ as similar arguments apply for $\kappa = \frac{1}{2}$. Consider the open ASEP $(\eta^2_t)_{t\geq 0}$ started from the Bernoulli-$\frac{1}{2}$-product measure μ^1_N. Then by the same arguments as in Lemma 4.14 and Proposition 4.15, there exist constants $c_0, C_0 > 0$ such that for all $1 \leq x \leq N^{\kappa'}$, the current of $(\eta^2_t)_{t\geq 0}$ satisfies

$$\mathbb{P}\Big(\big|\mathcal{J}^{(2)}_{N^{3/2}(1-q)^{-1}} - \frac{1}{4}N^{3/2}\big| > x\sqrt{N}\,\Big|\,\eta^2_0 \sim \mu^1_N\Big) \leq C_0 \exp(-c_0 x) \tag{4.81}$$

for all N large enough. Observe that due to assumption (4.78) and Lemma 2.4, there exists a coupling $\bar{\mathbf{P}}$ between $\tilde{\eta}^1_0 \sim \mu^1_N$ and $\tilde{\eta}^2_0 \sim \mu^2_N$ and constants $\tilde{c}_0, \tilde{C}_0 > 0$ such that for all $1 \leq y \leq N^{\kappa'}$ and N large enough, by a standard moderate deviation estimate,

$$\bar{\mathbf{P}}\Big(\Big|\sum_{v\in[\![N]\!]} \tilde{\eta}^2_0(v) - \sum_{v\in[\![N]\!]} \tilde{\eta}^1_0(v)\Big| \geq y\sqrt{N}\Big) \leq \exp(-c_0 y^2). \tag{4.82}$$

Thus, using the basic coupling for $(\eta^2_t)_{t\geq 0}$ started from $\tilde{\eta}^1_0$ and $\tilde{\eta}^2_0$, respectively, we see that (4.81) holds true for $\eta^2_0 \sim \mu^2_N$. Using now the same arguments as in Corollary 4.13 to convert the moderate deviations estimates to a variance bound, we conclude. ☐

REMARK **4.18.** *Proving a similar bound as (4.82) for general boundary parameters in the maximal current phase (using for example the results from [27, 29]), we conjecture that Lemma 4.17 can be extended to the entire maximal current phase of the open ASEP.*

4.7. **Current bounds for the open ASEP.** In this subsection, we record several consequences of the bounds on the stationary current and the moderate deviations for the current of the open ASEP. We start with the weakly high density phase and $\kappa < \frac{1}{2}$.

LEMMA **4.19.** *Let q satisfy (1.5) for $\kappa < \frac{1}{2}$. For all $N \in \mathbb{N}$, let $n \in [\![\frac{1}{2}\log_2(N) - 1]\!]$ and*

$$T = 2^n N(1-q)^{-1}. \tag{4.83}$$

Recall κ' from (4.23). Consider two stationary open ASEPs $(\eta^1_t)_{t\geq 0}$ and $(\eta^2_t)_{t\geq 0}$ such that the respective effective densities of the two processes satisfy

$$\rho^{(1)}_{\mathsf{L}} = \rho^{(1)}_N \quad \text{and} \quad \rho^{(1)}_{\mathsf{R}} = \rho^{(1)}_N \quad \text{where} \quad \rho^{(1)}_N = \frac{1}{2} + 2^{-n},$$
$$\rho^{(2)}_{\mathsf{L}} = \rho^{(1)}_N \quad \text{and} \quad \rho^{(2)}_{\mathsf{R}} = \rho^{(2)}_N \quad \text{where} \quad \rho^{(2)}_N = \frac{1}{2} + 2^{-(n+1)}.$$

For any coupling $\bar{\mathbf{P}}$ of $(\eta^1_t)_{t\geq 0}$ and $(\eta^2_t)_{t\geq 0}$, the currents $(\mathcal{J}^{(1)}_t)_{t\geq 0}$ and $(\mathcal{J}^{(2)}_t)_{t\geq 0}$ satisfy

$$\bar{\mathbf{P}}\left(\mathcal{J}^{(2)}_T \geq \mathcal{J}^{(1)}_T + 2^{-(n+2)}N\right) \geq 1 - C_0 \exp\big(-c_0 \min(2^{-n}N^{1/2}, N^{\kappa'})\big) \tag{4.84}$$

for some constants $c_0, C_0 > 0$, depending only on κ, and for all N large enough.

Proof. We get from Lemma 2.11 that the stationary currents satisfy

$$\mathbb{E}\big[\mathcal{J}^{(2)}_T\big] - \mathbb{E}\big[\mathcal{J}^{(1)}_T\big] \geq T(2^{-2n} - 2^{-2(n+1)})(1-q) = 2^{-(n+1)}N.$$

Moreover, we have that

$$\begin{aligned}\bar{\mathbf{P}}\left(\mathcal{J}^{(2)}_T \geq \mathcal{J}^{(1)}_T + 2^{-(n+2)}N\right) \geq 1 &- \mathbb{P}\left(\mathcal{J}^{(2)}_T \leq \mathbb{E}\big[\mathcal{J}^{(2)}_T\big] - 2^{-(n+3)}N\right) \\ &- \mathbb{P}\left(\mathcal{J}^{(1)}_T \geq \mathbb{E}\big[\mathcal{J}^{(1)}_T\big] + 2^{-(n+3)}N\right),\end{aligned}$$

and we thus conclude (4.84) using the moderate deviations in Proposition 4.15 to bound the fluctuations of the current of $(\eta^1_t)_{t\geq 0}$ and $(\eta^2_t)_{t\geq 0}$, respectively. $\square$

We have the following result on the current in the maximal current phase of the open ASEP when $\kappa < \frac{1}{2}$, which is similar to Lemma 4.19.

Lemma 4.20. *Let q satisfy (1.5) for some $\kappa < \frac{1}{2}$ and $\psi > 0$. Fix some $m \in \mathbb{N}$ and let*

$$T = mN^{3/2}(1-q)^{-1}. \tag{4.85}$$

Consider two stationary open ASEPs $(\eta^1_t)_{t\geq 0}$ and $(\eta^2_t)_{t\geq 0}$ such that $(\eta^1_t)_{t\geq 0}$ satisfies assumptions (1.6) and (1.9) for some $(\alpha,\beta,\gamma,\delta,q)$. Let $C_L, C_R, c_0, c_1 > 0$ be constants depending only on κ and $\psi > 0$, such that $(\eta^2_t)_{t\geq 0}$ with the same γ and δ, increasing α to α' and β to β', respectively, has effective densities

$$\rho^{(2)}_{\mathsf{L}} = \frac{1}{2} + C_L \frac{1}{\sqrt{N}} \quad \text{and} \quad \rho^{(2)}_{\mathsf{R}} = \frac{1}{2} - C_R \frac{1}{\sqrt{N}}.$$

Then under the basic coupling $\mathbf{P}$, the respective currents $(\mathcal{J}^{(1)}_t)_{t\geq 0}$ and $(\mathcal{J}^{(2)}_t)_{t\geq 0}$ satisfy

$$\mathbf{P}\left(\mathcal{J}^{(2)}_T \geq \mathcal{J}^{(1)}_T + mc_0\sqrt{N}\right) \geq \frac{c_1}{m^2} \tag{4.86}$$

for all $m \in \mathbb{N}$, and all N large enough.

Proof. From Proposition 3.1, verifying that $B(\beta,\delta,q)$ and $D(\alpha,\gamma,q)$ as well as $B(\beta',\delta,q)$ and $D(\alpha',\gamma,q)$ have the desired form, and Lemma 3.12 for strict monotonicity, we get that there exist some constants $C_L, C_R, c_2, C_2 > 0$ such that for any $m \in \mathbb{N}$

$$mc_2\sqrt{N} \leq \mathbb{E}\big[\mathcal{J}^{(2)}_T\big] - \mathbb{E}\big[\mathcal{J}^{(1)}_T\big] \leq mC_2\sqrt{N}$$

and all N large enough. Note that under the basic coupling, we can ensure that

$$\mathbf{P}\left(\mathcal{J}^{(2)}_T \geq \mathcal{J}^{(1)}_T\right) = 1.$$

A standard fact for (not necessarily independent) random variables $(X_i)_{i\in[\![m]\!]}$ and $(Y_i)_{i\in[\![m]\!]}$ is that by Cauchy–Schwarz

$$\operatorname{Var}\Big(\sum_{i\in[\![m]\!]}(X_i - Y_i)\Big) \leq m^4 \max_{i\in[\![m]\!]}(\max(\operatorname{Var}(X_i), \operatorname{Var}(Y_i))). \tag{4.87}$$

Hence, writing the current by time increments, we get that for all N large enough

$$\operatorname{Var}(\mathcal{J}^{(2)}_T - \mathcal{J}^{(1)}_T) \leq m^4 \max_{i\leq[\![m]\!], j\in\{1,2\}} \operatorname{Var}\left(\mathcal{J}^{(j)}_{iT/m} - \mathcal{J}^{(j)}_{(i-1)T/m}\right) \leq m^4 C_1 N$$

with some constant $C_1 > 0$. Using the Paley–Zygmund inequality, noting that $\mathcal{J}^{(2)}_T \geq \mathcal{J}^{(1)}_T$ almost surely under the basic coupling $\mathbf{P}$, we get that

$$\mathbf{P}\left(\mathcal{J}^{(2)}_T \geq \mathcal{J}^{(1)}_T + \frac{mc_2}{2}\sqrt{N}\right) \geq \frac{\mathbb{E}\big[\mathcal{J}^{(2)}_T - \mathcal{J}^{(1)}_T\big]^2}{4\operatorname{Var}\left(\mathcal{J}^{(2)}_T - \mathcal{J}^{(1)}_T\right) + 4\mathbb{E}\big[\mathcal{J}^{(2)}_T - \mathcal{J}^{(1)}_T\big]^2} \geq c_3 m^{-2} \tag{4.88}$$

for some constant $c_3 > 0$, which finishes the proof. $\square$

In the case where q satisfies (1.5) with $\kappa = \frac{1}{2}$, we have the following result on the current of the open ASEP in the high density phase.

