Exact decay of the persistence probability in the Airy_1 process

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Abstract

We consider the Airy_1 process, which is the limit process in KPZ growth models with flat and non-random initial conditions. We study the persistence probability, namely the probability that the process stays below a given threshold c for a time span of length L. This is expected to decay as $e^{-\kappa(c)L}$. We determine an analytic expression for $\kappa(c)$ for all $c \geq 3/2$ starting with the continuum statistics formula for the persistence probability. As the formula is analytic only for $c > 0$, we determine an analytic continuation of $\kappa(c)$ and numerically verify the validity for $c < 0$ as well.

1 Introduction and main results

For stochastic growth models in the Kardar-Parisi-Zhang (KPZ) universality class, the large time limit process of the interface depends on the initial and boundary conditions. In the one-dimensional case, several processes are known. When the limit shape is curved, the limit process is the Airy_2 process [20, 26] (see also [12, 32] for non-determinantal models). On the other hand, when the limit shape is flat and the initial condition is non-random one observes the Airy_1 process [8, 9, 34] (see also [32, 41] for convergence to the KPZ fixed point [22] for non-determinantal models). This is still the case for random initial conditions, provided that the initial height function under diffusive scaling goes to zero as first discussed by Quastel and Remenik in [31].

In this paper we focus on the Airy_1 process, $\mathcal{A}_1$, discovered by Sasamoto in [34]. The one-point distribution is given by [2, 18]

$$\mathbb{P}(\mathcal{A}_1(t) \leq s) = F_1(2s), \tag{1.1}$$

where F_1 is GOE Tracy-Widom distribution [40], while the m-point joint distribution is given by a Fredholm determinant (see [9, 34] for explicit expressions). In

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this paper we study another observable of the Airy_1 process, namely the persistence probability, which is the probability that the Airy_1 process stays below a given threshold over a time span $[0, L]$. One expects that

$$P(c, L) = \mathbb{P}(\mathcal{A}_1(s) \leq s \text{ for all } s \in [0, L]) \sim Ce^{-\kappa(c)L} \tag{1.2}$$

for large L. Numerical computations using the method in [4] for small values of L indicates that the exponential form is quite accurate already for small values of L [16]. This is probably due to the fast (super-exponential) decay of the correlation of the Airy_1 process as noticed first numerically in [5] and recently proven in [3].

The starting point is of our analysis is the continuum statistics formula of the probability in (1.2) obtained by Quastel and Remenik [28] (see Theorem 2.1 below). For the Airy_2 process such a formulation was obtained by Corwin, Quastel and Remenik in [11], where they started by the expression of the joint distribution as a Fredholm determinant on a fixed space as in original paper of Prähofer and Spohn [26] (this is referred as path integral formula), see also [7, 22] for a general scheme to connect the two representations for other limit processes in the KPZ class. The formula for the joint distribution in terms of an extended kernel follows from a biorthogonalization procedure [9], which could be made explicit in [22], see also [23] for extensions. The continuum statistics occurred to be very useful to determine properties of the Airy processes [1, 28–30], but also in discrete analogues [19, 23].

In 2010, Takeuchi and Sano were able to verify experimentally the KPZ predictions in an experiment with turbulent nematic liquid crystals [37, 39], in particular for the distribution functions and covariances. In [38], they also measure the $\kappa(c)$ with respect to the threshold given by the average of the process. Later, applying the numerical method in [4] on continuum statistics of Airy_1, Ferrari and Frings [16] measured $\kappa(c)$ for more general c, but they could not provide any analytic results for $\kappa(c)$. The main result of this work is an analytic formula for $\kappa(c)$ which is the following theorem.

Theorem 1.1. *For any $c \geq \frac{3}{2}$ and L large enough, it holds*

$$\mathbb{P}\left(\mathcal{A}_1(s) \leq c,\ \ s \in [0, L]\right) = Ce^{-\kappa(c)L + \mathcal{O}(e^{-L})}, \tag{1.3}$$

where C does not depend on L and

$$\kappa(c) = -2\sum_{n=1}^{\infty} n^{-5/3}\mathrm{Ai}'(2n^{2/3}c), \tag{1.4}$$

where Ai' is the derivative of the Airy function Ai.

Using the fact that the Airy_1 process is a limit of the last passage percolation, where a FKG inequality can be applied, we can also show that $\kappa(c)$ exists for all values of c.

Proposition 1.2 (Existence of $\kappa(c)$)**.** *For any $c \in \mathbb{R}$,*

$$\kappa(c) = -\lim_{L\to\infty} \frac{\ln\left(\mathbb{P}\left(\mathcal{A}_1(t) \leq c, \forall t \in [0, L]\right)\right)}{L} \tag{1.5}$$

exists.

The lower bound on c in Theorem 1.1 is purely technical and it could potentially be slightly improved with the approach of this paper, however not below $c = 0$. Thus we did not pursue this aspect. The formula we obtain is analytic for all $c > 0$, but not at 0 or below. Denoting by $\tilde{\kappa}$ the analytic continuation of (1.4), we have the following result.

Proposition 1.3 (Analytic continuation of $\kappa(c)$). *The analytic continuation is given by*

$$\tilde{\kappa}(c) = \begin{cases} \kappa(c), & \text{if } c \geq 0, \\ \kappa(0) - \int_c^0 dx f(x) - 6c - \frac{48}{7}\sum_{n\geq 1}(c - c(n))\mathbf{1}_{c<c(n)}, & \text{if } c < 0, \end{cases} \tag{1.6}$$

where $c(n) = -(2n\pi/3)^{2/3}$ *for any* $n \in \mathbb{Z}_{\geq 1}$, $\kappa(c)$ *is define in* (1.4) *and*

$$f(x) = \frac{2}{\pi \mathrm{i}} \int_\Gamma dw \frac{w^2 e^{\frac{w^3}{3} - 2wx}}{1 - e^{\frac{w^3}{3} - 2wx}} \tag{1.7}$$

with $\Gamma = \{|r| e^{\mathrm{sgn}(r)\pi \mathrm{i}/3} | r \in \mathbb{R}\}$ *oriented with increasing imaginary part*[1].