Lemma 4.21. *Let q satisfy (1.5) for $\kappa = \frac{1}{2}$ and $\psi > 0$. For all $N \in \mathbb{N}$, and $n \in \mathbb{N}$ set*

$$T = 2^n N(1-q)^{-1}. \tag{4.89}$$

Consider two stationary open ASEPs $(\eta_t^1)_{t\geq 0}$ and $(\eta_t^2)_{t\geq 0}$ such that the respective effective densities which satisfy

$$\rho_{\mathsf{L}}^{(1)} = \rho_N^{(1)} \quad \text{and} \quad \rho_{\mathsf{R}}^{(1)} = \rho_N^{(1)} \quad \text{where} \quad \rho_N^{(1)} = \frac{1}{2} + 2^{-n},$$
$$\rho_{\mathsf{L}}^{(2)} = \rho_N^{(1)} \quad \text{and} \quad \rho_{\mathsf{R}}^{(2)} = \rho_N^{(2)} \quad \text{where} \quad \rho_N^{(2)} = \frac{1}{2} + 2^{-(n+1)}.$$

Then there exists a constant $c_2 > 0$ such that for all n with

$$n \in \left[\log_2(c_1^{-1}\log(N)), c_2 \log_2(\sqrt{N}\log^{-1}(N))\right], \tag{4.90}$$

where the constant $c_1 > 0$ is taken from Lemma 4.6, and for all N large enough,

$$\mathbf{P}\left(\mathcal{J}_T^{(2)} \geq \mathcal{J}_T^{(1)} + 2^{-(n+2)}N\right) \geq 1 - N^{-9}. \tag{4.91}$$

Proof. As for Lemma 4.19, we get from Lemma 2.11 that the stationary currents satisfy

$$\mathbb{E}\big[\mathcal{J}_T^{(2)}\big] - \mathbb{E}\big[\mathcal{J}_T^{(1)}\big] \geq T(2^{-2n} - 2^{-2(n+1)})(1-q) = 2^{-(n+1)}N.$$

Note that the current $\mathcal{J}_T^{(2)}$ is stochastically dominated by the current of a stationary open ASEP in a Bernoulli-$\rho_N^{(2)}$-product measure. The result now follows from the same arguments as the second part of Proposition 4.15. □

Similarly, we can estimate the current in the maximal current phase when $\kappa = \frac{1}{2}$.

Lemma 4.22. *Let q satisfy (1.5) for $\kappa = \frac{1}{2}$ and $\psi > 0$. For all $N \in \mathbb{N}$, consider two stationary open ASEPs $(\eta_t^1)_{t\geq 0}$ and $(\eta_t^2)_{t\geq 0}$ with effective densities*

$$\rho_{\mathsf{L}}^{(1)} = \frac{1}{2} + \frac{C_L}{\sqrt{N}} \quad \text{and} \quad \rho_{\mathsf{R}}^{(1)} = \frac{1}{2} + \frac{C_R}{\sqrt{N}},$$
$$\rho_{\mathsf{L}}^{(2)} = \frac{1}{2} + \frac{C_L'}{\sqrt{N}} \quad \text{and} \quad \rho_{\mathsf{R}}^{(2)} = \frac{1}{2} + \frac{C_R'}{\sqrt{N}}$$

such that (1.6) and (1.9) holds, as well as the condition $\tilde{B}, \tilde{D} > -\psi$ in Proposition 3.2 is satisfied for both sets of boundary parameters. Let $T = mN^2\log(N)$ for some $m \in \mathbb{N}$. For $C_L, C_R \in \mathbb{R}$ and $C_0 > 0$, we can choose $C_L', C_R' > 0$ such that

$$\mathbf{P}\left(\mathcal{J}_T^{(2)} \geq \mathcal{J}_T^{(1)} + c_0 m\sqrt{N}\log(N)\right) \geq \frac{c_1}{\log^2(N)m^2} \tag{4.92}$$

with some constants $c_0, c_1 > 0$, all $m \in \mathbb{N}$ fixed, and all N large enough.

Proof. By Proposition 3.2 and Lemma 3.14, there exist $c_1, C_1 > 0$ such that

$$c_1 m\sqrt{N}\log(N) \leq \mathbb{E}\big[\mathcal{J}_T^{(2)}\big] - \mathbb{E}\big[\mathcal{J}_T^{(1)}\big] \leq C_1 m\sqrt{N}\log(N)$$

for all $N \in \mathbb{N}$ large enough. Here, we choose C_L', C_R' large enough by increasing α and β while decreasing γ and δ to meet the assumptions of Proposition 3.2. By Lemma 4.17,

$$\operatorname{Var}(\mathcal{J}_T^{(1)} - \mathcal{J}_T^{(2)}) \leq 4m^4 \max_{i \leq [\![m]\!], j \in \{1,2\}} \operatorname{Var}\left(\mathcal{J}_{iT/m}^{(j)} - \mathcal{J}_{(i-1)T/m}^{(j)}\right) \leq C_2 m^4 N \log^4(N)$$

for some constant $C_2 > 0$ and all N large enough. As in Lemma 4.20, we apply the Paley–Zygmund inequality to conclude. $\square$

5. Mixing times in the weakly high and low density phase

In this section, we prove estimates on the mixing times for the weakly high and low density phase. To do so, we provide an iterative scheme to bound the number of second class particles in the segment over time. This follows ideas from Section 7 of [48] for mixing times in the high and low density phase with constant boundary and bias parameters.

5.1. **Iterative bounds on the mixing time.** Throughout this section, we assume that $A > \max(1, C)$ as well as that q satisfies (1.5) for some $\kappa \in [0, \frac{1}{2}]$. We have the following result on the mixing time of the open ASEP in the weakly high density phase.

Proposition **5.1.** *Suppose that $\kappa < \frac{1}{2}$ and assume that $\mu_N = \mathrm{Ber}(\rho_N)$ such that*

$$\rho_N = \frac{1}{2} + 2^{-n} \tag{5.1}$$

for some $n \in [\![\frac{1}{2}\log_2(N)]\!]$. There exist constants $C_0, c_1, C_1 > 0$ such that the ε-mixing time of the open ASEP satisfies for all N large enough

$$\frac{t^N_{\mathrm{mix}}(\varepsilon)}{(1-q)^{-1}2^n N} \leq C_0 \tag{5.2}$$

for all $\varepsilon \in (0,1)$ with $\varepsilon \geq C_1 \exp(-c_1 \min(2^{-n}N^{1/2}, N^{\kappa'}))$, where we recall κ' from (4.23).

In order to show Proposition 5.1, we will prove a recursion on the **coupling time** $t_{\mathrm{couple}} = t^{N,n}_{\mathrm{couple}}(\varepsilon)$, where we set for all $\varepsilon \in (0,1)$

$$t^{N,n}_{\mathrm{couple}}(\varepsilon) := \inf\Big\{t \geq 0 \colon \mathbf{P}\big(\tau^{N,n}_{\mathrm{coal}} \leq t\big) \geq 1 - \varepsilon\Big\} \tag{5.3}$$

as the first time t such that the probability under the basic coupling that the open ASEP from any pair of initial states has coalesced by time t is larger than $1-\varepsilon$. Here, $\tau^{N,n}_{\mathrm{coal}}$ denotes the coalescence time of two open ASEPs under the basic coupling with the same parameters and stationary distribution $\mu_N = \mathrm{Ber}_N(\rho_N)$, started from the extremal configurations $\mathbf{1}$ and $\mathbf{0}$, respectively. Note that $t^{N,n}_{\mathrm{couple}}(\varepsilon) \geq t^N_{\mathrm{mix}}(\varepsilon)$ using the coupling representation of the total variation distance – see Lemma 2.2 in [48] – and that $t^{N,n}_{\mathrm{couple}}(\varepsilon)$ is decreasing in ε.

Lemma **5.2.** *Suppose that $\kappa < \frac{1}{2}$ and let $\mu_N = \mathrm{Ber}_N(\rho_N)$ be such that*

$$\rho_N = \frac{1}{2} + 2^{-n} \tag{5.4}$$

for some $n \in [\![\frac{1}{2}\log_2(N)]\!]$. Then there exist constants $c_0, C_0 > 0$ such that the coupling time of the open ASEP satisfies for any $\varepsilon \in (0,1)$

$$t^{N,n}_{\mathrm{couple}}\Big(2\varepsilon + C_0\exp(-c_0\min(2^{-n}N^{1/2}, N^{\kappa'}))\Big) \leq t^{N,n-1}_{\mathrm{couple}}(\varepsilon) + 4(1-q)^{-1}2^n N \tag{5.5}$$

for all N large enough, Moreover, there exist constants $C_1, C_2, c_2 > 0$ such that

$$t^{N,1}_{\mathrm{couple}}(\varepsilon) \leq C_1(1-q)^{-1}N \tag{5.6}$$

for all $\varepsilon \geq C_2\exp(-c_2 N^{\kappa'})$, and all N large enough.

REMARK **5.3.** *Using the particle-hole duality, the same result holds for the coupling time and the mixing time of the open ASEP in the low density phase, where the densities ρ_N take the form $\rho_N = \frac{1}{2} - 2^{-n}$ with some $n \in [\![\frac{1}{2}\log_2(N)]\!]$.*

Using the recursion on the coupling time for the open ASEP in the high density phase, we can deduce Proposition 5.1.

Proof of Proposition 5.1 using Lemma 5.2. Let $f(n) := C_0 \exp(-c_0 \min(2^{-n}N^{1/2}, N^{\kappa'}))$ for all $n \in \mathbb{N}$. Then we can rewrite (5.5) as

$$t^{N,n}_{\text{couple}}(\varepsilon) \leq t^{N,n-1}_{\text{couple}}\big(2^{-1}\varepsilon - 2^{-1}f(n)\big) + 4(1-q)^{-1}2^n N. \tag{5.7}$$

Iterating (5.7), we get that

$$t^{N,n}_{\text{couple}}(\varepsilon) \leq t^{N,1}_{\text{couple}}\big(2^{-n}\varepsilon - F(n)\big) + 8(1-q)^{-1}2^n N, \tag{5.8}$$

where we set $F(n) := \sum_{k=1}^{n} 2^{-k} f(k)$. Choosing now the constants $c_1, C_1 > 0$ accordingly, depending on the choice of $c_2, C_2 > 0$ in Lemma 5.2, a simple computation shows that for any $n \in [\![\frac{1}{2}\log_2(N)]\!]$, any $\varepsilon \geq C_2 \exp(-c_2 \min(2^{-n}N^{1/2}, N^{\kappa'}))$, and all N large enough,

$$2^{-n}\varepsilon - F(n) \geq C_1 \exp\big(-c_1 \min(2^{-n}N^{1/2}, N^{\kappa'})\big). \tag{5.9}$$

Using (5.6) and the fact that $t^{N,n}_{\text{couple}}(\varepsilon) \geq t^{N,n}_{\text{mix}}(\varepsilon)$ for all $\varepsilon \in (0,1)$, we conclude. □

Before we give the proof of Lemma 5.2, we state a corresponding result for the case where q from (1.5) satisfies $\kappa = \frac{1}{2}$. Again, note that the results stated for the high density phase also apply in the low density phase using the particle hole symmetry.

PROPOSITION **5.4.** *Let q from (1.5) satisfy $\kappa = \frac{1}{2}$. Let $\mu_N = \mathrm{Ber}_N(\rho_N)$ be such that we have $\rho_N = \frac{1}{2} + 2^{-n}$ for some n with*

$$n \in \left[\!\left[\log_2(c_1^{-1}\log(N)), \log_2(c_2\sqrt{N}\log^{-1}(N))\right]\!\right] \tag{5.10}$$

and some constant $c_2 > 0$, where the constant $c_1 > 0$ is taken from Lemma 4.6. Then there exists a constant $C_0 > 0$ such that the mixing time of the open ASEP satisfies for all $\varepsilon \geq N^{-8}$

$$\limsup_{N\to\infty} \frac{t^N_{\text{mix}}(\varepsilon)}{\max(2^n, n\log(N))N^{3/2}} \leq C_0. \tag{5.11}$$

As for $\kappa < \frac{1}{2}$, we obtain Proposition 5.4 from the following recursion on the coupling time $t^{N,n}_{\text{couple}}(\varepsilon)$ for the open ASEP in the high density phase (and hence omit the proof).

LEMMA **5.5.** *Let q from (1.5) satisfy $\kappa = \frac{1}{2}$. Let $\mu_N = \mathrm{Ber}(\rho_N)$ for $\rho_N = \frac{1}{2} + 2^{-n}$ be such that $n-1$ and n satisfies (5.10). Then there exists a constant $C_0 > 0$ such that the coupling time of the open ASEP satisfies*

$$t^{N,n}_{\text{couple}}\left(2\varepsilon + C_0 N^{-9}\right) \leq t^{N,n-1}_{\text{couple}}(\varepsilon) + (1-q)^{-1}2^{n+6}N \tag{5.12}$$

for all N large enough. Moreover, there exists a constant $C_1 > 0$ such that for $n = \lceil \log_2(c_1^{-1}\log(N)) \rceil$, we get that

$$t^{N,n}_{\text{couple}}(\varepsilon) \leq C_1 N^{3/2}\log(N) \tag{5.13}$$

for all $\varepsilon \geq 2N^{-9}$, and all N large enough.

5.2. **A multi-species open ASEP with partial ordering.** We deduce Lemma 5.2 and Lemma 5.5 using similar a setup as for the mixing time of the open ASEP in the high and low density phase in [48]. To this end, we define the partially ordered multi-species open ASEP and a corresponding diminished exclusion process. The arguments are similar to [48], except that we require refined estimates on the current established in Section 4.7 – see also Remark 7.4 in [48] – and more classes of particles. Let us stress that this multi-species open ASEP will be different from the multi-species ASEP on the integers defined in Section 2.1 as the different particle types only satisfy a partial order.

In order to define this multi-species ASEP, we couple four exclusion processes on $\llbracket N \rrbracket$ according to the basic coupling. Adapting the notation from Section 7 in [48], for $j \in \llbracket 4 \rrbracket$, we define the open ASEPs $(\eta_t^j)_{t\geq 0}$. The processes $(\eta_t^1)_{t\geq 0}$, $(\eta_t^2)_{t\geq 0}$ and $(\eta_t^3)_{t\geq 0}$ are defined with respect to parameters $(q,\alpha,\beta,\gamma,\delta)$ and started from (random) configurations $\eta^{(i)}$ such that under the basic coupling

$$\mathbf{P}\Big(\eta_t^{(1)} \succeq_{\mathrm{c}} \eta_t^{(3)} \succeq_{\mathrm{c}} \eta_t^{(2)} \text{ for all } t \geq 0\Big) = 1, \tag{5.14}$$

and such that $(\eta_t^3)_{t\geq 0}$ is stationary. We define the fourth open ASEP $(\eta_t^4)_{t\geq 0}$ with respect to the same q as the other three processes, but parameters $(\alpha',\beta',\gamma',\delta')$ for some $\alpha' \geq \alpha$ and $\beta' \geq \beta$, as well as $\gamma \geq \gamma'$ and $\delta \geq \delta'$ specified later on, and with stationary initial condition. In the following, let $(\xi_t)_{t\geq 0}$ denote the disagreement process between $(\eta_t^1)_{t\geq 0}$ and $(\eta_t^2)_{t\geq 0}$ and let $(\zeta_t)_{t\geq 0}$ be the disagreement process between $(\eta_t^3)_{t\geq 0}$ and $(\eta_t^4)_{t\geq 0}$. Recall from Remark 2.2 that we can interpret $(\xi_t)_{t\geq 0}$ and $(\zeta_t)_{t\geq 0}$ as Markov processes on $\{0,1,2\}^N$ and $\{0,1,\mathsf{A},\mathsf{B},\mathsf{A}',\mathsf{B}'\}^N$, respectively, under the basic coupling, using the extended disagreement process from Definition 4.1 for $(\zeta_t)_{t\geq 0}$. Since $(\eta_t^3)_{t\geq 0}$ and $(\eta_t^4)_{t\geq 0}$ are stationary, we can assume without loss of generality that the process $(\zeta_t)_{t\geq 0}$ is stationary as well. Our key observation is that $\zeta_t(x)=0$ implies that $\xi_t(x) \in \{0,2\}$ due to the assumption (5.14), and similarly, $\zeta_t(x)=1$ implies that $\xi_t(x)\in\{1,2\}$ for any $t\geq 0$ and $x \in \llbracket N \rrbracket$. In particular, $(\zeta_t,\xi_t)_{t\geq 0}$ does not attain the values $(0,1)$ and $(1,0)$.

Definition **5.6** (Partially ordered multi-species open ASEP)**.** *Consider the combined disagreement process $(\zeta_t,\xi_t)_{t\geq 0}$, which is a Markov process on $(\{0,1,\mathsf{A},\mathsf{B},\mathsf{A}',\mathsf{B}'\}\times\{0,1,2\})^N$. We denote by $(\chi_t)_{t\geq 0}$ a partially ordered multi-species exclusion process on*

$$\tilde{\Omega}_N := \{1, 2_{-1}, 2_0, 2_1, 2_2, 2_3, 2_4, 2_5, 0\}^N, \tag{5.15}$$

*and refer to 2_i as a **second class particle of type i**. Let $\chi_0(x)$ for all $x \in \llbracket N \rrbracket$ be given by*

$$\chi_0(x) := \begin{cases} 0 & \text{if } \xi_0(x)=0 \text{ and } \zeta_0(x)=0,\\ 2_0 & \text{if } \xi_0(x)=0 \text{ and } \zeta_0(x)\in\{\mathsf{A},\mathsf{B},\mathsf{A}',\mathsf{B}'\},\\ 2_1 & \text{if } \xi_0(x)=2 \text{ and } \zeta_0(x)=0,\\ 2_2 & \text{if } \xi_0(x)=2 \text{ and } \zeta_0(x)\in\{\mathsf{A},\mathsf{B},\mathsf{A}',\mathsf{B}'\},\\ 2_3 & \text{if } \xi_0(x)=2 \text{ and } \zeta_0(x)=1,\\ 2_4 & \text{if } \xi_0(x)=1 \text{ and } \zeta_0(x)\in\{\mathsf{A},\mathsf{B},\mathsf{A}',\mathsf{B}'\},\\ 1 & \text{if } \xi_0(x)=1 \text{ and } \zeta_0(x)=1. \end{cases} \tag{5.16}$$

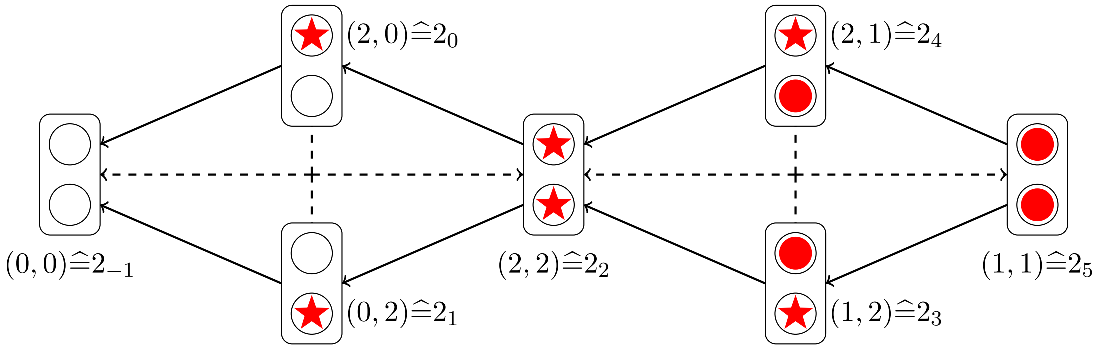

FIGURE 6. Illustration of the possible pairs for $(\zeta_t, \xi_t)_{t\geq 0}$ in Definition 5.6 and as well as their corresponding values in $(\chi_t)_{t\geq 0}$. The partial ordering is indicated by the directions of the (solid) arrows. The vertical dashed line indicates which pairs are not comparable under the partial ordering, and the horizontal dashed arrows indicate the outcome when an incomparable pair receives an update. Here, 2 for ζ_t stands for any value in the set $\{\mathsf{A}, \mathsf{A}', \mathsf{B}, \mathsf{B}'\}$.

We obtain the process $(\chi_t)_{t\geq 0}$ by following the updates of $(\zeta_t, \xi_t)_{t\geq 0}$ as a multi-species exclusion process under the basic coupling, i.e., along each edge we sort the states at rate 1 in increasing order, and at rate q in decreasing order. However, we have the following exceptions: Whenever under the basic coupling we set $\zeta_t(N) = \mathsf{B}$ and $\xi_t(N) = 1$ at some time t, we place a second class particle of type 4 in $(\chi_t)_{t\geq 0}$ at site N, i.e., $\chi_t(N) = 2_4$. Similarly, whenever we set $\zeta_t(1) = \mathsf{A}$ and $\xi_t(N) = 0$ at some time t, we place a second class particle of type 0 in $(\chi_t)_{t\geq 0}$ at site 1, i.e., $\chi_t(N) = 2_0$. Moreover, when a type 3 and a type 4 second class particle receive an update, we turn the type 3 particle into type 2, and the type 4 particle into type 5. Similarly, when a type 0 and a type 1 second class particle are updated, we turn the type 0 particle into type -1, and the type 1 particle into type 2.