Since we do not have an analytic proof that $\kappa(c)$ is analytic, we test whether the analytic continuation $\tilde{\kappa}(c)$ fits with the data obtained by numerical computations. As mentioned in [16] the kernel $K_{L,c}$ does not behaves well for large L: there are some off-diagonal entries which diverges super-exponentially in L. On top of it, some parts of the entries are highly oscillating. These two effects restrict very much the numerical implementation of the Fredholm determinant computation of [4], namely the values of L which can be simulated is (depending on the values of c) usually not more than $L = 3$. On the other hand, probably due to the fast decorrelation decay of the Airy_1 process [3], already for small values of L the logarithm of the persistence probability is already almost a perfect straight line, see for example Figure 1.

In [16] it was derived that, for $c \in \mathbb{R}$,

$$P(c, L) = \det\left(\mathbb{1} - B_0 + \Lambda_{L,c} e^{-L\Delta} B_0\right)_{L^2(\mathbb{R})}, \tag{1.8}$$

see Proposition 2.2 below for details. We numerically compute $P(c, L_n)$ for $L_n = 0.05n$ with $n \in \{1, 2, \ldots, 40\}$. Interpolating the obtained data $(L_n, \log P(c, L_n))$, we get a numerical estimate for the persistence exponent, $\hat{\kappa}(c)$, see Figure 1 for $c = 1$. For more values of $\hat{\kappa}(c)$ and details on the numerical issues regarding to the calculation of persistence probability, we refer the reader to Section 4 in [16].

In Figure 2 we compare $\hat{\kappa}(c)$ and analytic continuation of the persistence exponent $\tilde{\kappa}(c)$. The result indicates that the analytic continuation is likely to be the correct function. More information on the numerical data, see [17].

[1] In principle one can integrate explicitly $\int_c^0 dx f(x)$ and get a logarithm, see also (4.2). However, if one is not careful with the branch-cut of the logarithm, a numerical evaluation with Mathematica or similar programs leads to a non-smooth plot, unlike (1.6).

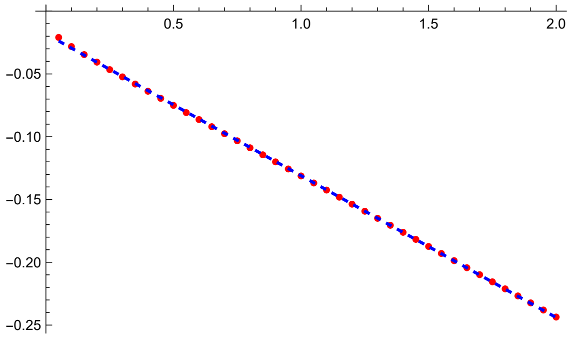

Figure 1: The red points are $(L_n, \log P(c, L_n))$ with $c = 1$ and $L_n = 0.05n$ with $n \in \{1, 2, \ldots, 40\}$, where $P(c, L_n)$ is calculated numerically. The dashed blue line is the reference line with slope -0.112, we refer the slope obtained in this way as $\hat{\kappa}(c)$.

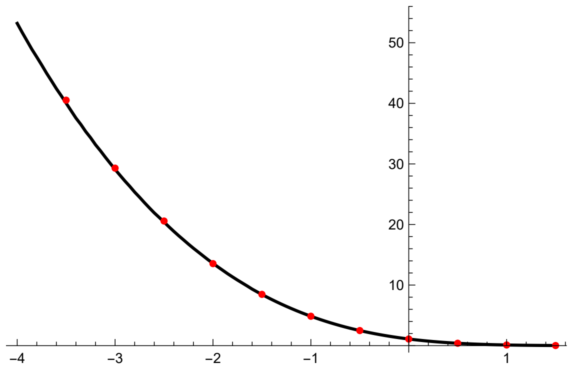

Figure 2: Comparison between numerical and theoretical exponent. The black line is the graph of $(c, \tilde{\kappa}(c))$ with $c \in (-4, 1.5)$ and the red points are the $(c, \hat{\kappa}(c))$ with $c \in \{-3.5, -3, \ldots, 1.5\}$ obtained by numerical calculation. We can see that the experimental data fits the theoretic data very well.

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2 Strategy and proof of Theorem 1.1

The starting point of the analysis is the result on the continuum statistics of Quastel and Remenik [28].

Theorem 2.1 (Theorem 4 of [28]). *It holds*

$$\mathbb{P}\left(\mathcal{A}_1(t) \leq g(t), 0 \leq t \leq L\right) = \det\left(\mathbb{1} - B_0 + \Lambda_{L,g} e^{-L\Delta} B_0\right)_{L^2(\mathbb{R})} \tag{2.1}$$

where g is a function in $H^1([0, L])$, Δ is the Laplacian, $B_0(x, y) = \mathrm{Ai}(x + y)$, and

$$\Lambda_{L,g}(x, y) = \frac{e^{-(x-y)^2/(4L)}}{\sqrt{4\pi L}} \mathbb{P}_{b(0)=x, b(L)=y}(b(s) \leq g(s), 0 \leq s \leq L) \tag{2.2}$$

with b a Brownian Bridge from x at time 0 to y at time L and with diffusion coefficient 2.

In order to state the persistence probability of a constant threshold c, we denote by P_0 the projection onto the interval $(-\infty, 0]$ and $\bar{P}_0 = \mathbb{1} - P_0$, furthermore, we define a new operator $e^{L\tilde{\Delta}}$ with kernel $e^{L\tilde{\Delta}}(x,y) = e^{L\Delta}(-x,y)$. For a constant threshold c, this was computed in [16] with the following result.

Proposition 2.2 (Proposition 2.1 of [16])**.** *For $c \in \mathbb{R}$ and $L > 0$, it holds*

$$\mathbb{P}(\mathcal{A}_1(s) \le c,\ \ s \in [0,L]) = \det\left(\mathbb{1} - K_{L,c}\right)_{L^2(\mathbb{R})}, \tag{2.3}$$

where $K_{L,c} = P_0 B_{0,c} + \tilde{K}_{L,c} + \hat{K}_{L,c}$ with $B_{0,c}(x,y) = \mathrm{Ai}(x+y+2c)$ and

$$\tilde{K}_{L,c} = \bar{P}_0 e^{L\Delta} P_0 e^{-L\Delta} B_{0,c}, \quad \hat{K}_{L,c} = \bar{P}_0 e^{L\tilde{\Delta}} \bar{P}_0 e^{-L\Delta} B_{0,c}. \tag{2.4}$$

Note that although the coefficient in front of Laplacian is negative, the operator $e^{-L\Delta} B_{0,c}$ is well-defined with kernel given by (see for instance [8,9,28,35])

$$e^{-L\Delta} B_{0,c}(x,y) = e^{-2L^3/3} e^{-L(x+y+2c)} \mathrm{Ai}(L^2 + x + y + 2c). \tag{2.5}$$

For later use, we also set

$$\tilde{B}_{0,c}(x,y) = \mathrm{Ai}(y - x + 2c), \quad \hat{B}_{0,c}(x,y) = \mathrm{Ai}(x - y + 2c). \tag{2.6}$$

Since the probability we are interested in goes to 0 as $L \to \infty$, the Fredholm determinant goes to 0 and as usual in these cases is the Fredholm series expansion not a good representation for the analysis. Instead, we are considering directly the logarithm of the persistence probability and use the trace expansion, namely

$$\ln(\det(\mathbb{1} - K_{L,c})) = -\sum_{n=1}^{\infty} \frac{\mathrm{Tr}(K_{L,c}^n)}{n}. \tag{2.7}$$

The strategy is to single out the terms in (2.7) which are linear in L (their sum will give $\kappa(c)$) and control all other terms. These terms will be bounded by using different norms after having multiplied by appropriate conjugations. These will be given by multiplication operators $U_r : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ with $U_r f(x) = e^{rx} f(x)$ for $r > 0$. The following standard results, see e.g. [36], will be constantly used throughout this work.

Theorem 2.3. *Let A, B be two operators. Then it holds*

1. $\mathrm{Tr}(U_r^{-1} A U_r) = \mathrm{Tr}(A)$,
2. $|\mathrm{Tr}(A)| \le \|A\|_1$,
3. $\max\{\|AB\|_1, \|AB\|_{\mathrm{HS}}\} \le \|A\|_{\mathrm{HS}} \|B\|_{\mathrm{HS}}$,
4. $\|AB\|_{\mathrm{HS}} \le \|A\|_{\mathrm{HS}} \|B\|_{\mathrm{op}}$,
5. $\|A\|_{\mathrm{op}} \le \|A\|_{\mathrm{HS}} \le \|A\|_1$,

whenever the r.h.s. are well-defined.

With those preliminary results in hand, we are able to state the main idea of how to get Theorem 1.1. For each $n \geq 1$, we will calculate $\operatorname{Tr}(K_{L,c}^n)$ with $K_{L,c}$ given in Proposition 2.2. In particular, the resulting terms can be divided into three categories: the one independent of L (for instance $\operatorname{Tr}((P_0 B_{0,c})^n)$, the one providing $\mathcal{O}(L)$ term (which comes from $\operatorname{Tr}(\hat{K}_{L,c}^n)$) and the rest error terms. Since as n goes to infinity, the number of error terms will also go to infinite, we need to make sure that their sum is finite. This is done by applying various inequalities of Theorem 2.3, more precisely, suppose A is an error term. Then we will find two other operators A_1, A_2 such that $\operatorname{Tr}(A) = \operatorname{Tr}(U_r A_1 U_r^{-1} \cdot U_r A_2 U_r^{-1})$, hence, we can apply Theorem 2.3 to obtain an upper bound for $|\operatorname{Tr}(A)|$.

Throughout this work we will consider $1 \leq r^2 \leq 2c$ with $r > 0$ and also define

$$\beta = \max\{2e^{r^3/3-2rc}, e^{(r-1/7)^3/3-2(r-1/7)c}\}. \tag{2.8}$$

Remark 2.4. *We will see that the absolute value of sum of all error terms is bounded by* $\sum_{n=1}^{\infty} \frac{7^n \beta^n}{n}$. *If we set* $r = \sqrt{2c}$, *then* $\beta \leq 2e^{-\frac{4\sqrt{2}c^{3/2}}{3}} < \frac{1}{7}$ *for* $c \geq \frac{3}{2}$, *which explains why we choose* $c \geq \frac{3}{2}$ *in Theorem 1.1.*

The term $e^{(r-1/7)^3/3-2(r-1/7)c}$ *and the prefactor 2 are purely technical. With a more delicate method, one can improve slightly this term, but it will not reduce the lower bound* $c \geq \frac{3}{2}$ *significantly, so we will not pursue in this aspect. For the estimate in the proof, the term* $e^{r^3/3-2rc}$ *is and the restriction* $0 < r^2 \leq 2c$ *are essentially. They are the main obstacle to generalizing our method to negative real number.*

2.1 Case $n = 1$

In order to illustrate the idea, let's first consider the $\operatorname{Tr}(K_{L,c}^n)$ with $n = 1$ and $L \geq 1$. We have

$$\operatorname{Tr}(K_{L,c}) = \operatorname{Tr}(P_0 B_{0,c}) + \operatorname{Tr}(\tilde{K}_{L,c}) + \operatorname{Tr}(\hat{K}_{L,c}) \tag{2.9}$$

with $\tilde{K}_{L,c}$ and $\hat{K}_{L,c}$ given in (2.4). Clearly, $\operatorname{Tr}(P_0 B_{0,c})$ does not depend on L and is finite by $0 \leq \operatorname{Ai}(x) \leq e^{-\frac{2x^{3/2}}{3}}$ for $x \geq 0$. Replacing $\bar{P}_0$ by $\mathbb{1} - P_0$ in the second term we have

$$\operatorname{Tr}(\tilde{K}_{L,c}) = \operatorname{Tr}(P_0 B_{0,c}) - \operatorname{Tr}(P_0 e^{L\Delta} P_0 e^{-L\Delta} B_{0,c}), \tag{2.10}$$

where we use cyclic property and the fact $e^{-L\Delta} B_{0,c} e^{L\Delta} = B_{0,c}$. The error term is then given by the second term on the right hand side of (2.10). In order to bound the error term, we define $A_1 = P_0 e^{L\Delta} P_0$ and $A_2 = P_0 e^{-L\Delta} B_{0,c} P_0$. Applying now $\|U_r^{-1} A_1 U_r^{-1}\|_{\mathrm{HS}} \leq \frac{1}{\sqrt{L}} \leq 1$ (by (A.24)), $\|U_r A_2 U_r\|_{\mathrm{HS}} \leq \beta e^{-\frac{4L^3}{3}} e^{-2Lc}$ (by (A.38)) and 3 of Theorem 2.3, we have

$$\begin{aligned} &|\operatorname{Tr}(P_0 e^{L\Delta} P_0 e^{-L\Delta} B_{0,c})| = |\operatorname{Tr}(U_r^{-1} A_1 U_r^{-1} U_r A_2 U_r)| \\ \leq &\|U_r^{-1} A_1 U_r^{-1}\|_{\mathrm{HS}} \|U_r A_2 U_r\|_{\mathrm{HS}} \leq \beta e^{-\frac{4L^3}{3}}. \end{aligned} \tag{2.11}$$