While Definition 5.6 may seem daunting at first glance, it is a key tool to relate the exit time of second class particles to a stationary system. Since $(\zeta_t, \xi_t)_{t\geq 0}$ can not attain the values $(1,0)$ and $(0,1)$ by construction, $(\zeta_t, \xi_t)_{t\geq 0}$ and $(\chi_t)_{t\geq 0}$ are in one-to-one correspondence after projecting type 2_5 to first class particles, type 2_{-1} to empty sites, and types $\mathsf{A}, \mathsf{B}, \mathsf{A}', \mathsf{B}'$ to second class particles. Observe that the second class particles of types -1 to 5 obey the partial ordering indicated in Figure 6. Note that all second class particles which enter at site N must have type 4 and all second class particles which enter at sites 1 must have type 0.
We will now follow the construction of a diminished process $(\chi_t^\star)_{t\geq 0}$ as in the proof of Lemma 7.2 of [48], which uses the partially ordered multi-species open ASEP $(\chi_t)_{t\geq 0}$ (however, in contrast to [48], we will allow in the partially ordered multi-species open ASEP $(\chi_t)_{t\geq 0}$ also for type -1 and type 0 particles). In this process $(\chi_t^\star)_{t\geq 0}$, we delete all sites which are either empty, occupied by a first class particle or of type 0 or -1 in χ_t, merging edges if necessary. Then replace all type $1,2,3$ second class particles by empty sites, and all type 4 and type 5 second class particles by first class particles. Finally, we extended the configuration to an element of $\{0,1\}^{\mathbb{Z}}$. We will now formalize this construction.

Definition 5.7 (Diminished partially ordered multi-species ASEP). *Given the process $(\chi_t)_{t\geq 0}$, we start by constructing a family of vectors $(v_t)_{t\geq 0}$, where $v_t \in \{0,1\}^k$ with some $k = k(t) \in \mathbb{N}\cup\{0\}$. For all $t \geq 0$, let v_t denote the vector of type 1 to 5 second class particles which have left the segment at the site 1 by time t. More precisely, we place*

- *1 at position $k+1-i$ if the i^{th} second class particle which exited is of type 4 or 5,*
- *0 at position $k+1-i$ if the i^{th} second class particle which exited is of type 1, 2, 3.*

Let us stress that we only consider the type 1 to 5 second class particles exiting for $(v_t)_{t\geq 0}$. For all $t \geq 0$, we assign a configuration $\chi^\star_t = \chi^\star_t(v_t) \in \{0,1\}^{\mathbb{Z}}$ as follows.

- *Delete all vertices in χ_t which are empty, contain a first class particle or a type -1 or type 0 second class particle to obtain a configuration χ'_t.*
- *Concatenate the vector v_t at the left-hand side of the diminished configuration χ'_t.*
- *Turn all second class particles in (v_t, χ'_t) to empty sites if they are of type 1, 2 or 3 and turn them into first class particles if they are of type 4 or 5 to get an element χ''_t.*
- *Extend χ''_t to a configuration $\chi^\star_t \in \{0,1\}^{\mathbb{Z}}$ by adding empty sites at the left-hand side and first class particles at the right-hand side of the segment.*

Note that so far, $\chi^\star_t$ is only defined up to translations on $\mathbb{Z}$. We define the process $(\chi^\star_t)_{t\geq 0}$ from $(\chi_t)_{t\geq 0}$ such that $\chi^\star_0 \in \mathcal{A}_0$, for $\mathcal{A}_0$ defined in (2.8). For $t > 0$, suppose that $\chi^\star_{t\_} \in \mathcal{A}_m$ holds for some $m \in \mathbb{Z}$. If at time t a second class particle of type 1, 2 or 3 exits at the right-hand side boundary in χ_t, we choose the updated configuration such that $\chi^\star_t \in \mathcal{A}_{m-1}$ holds. In all other cases, we choose $\chi^\star_t \in \mathcal{A}_m$.

A visualization of this construction (without type -1 and type 0 particles) can be found as Figure 8 in [48]. Note that by construction, the process $(\chi^\star_t)_{t\geq 0}$ is supported on $\bigcup_{m\in\mathbb{Z}} \mathcal{A}_m$, with $\mathcal{A}_m$ from (2.8), and thus has almost surely a finite left-most particle $(L^\star_t)_{t\geq 0}$ and right-most empty site $(R^\star_t)_{t\geq 0}$.

Lemma 5.8. *Let q satisfy (1.5) for some $\kappa \in [0, \frac{1}{2}]$ and $\psi > 0$. Suppose that ξ_0 contains at most y many second class particles, and hence χ_0 at most y particles of types 1, 2, 3. Then there exist constants $c_0, C_0, C_1 > 0$ such that for all $x \geq C_1 \log(N)$, and all N large enough,*

$$\mathbb{P}\left(L^\star_t, R^\star_t \in [-xN^\kappa - y, xN^\kappa] \text{ for all } 0 \leq t \leq N^3\right) \geq 1 - C_0 \exp(-c_0 x). \tag{5.17}$$

Proof. Recall the partial order $\succeq_{\mathrm{h}}$ from (2.12) for a simple exclusion process on $\bigcup_{m\in\mathbb{Z}} \mathcal{A}_m$. Verifying the marginal transition rates, we observe that the process $(\chi^\star_t)_{t\geq 0}$ has the law of an ASEP on the integers with censoring, where the rightmost empty site in $\chi^\star_s$ is replaced by a first class particle whenever a second class particle of type 1, 2, 3 exits from $(\chi_t)_{t\geq 0}$ from site N at time s. An edge e is in the censoring scheme $\mathcal{C}$ for $(\chi^\star_t)_{t\geq 0}$ at time t if and only if it was merged in χ_t in the deletion step, or if one of its endpoints is occupied by a particle which is not present in χ_t, and thus was only added in the construction when extending the configuration to $\mathbb{Z}$. To see that $\mathcal{C}$ is a genuine censoring scheme, observe that $(\hat{\chi}_t)_{t\geq 0}$ with

$$\hat{\chi}_t(x) := \begin{cases} 1 & \text{if } \chi_t(x) = 1 \\ 2 & \text{if } \chi_t(x) \in \{2_1, 2_2, 2_3, 2_4, 2_5\} \\ 0 & \text{if } \chi_t(x) \in \{2_0, 2_{-1}, 0\} \end{cases} \tag{5.18}$$

for all $x \in [\![N]\!]$ and $t \geq 0$ is again a continuous-time Markov chain, noting in particular that an update of a $\{2_0, 2_1\}$ pair to $\{2_{-1}, 2_2\}$ in the definition of $(\chi_t)_{t\geq 0}$ is consistent with the projection $(\hat{\chi}_t)_{t\geq 0}$. Recall the configurations ϑ_m from (2.13). Let $(\eta_t^{-y})_{t\geq 0}$ and $(\eta_t^0)_{t\geq 0}$ be two ASEPs on $\mathcal{A}_{-y}$ and $\mathcal{A}_0$, started from ϑ_0 and ϑ_{-y}, and using the same censoring scheme $\mathcal{C}$ as $(\chi_t^\star)_{t\geq 0}$, respectively. Observe that since $\chi_0^\star = \vartheta_0$, and χ_0 contains at most y many type $1,2,3$ particles by our assumptions, the basic coupling ensures that

$$\mathbf{P}(\eta_t^{-y} \succeq_{\mathrm{h}} \chi_t^\star \succeq_{\mathrm{h}} \eta_t^0 \text{ for all } t \geq 0) = 1. \tag{5.19}$$

The result follows from Lemma 2.9 for the processes $(\eta_t^{-y})_{t\geq 0}$ on $\mathcal{A}_{-y}$ and $(\eta_t^0)_{t\geq 0}$ on $\mathcal{A}_0$. □

Remark **5.9.** *In the same way as in Definition 5.7, we can define a second diminished process $(\bar{\chi}_t^\star)_{t\geq 0}$, swapping the roles of type $4,5$ and type $-1,0$ second class particles: Let $(w_t)_{t\geq 0}$ be a family of $\{0,1\}$-valued vectors, denoting the second class particles of types -1 to 3 which have left the segment at the site N by time t. For all $t \geq 0$, we assign a configuration $\chi_t^\star = \chi_t^\star(v_t) \in \{0,1\}^{\mathbb{Z}}$ by first deleting all vertices in χ_t which are empty, contain a first class particle or a type 4 or type 5 second class particle. We then concatenate the vector w_t at the right-hand side of the diminished segment and turn all second class particles to first class particles if they are of type $1,2$ or 3 and into empty sites if they are of type -1 or 0. Finally, we extend the segment to a configuration $\bar{\chi}_t^\star \in \{0,1\}^{\mathbb{Z}}$ in the same way as in Definition 5.7 to ensure that $(\bar{\chi}_t^\star)_{t\geq 0}$ can be interpreted as an ASEP on the integers with censoring, where whenever a type $1,2$ or 3 second class particles leaves at site N, the left-most particle is replaced by an empty site. It is straightforward to verify that a version of Lemma 5.8 continues to hold for the diminished process $(\bar{\chi}_t^\star)_{t\geq 0}$.*

5.3. **Exit time of second class particles.** We will now utilize the processes $(\chi_t)_{t\geq 0}$ and $(\chi_t^\star)_{t\geq 0}$ to establish Lemma 5.2 and Lemma 5.5 on the open ASEP in the high density phase. The processes $(\eta_t^1)_{t\geq 0}$, $(\eta_t^2)_{t\geq 0}$ and $(\eta_t^3)_{t\geq 0}$ are defined with respect to parameters $(q,\alpha,\beta,\gamma,\delta)$ such that the corresponding boundary densities $\rho_{\mathsf{L}}^{(j)}$ and $\rho_{\mathsf{R}}^{(j)}$ for $j \in [\![3]\!]$ satisfy

$$\rho_{\mathsf{L}}^{(j)} = \rho_{\mathsf{R}}^{(j)} = \frac{1}{2} + 2^{-n}$$

with some $n \in [\![\frac{1}{2}\log_2(N)]\!]$. Recall the componentwise order $\succeq_{\mathrm{c}}$. We assume that

$$\eta_0^1 \preceq_{\mathrm{c}} \eta_0^{\mathrm{up}} \sim \mathrm{Ber}_N\Big(\frac{1}{2} + 2^{-(n-1)}\Big) \quad \text{and} \quad \eta_0^2 \succeq_{\mathrm{c}} \eta_0^{\mathrm{low}} \sim \mathrm{Ber}_N\Big(\frac{1}{2} - 2^{-(n-1)}\Big), \tag{5.20}$$

as well as that $\eta_0^1 \succeq_{\mathrm{c}} \eta_0^2$. For the process $(\eta_t^4)_{t\geq 0}$, we will set in the following $\alpha' = \alpha$ and $\gamma' = \gamma$, and hence $\rho_{\mathsf{L}}^{(4)} = \rho_{\mathsf{L}}^{(1)}$, while we choose $\beta' > \beta$ and $\delta' = \delta$ such that $\rho_{\mathsf{R}}^{(4)}$ satisfies

$$\rho_{\mathsf{R}}^{(4)} = \frac{1}{2} + 2^{-(n+1)} < \rho_{\mathsf{R}}^{(1)}.$$

Note that $(\eta_t^4)_{t\geq 0}$ is by construction an open ASEP in the fan region of the high density phase, and that $(\chi_t)_{t\geq 0}$ contains no second class particles of types 0 or -1. Observe that due to Lemma 2.4, we can choose the initial configurations of $(\eta_t^3)_{t\geq 0}$ and $(\eta_t^4)_{t\geq 0}$ such that

$$\mathbf{P}\left(\eta_t^3 \succeq_{\mathrm{c}} \eta_t^4 \ \text{ for all } t \geq 0\right) = 1. \tag{5.21}$$

In particular, all second class particles in $(\zeta_t)_{t\geq 0}$ are of type B. We start with the case where $\kappa < \frac{1}{2}$ for q from (1.5), and study the time τ_{exit}

$$\tau_{\text{exit}} := \inf\{t \geq 0 \colon \xi_t(x) \in \{0,1\} \text{ for all } x \in [\![N]\!]\} \tag{5.22}$$

when all second class particles left the disagreement process $(\xi_t)_{t\geq 0}$ for $(\eta^1_t)_{t\geq 0}$ and $(\eta^2_t)_{t\geq 0}$.