It remains to deal with $\mathrm{Tr}(\hat{K}_{L,c})$. Recall that heat kernel has the following integral representation: for any $L>0$,

$$e^{L\Delta}(x,y)=\frac{1}{2\pi i}\int_{i\mathbb{R}+\sigma} dv e^{Lv^2+v(x-y)} \tag{2.12}$$

where $\sigma\in\mathbb{R}$ can arbitrarily be chosen. Furthermore, using the integral representation of Airy function on (2.5), we obtain

$$e^{-L\Delta}B_{0,c}(x,y)=\frac{1}{2\pi i}\int_{i\mathbb{R}+\mu_1} dw e^{\frac{w^3}{3}-Lw^2-w(x+y+2c)} \tag{2.13}$$

under the condition $\mu_1>L$. A simple computation gives

$$\mathrm{Tr}(\bar{P}_0e^{L\tilde{\Delta}}\bar{P}_0e^{-L\Delta}B_{0,c})=\frac{1}{(2\pi i)^2}\int_{i\mathbb{R}+\mu_1}dw\int_{i\mathbb{R}+\mu_2}dv\frac{e^{\frac{w^3}{3}-Lw^2+Lv^2}}{(v-w)^2} \tag{2.14}$$

whenever $\mu_2>\mu_1>L$. We then deform the contour v to i$\mathbb{R}$ and taking care of the pole at $v=w$ by Cauchy's residue theorem, we obtain

$$\begin{aligned}\mathrm{Tr}(\bar{P}_0e^{L\tilde{\Delta}}\bar{P}_0e^{-L\Delta}B_{0,c})&=\frac{1}{2\pi i}\int_{i\mathbb{R}+2L}dw\mathrm{Res}\Big(\frac{e^{\frac{w^3}{3}-Lw^2+Lv^2}-2wc}{(v-w)^2}\Big|w=v\Big)\\&\quad+\frac{1}{(2\pi i)^2}\int_{i\mathbb{R}+2L}dw\int_{i\mathbb{R}}dv\frac{e^{\frac{w^3}{3}-Lw^2+Lv^2-2wc}}{(v-w)^2}\\&=-2L\mathrm{Ai}'(2c)+\mathrm{Tr}(P_0e^{L\tilde{\Delta}}P_0e^{-L\Delta}B_{0,c}).\end{aligned} \tag{2.15}$$

For the error term $\mathrm{Tr}(P_0e^{L\tilde{\Delta}}P_0e^{-L\Delta}B_{0,c})$, similarly as (2.11), we get

$$|\mathrm{Tr}(P_0e^{L\tilde{\Delta}}P_0e^{-L\Delta}B_{0,c})|\le e^{-\frac{4L^3}{3}}. \tag{2.16}$$

Summarizing, for $n=1$ we have obtained the following result.

Proposition 2.5. *For any $c,r>0$ with $1\le r^2\le 2c$, we have*

$$|\mathrm{Tr}(K_{L,c})-2\mathrm{Tr}(P_0B_{0,c})+2L\mathrm{Ai}'(2c)|\le e^{-\frac{4L^3}{3}},\quad \forall L\ge 1. \tag{2.17}$$

2.2 Leading term for $n\ge 2$ case

In the decomposition that we will do below of $\mathrm{Tr}(K_{L,c}^n)$ with general n, there will be one term given by $\mathrm{Tr}(\bar{P}_0e^{L\tilde{\Delta}}\hat{B}_{0,c}^n\bar{P}_0e^{-L\Delta}B_{0,c})$, which is up to error terms the term appearing in $\kappa(c)$. Since the decomposition is a bit lengthly, we first get a control on this term.

Lemma 2.6. *Let $n\in\mathbb{Z}_{\ge1}$, $L,r\ge 1$ and $r^2\le 2c$, it holds*

$$\big|\mathrm{Tr}(\bar{P}_0e^{L\tilde{\Delta}}\hat{B}_{0,c}^n\bar{P}_0e^{-L\Delta}B_{0,c})+2(n+1)^{-2/3}L\mathrm{Ai}'(2(n+1)^{2/3}c)\big|\le\beta^{n+1}e^{-\frac{4L^3}{3}}. \tag{2.18}$$

Proof. Using the integral representation of Airy function, we have (see (A.18))

$$e^{L\tilde{\Delta}}\hat{B}^n_{0,c}(x,y)=\frac{1}{2\pi i}\int_{i\mathbb{R}+\mu_2}dwe^{\frac{w^3}{3}+wn^{-1/3}(x+y)+n^{-2/3}Lw^2-2n^{2/3}c} \tag{2.19}$$

with arbitrary $\mu_2>0$. Together with (2.13), we obtain

$$\begin{aligned}&\operatorname{Tr}(\bar{P}_0e^{L\tilde{\Delta}}\hat{B}^n_{0,c}\bar{P}_0e^{-L\Delta}B_{0,c})\\ =&\frac{1}{(2\pi i)^2}\int_{i\mathbb{R}+\mu_1}dw_1\int_{i\mathbb{R}+\mu_2}dw_2\frac{e^{\frac{nw_1^3+w_2^3}{3}+L(w_1^2-w_2^2)-2w_1(n+1)c-2w_2c}}{(w_1-w_2)^2},\end{aligned} \tag{2.20}$$

with $\mu_1>\mu_2>0$. Deforming the contours to satisfy $\mu_2=2L$ and $\mu_1=0$ we get

$$\begin{aligned}&\operatorname{Tr}(\bar{P}_0e^{L\tilde{\Delta}}(\hat{B}_{0,c})^n\bar{P}_0e^{-L\Delta}B_{0,c})\\ &\quad=-2(n+1)^{-2/3}\mathrm{Ai}'(2(n+1)^{2/3}c)L+\operatorname{Tr}(P_0e^{L\tilde{\Delta}}\hat{B}^n_{0,c}P_0e^{-L\Delta}B_{0,c}),\end{aligned} \tag{2.21}$$

where the first term is coming from the residue at $w_1=w_2$ (in form of an integral representation of the derivative of the Airy function). Similarly as (2.11), we define $A_1=P_0e^{L\tilde{\Delta}}\hat{B}^n_{0,c}P_0$ and $A_2=P_0e^{-L\Delta}B_{0,c}P_0$. Applying now $\|U_r^{-1}A_1U_r^{-1}\|_{\mathrm{HS}}\le\beta^ne^{Lr^2}$ (by (A.41)), $\|U_rA_2U_r\|_{\mathrm{HS}}\le\beta e^{-\frac{4L^3}{3}}$ (by (A.38)) and $r^2\le 2c$, we have

$$|\operatorname{Tr}(P_0e^{L\tilde{\Delta}}\hat{B}^n_{0,c}P_0e^{-L\Delta}B_{0,c})|\le\|U_r^{-1}A_1U_r^{-1}\|_{\mathrm{HS}}\|U_rA_2U_r\|_{\mathrm{HS}}\le\beta^{n+1}e^{-\frac{4L^3}{3}}. \tag{2.22}$$

$\square$

2.3 Full expansion of $\operatorname{Tr}(K^n_{L,c})$ for $n\ge 2$

Using the same idea, we can also deduce the asymptotic behavior of $\operatorname{Tr}(K^n_{L,c})$ with $n\ge 2$. Using $e^{L\tilde{\Delta}}e^{-L\Delta}B_{0,c}=\tilde{B}_{0,c}$ (by (A.8)), we can decompose

$$K_{L,c}=K_u+K_v+K_w+K_d-K_e \tag{2.23}$$

where

$$\begin{aligned}K_u&=P_0B_{0,c},\quad K_v=e^{L\Delta}P_0e^{-L\Delta}B_{0,c},\\ K_w&=\tilde{B}_{0,c}-P_0\tilde{B}_{0,c}-e^{L\tilde{\Delta}}P_0e^{-L\Delta}B_{0,c},\\ K_d&=P_0e^{L\tilde{\Delta}}P_0e^{-L\Delta}B_{0,c},\ K_e=P_0e^{L\Delta}P_0e^{-L\Delta}B_{0,c}.\end{aligned} \tag{2.24}$$

For a word σ_n of length n, we say that $\alpha\in\sigma_n$ if it exists $i\in\{1,\dots,n\}$ such that $\sigma_n(i)=\alpha$. Also we introduce the notation

$$\mathcal{S}^A_{\sigma_n}=(-1)^{\#\{i|\sigma_n(i)\in A\}}, \tag{2.25}$$

where A is a subset of the letters of σ_n. With the above definitions we can rewrite

$$\begin{aligned}\operatorname{Tr}(K^n_{L,c})=&\sum_{\sigma_n\in\{u,v,w,d,e\}^n}\mathcal{S}^e_{\sigma_n}\operatorname{Tr}\left(\prod_{i=1}^nK_{\sigma_n(i)}\right)\\ =&\sum_{\sigma_n\in\{u,v,w\}^n}\operatorname{Tr}\left(\prod_{i=1}^nK_{\sigma_n(i)}\right)+\sum_{\substack{\sigma_n\in\{u,v,w,d,e\}^n\\ d\ or\ e\in\sigma_n}}\mathcal{S}^e_{\sigma_n}\operatorname{Tr}\left(\prod_{i=1}^nK_{\sigma_n(i)}\right).\end{aligned} \tag{2.26}$$

Besides K_u, K_v, K_w, K_d and K_e, we introduce further the following operators:

$$K_a = \tilde{B}_{0,c}, \;\; K_b = P_0 \tilde{B}_{0,c}, \;\; K_c = e^{L\tilde{\Delta}} P_0 e^{-L\Delta} B_{0,c} \tag{2.27}$$

so that $K_w = K_a - K_b - K_c$. First we control the last term in (2.26) as follows.

Lemma 2.7. *Let $n \geq 2$, $L \geq 1, r^2 \leq 2c$ with $r \geq 1$ and β given in (2.8), it holds*

$$\sum_{\substack{\sigma_n \in \{u,v,w,d,e\}^n \\ d \text{ or } e \in \sigma_n}} \left| \operatorname{Tr}\left(\prod_{i=1}^n K_{\sigma_n(i)} \right) \right| \leq 7^n e^{-\frac{4L^3}{3n^2}} \beta^n. \tag{2.28}$$

Proof. Since $K_w = K_a - K_b - K_c$, we have then

$$\sum_{\substack{\sigma_n \in \{u,v,w,d,e\}^n \\ d \text{ or } e \in \sigma_n}} \operatorname{Tr}\left(\prod_{i=1}^n K_{\sigma_n(i)} \right) = \sum_{\substack{\sigma_n \in \{u,v,a,b,c,d,e\}^n \\ d \text{ or } e \in \sigma_n}} \mathcal{S}^{b,c}_{\sigma_n} \operatorname{Tr}\left(\prod_{i=1}^n K_{\sigma_n(i)} \right). \tag{2.29}$$

Note that there are in total $7^n - 5^n$ many summations appearing on the right hand side, that is, the cardinality of the set $\{a,b,c,d,e,u,v\}^n \setminus \{a,b,c,u,v\}^n$. Hence, in order to prove the claim, we only need to bound the summation on the right hand side, to this end, we choose arbitrary $\sigma_n \in \{u,v,a,b,c,d,e\}^n$ with $e \in \sigma_n$. Using the cyclic property of trace, we can assume $\sigma_n(1) = e$. Define now

$$\Phi = \underbrace{P_0 e^{L\Delta} P_0}_{=:\varphi_1} \cdot \underbrace{P_0 e^{-L\Delta} B_{0,c} \left[\prod_{i=2}^n K_{\sigma_n(i)} \right] P_0}_{=:\varphi_2}. \tag{2.30}$$