Lemma **5.10.** *Let q from (1.5) with $\kappa < \frac{1}{2}$ and $\psi > 0$. Let $n \in [\![\frac{1}{2}\log_2(N)]\!]$. Then there exist constants $c_0, C_0 > 0$ such that the exit time of second class particles for the disagreement process $(\xi_t)_{t\geq 0}$ satisfies*

$$\mathbf{P}\left(\tau_{\text{exit}} \geq 2^{n+6}N(1-q)^{-1}\right) \leq C_0 \exp(-c_0 \min(2^{-n}N^{1/2}, N^{\kappa'})) \tag{5.23}$$

for all N sufficiently large, where we recall κ' from (4.23).

Proof. Under the basic coupling $\mathbf{P}$, there exist constants $c_1, C_1 > 0$ such that for any $t \geq 0$

$$\mathbf{P}\left(\sum_{x\in[\![N]\!]} \mathbb{1}_{\{\eta^1_t(x)\neq\eta^2_t(x)\}} \geq 4N2^{-n}\right) \leq C_1 \exp\left(-c_1 2^{-n}N^{1/2}\right). \tag{5.24}$$

for all N large enough. Here, we use that second class particles can only exit the segment in $(\xi_t)_{t\geq 0}$, and assumption (5.20) for time 0 together with a tail bound for a Binomial-$(N, 2^{-n+1})$-random variable. Similarly, using Lemma 2.4 for the marginals of the stationary process $(\zeta_t)_{t\geq 0}$, there exist constants $c_2, C_2 > 0$ such that for any $t \geq 0$

$$\mathbf{P}\left(\sum_{x\in[\![N]\!]} \mathbb{1}_{\{\eta^3_t(x)\neq\eta^4_t(x)\}} \geq 2N2^{-n}\right) \leq C_2 \exp\left(-c_2 2^{-n}N^{1/2}\right) \tag{5.25}$$

for all N large enough. For all $t \geq 0$, let $M_4(t)$ denote the number of second class particles in the process $(\zeta_t)_{t\geq 0}$, which have left at site 1 by time t. Note that this agrees with the number of type 4 second class particles in the process $(\chi_t)_{t\geq 0}$, which have exited the segment at site 1 until time t. Recall κ' from (4.23). From Lemma 4.19 and (5.25) for times $t = 0$ and $t = 2^{n+6}N(1-q)^{-1}$, we get that

$$\mathbf{P}\left(M_4(2^{n+6}N(1-q)^{-1}) \geq 8N2^{-n}\right) \geq 1 - C_3 \exp\left(-c_3 \min(2^{-n}N^{1/2}, N^{\kappa'})\right) \tag{5.26}$$

for some constants $c_3, C_3 > 0$, and all N large enough. Recall from Lemma 5.8 that with probability at least $1 - C_4 \exp(-c_4 N^{\kappa'})$ for some constants $C_4, c_4 > 0$, every type $1,2,3$ second class particle in $(\chi_t)_{t\geq 0}$ has until time N^3 at most $N^{\kappa+\kappa'} \leq N^{1/2}$ many second class particles of type 4 or type 5 to its left, counting also all second class particles which have exited the segment. Since on the event in (5.24), the process $(\chi_t)_{t\geq 0}$ can create a total of at most $4N2^{-n}$ type 5 second class particles by coalescence of type 3 and type 4 second class particles, together with (5.25) and (5.26), this yields the desired bound on the exit time of type $1,2,3$ second class particles in $(\chi_t)_{t\geq 0}$. ☐

Similarly, we have the following bound when $\kappa = \frac{1}{2}$. We will only provide the necessary modifications in the proof of Lemma 5.10.

LEMMA **5.11.** *Let q from (1.5) with $\kappa = \frac{1}{2}$ and $\psi > 0$. Assume that n and $n-1$ satisfy (5.10). Then there exist constants $C_0, C_1 > 0$ such that the exit time of second class particles for the disagreement process $(\xi_t)_{t\geq 0}$ satisfies*

$$\mathbf{P}\left(\tau_{\text{exit}} \geq C_1 2^n N^{3/2}\right) \leq C_0 N^{-9} \tag{5.27}$$

for all N sufficiently large. Moreover, when $n = \lceil \log_2(c_1^{-1}\log(N))\rceil$ for the constant $c_1 > 0$ from Lemma 4.6, and $\eta_0^1 = \mathbf{1}$ as well as $\eta_0^2 = \mathbf{0}$, then we get that for some constants $C_2, C_3 > 0$ and all N large enough

$$\mathbf{P}\left(\tau_{\text{exit}} \geq C_3 \log(N) N^{3/2}\log(N)\right) \leq C_2 N^{-9}. \tag{5.28}$$

Sketch of proof. The first part (5.27) follows from the same arguments as Lemma 5.10 using Lemma 4.21 under assumption (5.10) instead of Lemma 4.19 for a lower bound on number of second class particles which have left the segment until time $C_1 2^n N^{3/2}$ at site 1 in $(\zeta_t)_{t\geq 0}$. Here, we choose $C_1 > 0$ sufficiently large so that the right-hand side in (5.26) is at least $1 - N^{-9}$ and the constant c_2 in (5.10) sufficiently small such that $2^n \geq C_1 \log(N)$ holds. For the second statement (5.28), for $n = \lceil \log_2(c_1^{-1}\log(N))\rceil$, we iterate the lower bound in (5.26) 2^n many times to ensure that at least $8N$ many second class particles of type 4 have exited at site 1 in $(\chi_t)_{t\geq 0}$ with probability at least $1 - N^{-8}$. Noting that the process $(\chi_t)_{t\geq 0}$ contains at most N many second class particles of types $1, 2, 3$, and hence at most N many second class particles of type 5 are created, we conclude. $\square$

Using Lemma 5.10 and Lemma 5.11, we obtain the desired bounds on the coupling time, and hence the mixing time of the open ASEP in the weakly high density phase.

Proof of Lemma 5.2 and Lemma 5.5. Consider open ASEPs $(\tilde{\eta}_t^{(j)})_{t\geq 0}$ for $j \in [\![4]\!]$ with respect to the same parameters γ, δ, q, and

$$\alpha^{(3)} \geq \alpha^{(1)} = \alpha^{(2)} \geq \alpha^{(4)} \quad \text{and} \quad \beta^{(4)} \geq \beta^{(1)} = \beta^{(2)} \geq \beta^{(3)}$$

such that the respective effective densities satisfy

$$\rho_{\mathsf{L}}^{(1)} = \rho_{\mathsf{R}}^{(1)} = \frac{1}{2} + 2^{-n}, \quad \rho_{\mathsf{L}}^{(3)} = \rho_{\mathsf{R}}^{(3)} = \frac{1}{2} + 2^{-(n-1)}, \quad \rho_{\mathsf{L}}^{(4)} = \rho_{\mathsf{R}}^{(4)} = \frac{1}{2} - 2^{-(n-1)}.$$

We let $\tilde{\eta}_0^{(1)} = \tilde{\eta}_0^{(3)} = \mathbf{1}$ and $\tilde{\eta}_0^{(2)} = \tilde{\eta}_0^{(4)} = \mathbf{0}$ almost surely, where $\mathbf{1}$ and $\mathbf{0}$ denote the all full and all empty configurations. Note that under the basic coupling for different boundary conditions – see for example Lemma 2.1 in [48] – we get that

$$\mathbf{P}\left(\tilde{\eta}_t^{(3)} \succeq_{\mathrm{c}} \tilde{\eta}_t^{(1)} \succeq_{\mathrm{c}} \tilde{\eta}_t^{(2)} \succeq_{\mathrm{c}} \tilde{\eta}_t^{(4)} \text{ for all } t \geq 0\right) = 1.$$

Recalling Remark 5.3, as well as η_0^{up} and η_0^{low} from (5.20), there exists a coupling $\tilde{\mathbf{P}}$ such that at time $T = t_{\text{couple}}^{N,n-1}(\varepsilon)$, we get that for all N large enough

$$\tilde{\mathbf{P}}\left(\tilde{\eta}_T^{(3)} = \eta_0^{\text{up}} \text{ and } \tilde{\eta}_T^{(4)} = \eta_0^{\text{low}}\right) \geq 1 - 2\varepsilon.$$

When $\kappa < \frac{1}{2}$, we apply Lemma 5.10 for $\eta_0^1 = \tilde{\eta}_T^{(1)}$ and $\eta_0^2 = \tilde{\eta}_T^{(2)}$ to conclude. Note that this includes the special case $n = 1$, which yields (5.6). For $\kappa = \frac{1}{2}$, we apply Lemma 5.10 for

$\eta_0^1 = \tilde{\eta}_T^{(1)}$ and $\eta_0^2 = \tilde{\eta}_T^{(2)}$ whenever $n-1$ and n satisfy (5.20), and the rough bound (5.28) otherwise, to obtain the statement (5.13). $\square$

6. Upper bounds on the mixing time at the triple point

Using a similar strategy as in Section 5, we will now establish the upper bounds on the mixing times in Theorem 1.1 and Theorem 1.2. We will start by recalling the setup of Section 5.2, suitably adapted for the open ASEP at the triple point.