Applying $\|U_r^{-1}\varphi_1 U_r^{-1}\|_{\mathrm{HS}} \leq \frac{1}{\sqrt{L}} \leq 1$ (by (A.24)) and $\|U_r \varphi_2 U_r\|_{\mathrm{HS}} \leq \beta^n e^{-\frac{4L^3}{3n^2}} e^{-2Lc}$ (by Corollary B.13), we have

$$|\operatorname{Tr}(\Phi)| = |\operatorname{Tr}(U_r^{-1}\Phi U_r)| \leq \|U_r^{-1}\varphi_1 U_r^{-1}\|_{\mathrm{HS}} \|U_r \varphi_2 U_r\|_{\mathrm{HS}} \leq \beta^n e^{-\frac{4L^3}{3n^2}}. \tag{2.31}$$

Similarly, we can also show the result for $d \in \sigma_n$, we only need to apply the transformation $P_0 e^{L\tilde{\Delta}} P_0 \mapsto U_r^{-1} P_0 e^{L\tilde{\Delta}} P_0 U_r^{-1}$ and use (A.24). ☐

Next we consider the first term of (2.26), namely

$$\begin{aligned}
&\sum_{\sigma_n \in \{u,v,w\}^n} \operatorname{Tr}\left(\prod_{i=1}^n K_{\sigma_n(i)} \right) = \operatorname{Tr}(K_u^n) + \operatorname{Tr}(K_v^n) + \operatorname{Tr}(K_w^n) \\
&+ \sum_{\substack{\sigma_n \in \{u,v\}^n \\ u,v \in \sigma_n}} \operatorname{Tr}\left(\prod_{i=1}^n K_{\sigma_n(i)} \right) + \sum_{\substack{\sigma_n \in \{u,w\}^n \\ u,w \in \sigma_n}} \left(\prod_{i=1}^n K_{\sigma_n(i)} \right) + \sum_{\substack{\sigma_n \in \{v,w\}^n \\ v,w \in \sigma_n}} \operatorname{Tr}\left(\prod_{i=1}^n K_{\sigma_n(i)} \right) \\
&+ \mathbb{1}_{n \geq 3} \sum_{\substack{\sigma_n \in \{u,v,w\}^n \\ u,v,w \in \sigma_n}} \operatorname{Tr}\left(\prod_{i=1}^n K_{\sigma_n(i)} \right).
\end{aligned} \tag{2.32}$$

In the next sections we analyze the terms in (2.32) one after the other.

2.3.1 Single terms

The first two terms in (2.32) do not depend on L and are easy to bound.

Lemma 2.8. *For any $n \in \mathbb{Z}_{\geq 2}$ we have $\operatorname{Tr}(K_u^n) = \operatorname{Tr}(K_v^n) = \operatorname{Tr}((P_0 B_{0,c})^n)$ and*

$$|\operatorname{Tr}((P_0 B_{0,c})^n)| \leq \beta^n. \tag{2.33}$$

Proof. By definition of K_u, we have $\operatorname{Tr}(K_u^n) = \operatorname{Tr}((P_0 B_{0,c})^n)$. As for $\operatorname{Tr}(K_v^n)$, using the definition of $K_v = e^{L\Delta} P_0 e^{-L\Delta} B_{0,c}$ and the fact that $B_{0,c}$ commute with $e^{L\Delta}$, we obtain $\operatorname{Tr}(K_v^n) = \operatorname{Tr}((P_0 B_{0,c})^n)$. It remains to prove the upper bound. By (A.20), we have $\|U_r P_0 B_{0,c} P_0 U_r^{-1}\|_{\mathrm{HS}} \leq \beta$. Since $n \geq 2$, we can apply Theorem 2.3 to deduce

$$|\operatorname{Tr}((P_0 B_{0,c})^n)| = |\operatorname{Tr}((U_r P_0 B_{0,c} P_0 U_r^{-1})^n)| \leq \|U_r P_0 B_{0,c} P_0 U_r^{-1}\|_{\mathrm{HS}}^n \leq \beta^n. \tag{2.34}$$

□

Now we need to consider $\operatorname{Tr}(K_w^n)$ with $n \geq 2$, which gives some terms of order 1 and some terms linear in L plus error terms as we will show in Proposition 2.13. Recall that

$$K_w = \tilde{B}_{0,c} - P_0 \tilde{B}_{0,c} - e^{L\tilde{\Delta}} P_0 e^{-L\Delta} B_{0,c}. \tag{2.35}$$

Using the cyclic property of the trace we get, for $n \geq 2$

$$\operatorname{Tr}(K_w^n) = \operatorname{Tr}(\bar{P}_0 \tilde{B}_{0,c} K_w^{n-1}) - \operatorname{Tr}(P_0 e^{-L\Delta} B_{0,c} K_w^{n-1} e^{L\tilde{\Delta}}). \tag{2.36}$$

We will start with the easy term, that is, the second term on the right hand side:

Lemma 2.9. *Let $n \in \mathbb{Z}_{\geq 2}, L, r \geq 1$ and $r^2 \leq 2c$, it holds*

$$|\operatorname{Tr}(P_0 e^{-L\Delta} B_{0,c} K_w^{n-1} e^{L\tilde{\Delta}}) - \operatorname{Tr}(P_0 \hat{B}_{0,c} (\bar{P}_0 \hat{B}_{0,c})^{n-1})| \leq 3^{n-1} \beta^n e^{-\frac{4L^3}{3n^2}} \tag{2.37}$$

and

$$|\operatorname{Tr}(P_0 \hat{B}_{0,c} (\bar{P}_0 \hat{B}_{0,c})^{n-1})| \leq \beta^n. \tag{2.38}$$

Proof. Let us start by deriving the bound (2.38). Applying (A.23) and Theorem (2.3), we have

$$\begin{aligned} &|\operatorname{Tr}(P_0 \hat{B}_{0,c} (\bar{P}_0 \hat{B}_{0,c})^{n-1})| \\ \leq &\|U_r P_0 \hat{B}_{0,c} \bar{P}_0 U_r^{-1}\|_{\mathrm{HS}} \|U_r \bar{P}_0 \hat{B}_{0,c} U_r^{-1}\|_{\mathrm{op}}^{n-2} \|U_r \bar{P}_0 \hat{B}_{0,c} P_0 U_r^{-1}\|_{\mathrm{HS}} \leq \beta^n. \end{aligned} \tag{2.39}$$

Next we show (2.37). Recall that $K_w = K_a - K_b - K_c$ and $K_a - K_c = e^{L\tilde{\Delta}} \bar{P}_0 e^{-L\Delta} B_{0,c}$. Hence,

$$\begin{aligned} &\operatorname{Tr}(P_0 e^{-L\Delta} B_{0,c} K_w^{n-1} e^{L\tilde{\Delta}} P_0) = \operatorname{Tr}(P_0 e^{-L\Delta} B_{0,c} (e^{L\tilde{\Delta}} \bar{P}_0 e^{-L\Delta} B_{0,c} - P_0 \tilde{B}_{0,c})^{n-1} e^{L\tilde{\Delta}} P_0) \\ &= \operatorname{Tr}(P_0 e^{-L\Delta} B_{0,c} (e^{L\tilde{\Delta}} \bar{P}_0 e^{-L\Delta} B_{0,c})^{n-1} e^{L\tilde{\Delta}} P_0) \\ &\quad + \sum_{\substack{\sigma_{n-1} \in \{a,b,c\}^{n-1} \\ b \in \sigma_{n-1}}} \mathcal{S}^{b,c}_{\sigma_{n-1}} \operatorname{Tr}\left(P_0 e^{-L\Delta} B_{0,c} \left[\prod_{i=1}^{n-1} K_{\sigma_{n-1}(i)} \right] e^{L\tilde{\Delta}} P_0 \right). \end{aligned} \tag{2.40}$$

Applying $e^{-L\Delta}B_{0,c}e^{L\tilde{\Delta}} = e^{-L\Delta}e^{L\Delta}\hat{B}_{0,c} = \hat{B}_{0,c}$ (by (A.10)), we have

$$\operatorname{Tr}(P_0 e^{-L\Delta}B_{0,c}(e^{L\tilde{\Delta}}\bar{P}_0 e^{-L\Delta}B_{0,c})^{n-1}e^{L\tilde{\Delta}}P_0) = \operatorname{Tr}(P_0\hat{B}_{0,c}(\bar{P}_0\hat{B}_{0,c})^{n-1}). \tag{2.41}$$

Now it remains to bound the sum in (2.40). Let now $\sigma_{n-1} \in \{a,b,c\}^n$ with $b \in \sigma_{n-1}$ and define

$$\begin{aligned}\Phi =& P_0 e^{-L\Delta}B_{0,c}\left[\prod_{i=1}^{n-1} K_{\sigma_{n-1}(i)}\right]e^{L\tilde{\Delta}}P_0\\ =& \underbrace{P_0 e^{-L\Delta}B_{0,c}\left[\prod_{i=1}^{\ell_b-1} K_{\sigma_{n-1}(i)}\right]P_0}_{=:\varphi_1} \cdot \underbrace{P_0\tilde{B}_{0,c}\left[\prod_{i=\ell_b+1}^{n-1} K_{\sigma_{n-1}(i)}\right]e^{L\tilde{\Delta}}P_0}_{=:\varphi_2},\end{aligned} \tag{2.42}$$

where $\ell_b = \max\{i|\sigma_{n-1}(i) = b\} \geq 1$. Applying $\|U_r\varphi_1 U_r\|_{\mathrm{HS}} \leq \beta^{\ell_b} e^{-\frac{4L^3}{3\ell_b^2}} e^{-2Lc}$ (by Corollary B.13), $\|U_r^{-1}\varphi_2 U_r^{-1}\|_{\mathrm{HS}} \leq \beta^{n-\ell_b}e^{Lr^2}$ (by Lemma B.3) and $r^2 \leq 2c$, we have

$$|\operatorname{Tr}(\Phi)| \leq \|U_r\varphi_1 U_r\|_{\mathrm{HS}}\|U_r^{-1}\varphi_2 U_r^{-1}\|_{\mathrm{HS}} \leq \beta^n e^{-\frac{4L^3}{3n^2}}. \tag{2.43}$$

Applying this and triangle inequality on (2.40), the result follows from the fact that there are in total $3^{n-1} - 2^{n-1}$ many summations. $\square$

It remains to consider $\operatorname{Tr}(\bar{P}_0\tilde{B}_{0,c}K_w^{n-1})$ in (2.36). Similarly as (2.40), we deduce

$$\begin{aligned}\operatorname{Tr}(\bar{P}_0\tilde{B}_{0,c}K_w^{n-1}) =& \operatorname{Tr}(\bar{P}_0\tilde{B}_{0,c}e^{L\tilde{\Delta}}(\bar{P}_0\hat{B}_{0,c})^{n-2}\bar{P}_0 e^{-L\Delta}B_{0,c})\\ &+ \sum_{\substack{\sigma_{n-1}\in\{a,b\}^{n-1}\\ b\in\sigma_{n-1}}} \mathcal{S}^b_{\sigma_{n-1}} \operatorname{Tr}\left(\bar{P}_0\tilde{B}_{0,c}\prod_{i=1}^{n-1}K_{\sigma_{n-1}(i)}\right)\\ &+ \sum_{\substack{\sigma_{n-1}\in\{a,b,c\}^{n-1}\\ b,c\in\sigma_{n-1}}} \mathcal{S}^{b,c}_{\sigma_{n-1}} \operatorname{Tr}\left(\bar{P}_0\tilde{B}_{0,c}\prod_{i=1}^{n-1}K_{\sigma_{n-1}(i)}\right).\end{aligned} \tag{2.44}$$

By definition, $K_a = \tilde{B}_{0,c}$ and $K_b = P_0\tilde{B}_{0,c}$, the term on the second line does not depend on L, hence for this term, it is enough to get an upper bound which is summable for $n \geq 1$. In Lemma 2.10 we get a bound for the terms in the second line, in Lemma 2.11 we bound the terms in the third line. Finally, in Lemma 2.12, we will show that the first term on the right hand side will provide $\mathcal{O}(L)$ term.