Recall the process $(\chi_t)_{t\geq 0}$ from Definition 5.6 with respect to open ASEPs $(\eta_t^j)_{t\geq 0}$ for $j \in [\![4]\!]$. For q from (1.5) with some $\kappa \in [0, \frac{1}{2}]$ and $\psi > 0$, assume that the boundary parameters $\alpha, \beta, \gamma, \delta$ satisfy the conditions (1.6) and (1.9) with respect to some constants $\tilde{A}, \tilde{C} \in \mathbb{R}$. Recall D and B from (3.1) and (3.2), respectively, where

$$B = -qe^{-\tilde{B}N^{-1/2}+o(N^{-1/2})} \quad \text{and} \quad D = -qe^{-\tilde{D}N^{-1/2}+o(N^{-1/2})} \tag{6.1}$$

for some constants $\tilde{B}, \tilde{D} \in \mathbb{R}$ when $\kappa < \frac{1}{2}$ and $\tilde{B} = \tilde{D} > -\psi$ for $\kappa = \frac{1}{2}$. Note that for $\gamma, \delta > 0$

$$CD = -\frac{\alpha}{\gamma} \quad \text{and} \quad AB = -\frac{\beta}{\delta}. \tag{6.2}$$

The processes $(\eta_t^1)_{t\geq 0}$, $(\eta_t^2)_{t\geq 0}$ and $(\eta_t^3)_{t\geq 0}$ are defined with respect to parameters $(q, \alpha, \beta, \gamma, \delta)$. Moreover, we set

$$\eta_0^1 \preceq_{\mathrm{c}} \eta_0^{\mathrm{up}} \sim \mathrm{Ber}_N\Big(\frac{1}{2} + 2^{-k}\Big) \quad \text{and} \quad \eta_0^2 \succeq_{\mathrm{c}} \eta_0^{\mathrm{low}} \sim \mathrm{Ber}_N\Big(\frac{1}{2} - 2^{-\ell}\Big) \tag{6.3}$$

for some $k, \ell \in \mathbb{N}$ specified later on, and assert that $\eta_0^1(x) \geq \eta_0^2(x)$ for all $x \in [\![N]\!]$. For the process $(\eta_t^4)_{t\geq 0}$, we will set in the following $\alpha' > \alpha$ and $\beta' > \beta$, while $\gamma' \leq \gamma$ and $\delta' \leq \delta$. In order to choose $\alpha', \beta', \gamma', \delta'$, recall the function F from (3.49) for $\kappa < \frac{1}{2}$, and the function $\tilde{F}$ from (3.50) for $\kappa = \frac{1}{2}$. We will choose the constants α' and β' sufficiently large such that

$$A' = A(\beta', \delta', q) = \exp(-\tilde{A}' N^{-1/2}) \quad \text{and} \quad C' = C(\alpha', \gamma', q) = \exp(-\tilde{C}' N^{-1/2}) \tag{6.4}$$

with respect to some constants $\tilde{A}', \tilde{C}'$ satisfy

$$\begin{aligned} F(\tilde{A}', \tilde{C}') - F(\tilde{A}, \tilde{C}) &\geq c_{\kappa, \tilde{A}, \tilde{C}} > 0 \ \text{ if } \kappa < \frac{1}{2} \\ \tilde{F}(\tilde{A}', \tilde{C}') - \tilde{F}(\tilde{A}, \tilde{C}) &\geq c_{\kappa, \tilde{A}, \tilde{C}} > 0 \ \text{ if } \kappa = \frac{1}{2} \end{aligned} \tag{6.5}$$

for some constant $c_{\kappa, \tilde{A}, \tilde{C}} > 0$, depending only on $\kappa \in [0, \frac{1}{2}]$ as well as on $\tilde{A}$ and $\tilde{C}$, while at the same time (2.3) holds with respect to $\alpha', \beta', \gamma', \delta', q$. Note that we can always find such α' and β' and a strictly positive constant $c_{\kappa, \tilde{A}, \tilde{C}}$ due to Lemma 3.12 for $\kappa < \frac{1}{2}$ and Lemma 3.14 for $\kappa = \frac{1}{2}$, and suitable γ', δ' due to (6.2) for $\kappa = \frac{1}{2}$. The processes $(\eta_t^3)_{t\geq 0}$ and $(\eta_t^4)_{t\geq 0}$ are both chosen to be stationary, satisfying

$$\mathbf{P}\left(\eta_t^3 \succeq_{\mathrm{c}} \eta_t^4 \ \text{ for all } t \geq 0\right) = 1 \tag{6.6}$$

with respect to the basic coupling $\mathbf{P}$.

6.1. **Exit time of second class particles.** In the following, we adapt the strategy from Section 5.3 for mixing times in the weakly high and low density phase. Recall from (5.22) that τ_{exit} denotes the exit time of the second class particles in the disagreement process $(\xi_t)_{t\geq 0}$ between $(\eta^1_t)_{t\geq 0}$ and $(\eta^2_t)_{t\geq 0}$ with initial conditions (6.3). We have the following result on the exit time of second class particles at the triple point when $\kappa < \frac{1}{2}$.

Lemma **6.1.** *Let q from (1.5) with $\kappa < \frac{1}{2}$ and $\psi > 0$. Assume that the parameters $\alpha, \beta, \gamma, \delta$ satisfy the conditions (1.6) and (1.9) for some constants $\tilde{A}, \tilde{C}$, and that there exist constants $C_{\text{up}}, C_{\text{low}} > 0$ such that k, ℓ from (6.3) satisfy*

$$\frac{C_{\text{up}}}{\sqrt{N}} \geq 2^{-k} \geq -2^{-\ell} \geq -\frac{C_{\text{low}}}{\sqrt{N}} \tag{6.7}$$

for all N large enough. Then there exist constants $C_0, c_0 > 0$, depending only $\tilde{A}, \tilde{C}, C_{\text{up}}, C_{\text{low}}$, such that we have

$$\mathbf{P}\left(\tau_{\text{exit}} \leq C_0 N^{3/2}(1-q)^{-1}\right) \geq c_0 \tag{6.8}$$

for all N sufficiently large.

Proof. We follow a similar strategy as for Lemma 5.10. Note that under the basic coupling $\mathbf{P}$, there exist constants $c_1, C_1 > 0$ such that for any $t \geq 0$ and $m \in \mathbb{N}$

$$\mathbf{P}\left(\sum_{x\in[\![N]\!]} \mathbb{1}_{\{\eta^1_t(x)\neq\eta^2_t(x)\}} \geq \sqrt{m}(C_{\text{up}}+C_{\text{low}})\sqrt{N}\right) \leq C_1 \exp(-c_1\sqrt{m}) \tag{6.9}$$

for all N large enough. Similarly, using Lemma 2.4 for the marginals of the stationary process $(\zeta_t)_{t\geq 0}$, and assumption (6.6), there exist constants $c_2, C_2, C_3 > 0$ such that for any $t \geq 0$ and $m \in \mathbb{N}$

$$\mathbf{P}\left(\sum_{x\in[\![N]\!]} \mathbb{1}_{\{\eta^3_t(x)\neq\eta^4_t(x)\}} \geq \sqrt{m}C_3\sqrt{N}\right) \leq C_2 \exp(-c_2\sqrt{m}) \tag{6.10}$$

for all N large enough. For the disagreement process $(\zeta_t)_{t\geq 0}$ between $(\eta^3_t)_{t\geq 0}$ and $(\eta^4_t)_{t\geq 0}$, let $\mathcal{J}^{\mathsf{A}}_t$ denote the number of type A second class particles which have exited the segment at site 1 by time $t \geq 0$. Similarly, let $\mathcal{J}^{\mathsf{B}}_t$ denote the number of type B second class particles which have exited the segment at site N by time $t \geq 0$. Set in the following

$$T = T(m) = mN^{3/2}(1-q)^{-1}.$$

Let $(\mathcal{J}^3_t)_{t\geq 0}$ and $(\mathcal{J}^4_t)_{t\geq 0}$ denote the current in $(\eta^3_t)_{t\geq 0}$ and $(\eta^4_t)_{t\geq 0}$ at site 1, respectively. Since in the disagreement process $(\zeta_t)_{t\geq 0}$ type A particles are only created at site N while type B particles are only created at site 1, we observe that

$$\mathcal{J}^4_t - \mathcal{J}^3_t = \mathcal{J}^{\mathsf{A}}_t + \mathcal{J}^{\mathsf{B}}_t + \sum_{x\in[\![N]\!]} \mathbb{1}_{\{\zeta_t(x)=\mathsf{B}\}} - \mathbb{1}_{\{\zeta_0(x)=\mathsf{B}\}} \tag{6.11}$$

almost surely for all $t \geq 0$. From Lemma 4.20 with our choice of the boundary parameters according to (6.5), and (6.10) at times $t = 0$ and $t = T$, we see that there exist constants

$c_4, c_5 > 0$ such that for all $m \in \mathbb{N}$

$$\mathbf{P}\left(\mathcal{J}_T^{\mathsf{A}} + \mathcal{J}_T^{\mathsf{B}} \geq mc_4\sqrt{N}\right) \geq \frac{c_5}{m^2}. \tag{6.12}$$

Let $M_0(s)$ denote the number of type 0 second class particles which have left in $(\chi_t)_{t\geq 0}$ until time $s \geq 0$ at site N. Similarly, let $M_4(s)$ denote the number of type 4 second class particles which have left in $(\chi_t)_{t\geq 0}$ until time $s \geq 0$ at site 1. We define the events

$$\mathcal{A}_1^m := \left\{M_4(T) \geq \frac{c_4}{4}m\sqrt{N}\right\} \quad \text{and} \quad \mathcal{A}_2^m := \left\{M_0(T) \geq \frac{c_4}{4}m\sqrt{N}\right\},$$

and recall that only type 4 second class particles are created at site N (paired with a type A second class particle in $(\zeta_t)_{t\geq 0}$) while only type 0 second class particles are created at site 1 (paired with a type B second class particle in $(\zeta_t)_{t\geq 0}$) in the process $(\chi_t)_{t\geq 0}$. Since the events in (6.9) and (6.10) ensure that there are at most $\sqrt{m}(C_3+C_{\text{up}}+C_{\text{low}})\sqrt{N}$ many second class particles of types A and B in the segment at times 0 and T, we get from (6.11) and (6.12) that there exists some $m_0 \in \mathbb{N}$ and $c_6 > 0$ such that

$$\begin{aligned} \max\left(\mathbf{P}(\mathcal{A}_1^m), \mathbf{P}(\mathcal{A}_2^m)\right) &= \max\left(\mathbf{P}\big(M_4(T) \geq \frac{c_4}{4}m\sqrt{N}\big), \mathbf{P}\big(M_0(T) \geq \frac{c_4}{4}m\sqrt{N}\big)\right) \\ &\geq \frac{1}{2}\mathbf{P}\left(\mathcal{J}_T^{\mathsf{A}} + \mathcal{J}_T^{\mathsf{B}} \geq mc_4\sqrt{N}\right) - 2C_2\exp(-c_2\sqrt{m}) \geq \frac{c_6}{4m^2} \end{aligned} \tag{6.13}$$

for all $m \geq m_0$, and all N large enough. The same arguments as in Lemma 5.10, using the diminished process $(\chi_t^\star)_{t\geq 0}$, guarantee that with probability at least $1 - C_7\exp(-c_7N^{\kappa'})$ for some constants $c_7, C_7 > 0$ and κ' from (4.23), there are at most $N^{\kappa+\kappa'} \leq N^{1/2}$ many second class particles of type 4 to the left of any type $1,2,3$ second class particle at time T. Similarly, using the diminished process $(\bar{\chi}_t^\star)_{t\geq 0}$ from Remark 5.9, we see that with probability at least $1 - C_8\exp(-c_8N^{\kappa'})$ for some constants $c_8, C_8 > 0$, there are at most $N^{\kappa+\kappa'}$ many second class particles of type 0 to the right of any type $1,2,3$ second class particle at time T. Hence, choosing $m = \max(8c_4^{-1}, m_0)$, either of the events $\mathcal{A}_1^m$ and $\mathcal{A}_2^m$ implies that τ_{exit} has occurred by time T with probability tending to 1 as $N \to \infty$, allowing us to conclude by (6.13). □

We have the following result when $\kappa = \frac{1}{2}$. Since we apply the same arguments as in the proof of Lemma 5.11, we only describe the necessary adjustments.

Lemma 6.2. *Let q from (1.5) with $\kappa = \frac{1}{2}$ and $\psi > 0$. Assume that the parameters $\alpha, \beta, \gamma, \delta$ satisfy the conditions (1.6) and (1.9) for some constants $\tilde{A}, \tilde{C}$, and that there exist constants $C_{\text{up}}, C_{\text{low}} > 0$ such that k, ℓ from (6.3) satisfy*

$$\frac{C_{\text{up}}\log(N)}{\sqrt{N}} \geq 2^{-k} \geq -2^{-\ell} \geq -\frac{C_{\text{low}}\log(N)}{\sqrt{N}} \tag{6.14}$$

for all N large enough. Then there exist constants $C_0, c_0 > 0$, depending only $\tilde{A}, \tilde{C}, C_{\text{up}}, C_{\text{low}}$, such that we have for all N sufficiently large

$$\mathbf{P}\left(\tau_{\text{exit}} \leq C_0N^2\log(N)\right) \geq \frac{c_0}{\log^2(N)}. \tag{6.15}$$

Sketch of proof. Note that by Lemma 2.4, there exists a constant $C_1 > 0$ such that

$$\mathbf{P}\Big(\sum_{x \in \llbracket N \rrbracket} \mathbb{1}_{\{\eta_t^3(x) \neq \eta_t^4(x)\}} + \mathbb{1}_{\{\eta_t^3(x) \neq \eta_t^4(x)\}} \geq C_1 \log(N)\sqrt{N} \Big) \leq N^{-9} \tag{6.16}$$

for all N large enough. Set $T = mN^2 \log(N)$ for some constant $m \in \mathbb{N}$ chosen later on. Recall that $M_4(s)$ and $M_0(s)$ denote the number of type 4 and type 0 second class particles, which have exited in $(\chi_t)_{t \geq 0}$ by time s at site 1 and site N, respectively. The same arguments as in Lemma 6.1 with a lower bound on the current of second class particles in $(\zeta_t)_{t \geq 0}$ by Lemma 4.22 yield that

$$\max\Big(\mathbf{P}\big(M_4(T) \geq mc_2 \log(N)\sqrt{N}\big), \mathbf{P}\big(M_0(T) \geq mc_2 \log(N)\sqrt{N}\big)\Big) \geq \frac{c_3}{\log^2(N)m^2} \tag{6.17}$$

for some constants $c_2, c_3 > 0$, for all m fixed, and all N large enough. Moreover, by Lemma 5.8 and Remark 5.9, there exists a constant $C_4 > 0$ such that with probability at least $1 - N^{-8}$ at most $C_4 \log(N)\sqrt{N}$ many type 4 second class particles are to the left of any type $1, 2, 3$ second class particle in χ_T, while at most $C_4 \log(N)\sqrt{N}$ many type 0 second class particles are to the right of any type $1, 2, 3$ second class particle in χ_T. Choosing now the constant m in (6.17) sufficiently large, we conclude. □

6.2. **Proof of the upper bound on the mixing time.** We have now all tools to establish an upper bound on the mixing time of the open ASEP at the triple point. We start with the case where q satisfies (1.5) with $\kappa < \frac{1}{2}$.

Proof of the upper bound in Theorem 1.1. Let $C_{\text{up}}, C_{\text{low}} > 0$ be two constants such that

$$\frac{1}{2} + \frac{C_{\text{up}}}{\sqrt{N}} \geq \max\left(\frac{A}{A+1}, \frac{1}{C+1}\right) \geq \min\left(\frac{A}{A+1}, \frac{1}{C+1}\right) \geq \frac{1}{2} - \frac{C_{\text{low}}}{\sqrt{N}} \tag{6.18}$$

for all N large enough, assuming without loss of generality that $\frac{C_{\text{up}}}{\sqrt{N}}$ and $\frac{C_{\text{low}}}{\sqrt{N}}$ are powers of 2. Recall the coupling time $t_{\text{couple}}(\varepsilon)$ from (5.3) for the open ASEP with respect to boundary parameters $\alpha, \beta, \gamma, \delta$ and consider four open ASEPs $(\tilde{\eta}_t^{(j)})_{t \geq 0}$ for $j \in \llbracket 4 \rrbracket$. We use the same parameters γ, δ, q as well as

$$\alpha^{(3)} \geq \alpha^{(1)} = \alpha^{(2)} \geq \alpha^{(4)} \quad \text{and} \quad \beta^{(4)} \geq \beta^{(1)} = \beta^{(2)} \geq \beta^{(3)}$$

such that the respective effective densities satisfy

$$\rho_{\mathsf{L}}^{(1)} = \frac{1}{C+1}, \quad \rho_{\mathsf{R}}^{(1)} = \frac{A}{A+1}, \quad \rho_{\mathsf{L}}^{(3)} = \rho_{\mathsf{R}}^{(3)} = \frac{1}{2} + \frac{C_{\text{up}}}{\sqrt{N}}, \quad \rho_{\mathsf{L}}^{(4)} = \rho_{\mathsf{R}}^{(4)} = \frac{1}{2} - \frac{C_{\text{low}}}{\sqrt{N}}. \tag{6.19}$$

We let $\tilde{\eta}_0^{(1)} = \tilde{\eta}_0^{(3)} = \mathbf{1}$ and $\tilde{\eta}_0^{(2)} = \tilde{\eta}_0^{(4)} = \mathbf{0}$ almost surely. Under the basic coupling, we get

$$\mathbf{P}\left(\tilde{\eta}_t^{(3)} \succeq_{\text{c}} \tilde{\eta}_t^{(1)} \succeq_{\text{c}} \tilde{\eta}_t^{(2)} \succeq_{\text{c}} \tilde{\eta}_t^{(4)} \text{ for all } t \geq 0\right) = 1. \tag{6.20}$$

We consider two random configurations

$$\eta_0^{\text{up}} \sim \text{Ber}_N\Big(\frac{1}{2} + \frac{C_{\text{up}}}{\sqrt{N}}\Big) \quad \text{and} \quad \eta_0^{\text{low}} \sim \text{Ber}_N\Big(\frac{1}{2} - \frac{C_{\text{low}}}{\sqrt{N}}\Big).$$

Then by Proposition 5.1, with $2^{-n} = C_{\rm up}/\sqrt{N}$, increasing the constants $C_{\rm up}$ and $C_{\rm low}$ from (6.18) if necessary, for every $\varepsilon > 0$, there exists some constant $C_0 > 0$ such that for all N large enough

$$\mathbf{P}\left(\tilde{\eta}^{(3)}_{C_0 N^{3/2}(1-q)^{-1}} = \eta_0^{\rm up} \text{ and } \tilde{\eta}^{(4)}_{C_0 N^{3/2}(1-q)^{-1}} = \eta_0^{\rm low} \text{ with } \eta_0^{\rm up} \succeq_{\rm c} \eta_0^{\rm low}\right) \geq 1 - \varepsilon.$$

Let $c_0 > 0$ be taken from Lemma 6.1 and set $\varepsilon = c_0/2$. Then by Lemma 6.1 and (6.20), there exists a constant $C_2 > 0$ such that

$$\mathbf{P}\left(\tilde{\eta}^{(1)}_{C_2 N^{3/2}(1-q)^{-1}} = \tilde{\eta}^{(2)}_{C_2 N^{3/2}(1-q)^{-1}}\right) \geq \varepsilon \tag{6.21}$$

for all N large enough. Since $\varepsilon > 0$ does not depend on N, iterating (6.21) order ε^{-1} many times gives the desired result. □

Using a similar argument, we obtain an upper bound on mixing time of the open ASEP when q satisfies (1.5) with $\kappa = \frac{1}{2}$.

Proof of the upper bound in Theorem 1.2. We use the same setup as in the proof of the upper bound of Theorem 1.1, but assert that

$$\rho_{\sf L}^{(3)} = \rho_{\sf R}^{(3)} = \frac{1}{2} + \frac{C_{\rm up}\log(N)}{\sqrt{N}} = \frac{1}{2} + 2^{-k}, \quad \rho_{\sf L}^{(4)} = \rho_{\sf R}^{(4)} = \frac{1}{2} - \frac{C_{\rm low}\log(N)}{\sqrt{N}} = \frac{1}{2} - 2^{-\ell}.$$

for some constants $C_{\rm up}, C_{\rm low} > 0$ and $k, \ell \in \mathbb{N}$. Then by Proposition 5.4, increasing the constants $C_{\rm up}, C_{\rm low}$ if necessary, there exists a constant $C_0 > 0$ such that

$$\mathbf{P}\left(\tilde{\eta}^{(3)}_{C_0 N^2 \log(N)^{-1}} = \eta_0^{\rm up} \text{ and } \tilde{\eta}^{(4)}_{C_0 N^2 \log(N)^{-1}} = \eta_0^{\rm low} \text{ with } \eta_0^{\rm up} \succeq_{\rm c} \eta_0^{\rm low}\right) \geq 1 - 2N^{-8}$$

for all N large enough. Now Lemma 6.2 yields that for some constant $C_1 > 0$

$$\liminf_{N\to\infty} \log^2(N)\mathbf{P}\left(\tilde{\eta}^{(1)}_{C_1 N^2 \log(N)} = \tilde{\eta}^{(2)}_{C_1 N^2 \log(N)}\right) > 0. \tag{6.22}$$

Iterating now (6.22) order $\log^2(N)$ many times gives the desired result. □

Remark **6.3.** *Observe that we apply the same line of arguments for both regimes $\kappa < \frac{1}{2}$ and $\kappa = \frac{1}{2}$. It is therefore natural to conjecture that the upper bound of order $N^{3/2+\kappa}$ on the mixing time can be extended to the case $\kappa = \frac{1}{2}$. However, this requires an improved bound on the variance of the current of the open ASEP of order N until time $T \asymp N^2$, which could for example be achieved by an improved bound on the speed of second class particles in Proposition 4.10 when $\kappa = \frac{1}{2}$. We leave this to future work.*

7. Lower bound on the mixing times

In this section, we provide lower bounds on the mixing times of the open ASEP at the triple point. We start with an auxiliary result on the speed of second class particles for an ASEP on the integers. Fix some $z > 0$ (specified later on), and let $\bar{\pi}_N^z$ denote the product measure on $\{0,1,2\}^{\mathbb{Z}}$ with marginals

$$\bar{\pi}_N^z(x) = \begin{cases} 1 & \text{with probability } \frac{1}{2} \\ 2 & \text{with probability } \frac{z}{\sqrt{N}} \\ \infty & \text{with probability } \frac{1}{2} - \frac{z}{\sqrt{N}} \end{cases} \tag{7.1}$$

for all $x \in [\frac{3}{8}N, \frac{5}{8}N]$ as well as

$$\bar{\pi}_N^z(x) = \begin{cases} 1 & \text{with probability } \frac{1}{2} \\ \infty & \text{with probability } \frac{1}{2} \end{cases} \tag{7.2}$$

for all $x \notin [\frac{3}{8}N, \frac{5}{8}N]$. With a slight abuse of notation, we can interpret $\bar{\pi}_N^z$ as a probability measure on the space $\{0,1,2\}^N$ for all N large enough. We have the following result on the location of second class particles in an ASEP on the integers started from $\bar{\pi}_N^z$.

Lemma **7.1.** *Let q satisfy (1.5) with some $\kappa < \frac{1}{2}$. Consider an ASEP on the integers $(\eta_t^{\mathbb{Z}})_{t\geq 0}$ with initial distribution $\bar{\pi}_N^z$. Then for every $\varepsilon, z > 0$, there exists some constant $c_1 > 0$ such that for all N large enough*

$$\mathbb{P}\left(\eta_t^{\mathbb{Z}}(x) \in \{0,1\} \text{ for all } t \in [0, c_1(1-q)^{-1}N^{3/2}] \text{ and } x \notin \left[\tfrac{1}{4}N, \tfrac{3}{4}N\right]\right) \geq 1-\varepsilon. \tag{7.3}$$

Similarly, if q satisfies (1.5) with $\kappa = \frac{1}{2}$, then for every $\varepsilon, z > 0$, there exists some constant $c_2 > 0$ such that for all N large enough

$$\mathbb{P}\left(\eta_t^{\mathbb{Z}}(x) \in \{0,1\} \text{ for all } t \in [0, c_2 N^2 \log^{-1}(N)] \text{ and } x \notin \left[\tfrac{1}{4}N, \tfrac{3}{4}N\right]\right) \geq 1-\varepsilon. \tag{7.4}$$

Proof. Let $\kappa < \frac{1}{2}$, and recall Lemma 4.8 on moderate deviations for a collection of second class particles. The claim in (7.3) follows now by the same arguments as for $\rho_N = 1/2$ in Lemma 4.8, shifting the lattice by $N/2$, and using the fact that $N^{\kappa+\kappa'+\frac{1}{2}} \leq N/8$ for all N large enough. The bound in (7.4) for the case $\kappa = \frac{1}{2}$ is similar using Lemma 4.9 instead of Lemma 4.8 for a moderate deviation estimate. $\square$

In order to show the lower bounds on the mixing time, we require a simple observation on the number of particles under the stationary distribution μ_N of the open ASEP.

Lemma **7.2.** *Let q satisfy (1.5) for some $\kappa \in [0, \frac{1}{2}]$ and assume that the boundary parameters satisfy the assumption (1.9). Then for every $\varepsilon > 0$, there exists some constant $C_0 > 0$ such that for all $i \in [\![4]\!]$*

$$\mu_N\left(\sum_{(i-1)N/4 < x \leq iN/4} \eta(x) \notin \left[\tfrac{1}{2}N - C_0\sqrt{N}, \tfrac{1}{2}N + C_0\sqrt{N}\right]\right) \leq \varepsilon \tag{7.5}$$

with $N \in \mathbb{N}$ large enough.

Proof. This statement is immediate from Lemma 2.4 which says that μ_N is stochastically dominated from below by a Bernoulli-($\frac{1}{2} - mN^{-1/2}$)-product measure and from above by a Bernoulli-($\frac{1}{2} + mN^{-1/2}$)-product measure with some constant $m > 0$. $\square$

We have now all tools to show the desired lower bounds on the mixing times.

Proof of the lower bound in Theorem 1.1 and Theorem 1.2. We let $(\eta_t)_{t\geq 0}$ denote an open ASEP on $\{0,1\}^N$ with respect to boundary parameters $\alpha, \beta, \gamma, \delta$ and q, which satisfy (1.9). We define $(\eta_t^{(1)})_{t\geq 0}$ and $(\eta_t^{(2)})_{t\geq 0}$ as two disagreement processes on $\{0,1,2\}^N$ with the same parameters γ, δ, q, but where α', β' are chosen such that the stationary distribution $\bar{\mu}_N$ of both processes satisfies

$$\bar{\mu}_N \sim \mathrm{Ber}_N\Big(\frac{1}{2}\Big).$$

We assert that $\eta_0 = \eta_0^{(1)} \sim \bar{\mu}_N$ while $\eta_0^{(2)} \sim \bar{\pi}_N^z$ for some $z > 0$ specified later on, as well as $\eta_0^{(2)} \succeq_{\mathrm{c}} \eta_0^{(1)}$, extending the componentwise ordering $\succeq_{\mathrm{c}}$ with respect to the ordering $1 \succeq 2 \succeq 0$ on $\{0,1,2\}^N$. Let $\kappa < \frac{1}{2}$ and $\varepsilon > 0$. Let $(\eta_t^{\mathbb{Z}})_{t\geq 0}$ be an ASEP on the integers with $\eta_0^{(2)}(x) = \eta_0^{\mathbb{Z}}(x)$ for all $x \in [\![N]\!]$, and a Bernoulli-$\frac{1}{2}$-product measure for all $x \notin [\![N]\!]$. Note that when projecting all second class particles to first class particles, the law of $(\eta_t^{\mathbb{Z}})_{t\geq 0}$ is stochastically dominated from above by a Bernoulli-$\big(\frac{1}{2} + \frac{z}{\sqrt{N}}\big)$-product measure. Thus, combining Proposition 4.10 and Lemma 4.14, there exists a constant $c_1 > 0$, depending only on ε and z, such that under the basic coupling between $(\eta_t^{(2)})_{t\geq 0}$ and $(\eta_t^{\mathbb{Z}})_{t\geq 0}$, we get that for all N large enough

$$\mathbf{P}\Big(\eta_t^{(2)}(x) = \eta_t^{\mathbb{Z}}(x) \text{ for all } t \in [0, c_1(1-q)^{-1}N^{3/2}] \text{ and } x \in \big[\tfrac{1}{4}N, \tfrac{3}{4}N\big]\Big) \geq 1 - \varepsilon. \tag{7.6}$$

From (7.6) and Lemma 7.1 to bound the motion of second class particles within $(\eta_t^{\mathbb{Z}})_{t\geq 0}$, we see that for every $\varepsilon, z > 0$, there exists some constant $c_2 > 0$ such that for all N large enough

$$\mathbf{P}\Big(\eta_t^{(1)}(x) = \eta_t^{(2)}(x) \text{ for all } t \in [0, c_2(1-q)^{-1}N^{3/2}] \text{ and } x \notin \big[\tfrac{1}{4}N, \tfrac{3}{4}N\big]\Big) \geq 1 - \varepsilon. \tag{7.7}$$

When $\kappa = \frac{1}{2}$, a similar argument yields that for every $\varepsilon, z > 0$, there exists some constant $c_3 > 0$ such that for all N large enough

$$\mathbf{P}\Big(\eta_t^{(1)}(x) = \eta_t^{(2)}(x) \text{ for all } t \in [0, c_3 N^2 \log^{-1}(N)] \text{ and } x \notin \big[\tfrac{1}{4}N, \tfrac{3}{4}N\big]\Big) \geq 1 - \varepsilon. \tag{7.8}$$

Using again Proposition 4.10 and Lemma 4.14 when $\kappa < \frac{1}{2}$, we see that for every $\varepsilon, z > 0$, there exists some constant $c_4 > 0$ such that for all N large enough

$$\mathbf{P}\Big(\eta_t^{(1)}(x) = \eta_t(x) \text{ for all } t \in [0, c_4(1-q)^{-1}N^{3/2}] \text{ and } x \in \big[\tfrac{1}{4}N, \tfrac{3}{4}N\big]\Big) \geq 1 - \varepsilon. \tag{7.9}$$

Similarly, when $\kappa = \frac{1}{2}$, we get that for every $\varepsilon, z > 0$, there exists some constant $c_5 > 0$ such that for all N large enough

$$\mathbf{P}\Big(\eta_t^{(1)}(x) = \eta_t(x) \text{ for all } t \in [0, c_5 N^2 \log^{-1}(N)] \text{ and } x \in \big[\tfrac{1}{4}N, \tfrac{3}{4}N\big]\Big) \geq 1 - \varepsilon. \tag{7.10}$$

Combining (7.7) and (7.9) for $\kappa < \frac{1}{2}$, with probability at least $1-2\varepsilon$, no second class particle in $(\eta_t)_{t\geq 0}$ started from $\bar{\pi}_N$ has left the segment by time $\min(c_2, c_4)N^{3/2}(1-q)^{-1}$. Similarly, combining (7.8) and (7.10) for $\kappa = \frac{1}{2}$, with probability at least $1 - 2\varepsilon$, no second class particle in $(\eta_t)_{t\geq 0}$ started from $\bar{\pi}_N$ has left the segment by time $\min(c_3, c_5)N^2 \log^{-1}(N)$. In view of Lemma 7.2, for any $\varepsilon > 0$, first taking $z > 0$ sufficiently large and then the constants $c_2, c_3, c_4, c_5 > 0$ sufficiently small, this yields the desired lower bounds on the mixing time in Theorem 1.1 and Theorem 1.2. $\square$

Acknowledgment. We are grateful to Amol Aggarwal and Ivan Corwin for valuable comments on moderate deviations for the current of the open ASEP. Moreover, we thank Márton Balázs and Guillaume Barraquand for helpful discussions. This work was partly funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy - GZ 2047/1, projekt-id 390685813 and by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - Projektnummer 211504053 - SFB 1060.

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