Lemma 2.10. *Let $n \in \mathbb{Z}_{\geq 2}$, $L, r \geq 1, r^2 \leq 2c$ and $\sigma_{n-1} \in \{a,b\}^{n-1}$ with $b \in \sigma_{n-1}$, then*

$$\left|\operatorname{Tr}\left(\bar{P}_0\tilde{B}_{0,c}\left[\prod_{i=1}^{n-1}K_{\sigma_{n-1}(i)}\right]\right)\right| \leq \beta^n. \tag{2.45}$$

Proof. Since $b \in \sigma_{n-1}$ we have $1 \leq \ell_b = \max\{i|\sigma_{n-1} = b\} \leq n-1$. This implies

$$\Phi = \bar{P}_0\tilde{B}_{0,c}\left[\prod_{i=1}^{n-1}K_{\sigma_{n-1}(i)}\right]\bar{P}_0 = \underbrace{\bar{P}_0\tilde{B}_{0,c}\left[\prod_{i=1}^{\ell_b-1}K_{\sigma_{n-1}(i)}\right]P_0}_{=:\varphi_1} \cdot \underbrace{P_0\tilde{B}_{0,c}^{n-\ell_b}\bar{P}_0}_{=:\varphi_2}. \tag{2.46}$$

Applying $\|U_r^{-1}\varphi_1 U_r\|_{\mathrm{HS}} \leq \beta^{\ell_b}$ (by Lemma B.2) and $\|U_r^{-1}\varphi_2 U_r\|_{\mathrm{HS}} \leq \beta^{n-\ell_b}$ (by (A.22)), we have

$$|\operatorname{Tr}(\Phi)| \leq \|U_r^{-1}\varphi_1 U_r\|_{\mathrm{HS}}\|U_r^{-1}\varphi_2 U_r\|_{\mathrm{HS}} \leq \beta^n. \tag{2.47}$$

□

Next we bound the traces of the terms on the last line of (2.44).

Lemma 2.11. *Let $n \in \mathbb{Z}_{\geq 2}$, $L, r \geq 1, r^2 \leq 2c$ and $\sigma_{n-1} \in \{a,b,c\}^{n-1}$ with $b, c \in \sigma_{n-1}$, then*

$$\left|\operatorname{Tr}\left(\bar{P}_0 \tilde{B}_{0,c}\left[\prod_{i=1}^{n-1} K_{\sigma_{n-1}(i)}\right]\right)\right| \leq 2\beta^n e^{-\frac{4L^3}{3n^2}}. \tag{2.48}$$

Proof. Replacing $\bar{P}_0 = \mathbb{1} - P_0$ we have the decomposition

$$\operatorname{Tr}\left(\bar{P}_0\tilde{B}_{0,c}\prod_{i=1}^{n-1} K_{\sigma_{n-1}(i)}\right) = \operatorname{Tr}\left(\tilde{B}_{0,c}\prod_{i=1}^{n-1} K_{\sigma_{n-1}(i)}\right) - \operatorname{Tr}\left(P_0\tilde{B}_{0,c}\prod_{i=1}^{n-1} K_{\sigma_{n-1}(i)}\right). \tag{2.49}$$

By assumption, there exists $i \in \{1,\dots,n-1\}$ such that $K_{\sigma_{n-1}(i)} = P_0\tilde{B}_{0,c}$, hence, due to the cyclic property of trace, it is enough to show that

$$\left|\operatorname{Tr}\left(P_0\tilde{B}_{0,c}\left[\prod_{i=1}^{n-1} K_{\sigma_{n-1}(i)}\right]P_0\right)\right| \leq \beta^n e^{-\frac{4L^3}{3n^2}} \tag{2.50}$$

for any $\sigma_{n-1} \in \{a,b,c\}^{n-1}$ with $c \in \sigma_{n-1}$. Define $\ell_c = \min\{i | \sigma_{n-1} = c\} \leq n-1$ and

$$\begin{aligned}\Phi =& P_0\tilde{B}_{0,c}\left[\prod_{i=1}^{n-1} K_{\sigma_{n-1}(i)}\right]P_0\\ =& \underbrace{P_0\tilde{B}_{0,c}\left[\prod_{i=1}^{\ell_c-1} K_{\sigma_{n-1}(i)}\right]e^{L\tilde{\Delta}}P_0}_{=:\varphi_1}\cdot\underbrace{P_0 e^{-L\Delta}B_{0,c}\left[\prod_{i=\ell_c+1}^{n-1} K_{\sigma_{n-1}(i)}\right]P_0}_{=:\varphi_2}.\end{aligned} \tag{2.51}$$

Applying $\|U_r^{-1}\varphi_1 U_r^{-1}\|_{\mathrm{HS}} \leq \beta^{\ell_c}e^{Lr^2}$ (by Lemma B.3), $\|U_r\varphi_2 U_r\|_{\mathrm{HS}} \leq \beta^{n-\ell_c}e^{-\frac{4L^3}{3(n-\ell_c)^2}}e^{-2Lc}$ (by Corollary B.13) and $r^2 \leq 2c$, we have

$$|\operatorname{Tr}(\Phi)| \leq \|U_r^{-1}\varphi_1 U_r^{-1}\|_{\mathrm{HS}}\|U_r\varphi_2 U_r\|_{\mathrm{HS}} \leq \beta^n e^{-\frac{4L^3}{3n^2}}. \tag{2.52}$$

Applying triangle inequality on (2.49), we then obtain the claimed result. □

It remains to consider the first term appearing on the right hand side of (2.44).

Lemma 2.12. *Let $n \in \mathbb{Z}_{\geq 2}, L, r \geq 1$ and $r^2 \leq 2c$, denote*

$$\tilde{\Phi} = \bar{P}_0\tilde{B}_{0,c}e^{L\tilde{\Delta}}(\bar{P}_0\hat{B}_{0,c})^{n-2}\bar{P}_0 e^{-L\Delta}B_{0,c}. \tag{2.53}$$

Then it holds

$$|\operatorname{Tr}(\tilde{\Phi}) - \kappa_n(c)L + \Psi_n^2| \le n^2\beta^n e^{-\frac{4L^3}{3n^2}}, \tag{2.54}$$

where $\kappa_n(c) = -2n^{-2/3}\mathrm{Ai}'(2n^{2/3}c)$ *and*

$$\Psi_n^2 = \mathbb{1}_{n\ge 3}\sum_{j=2}^{n-1}\operatorname{Tr}(P_0\hat{B}_{0,c}(\bar{P}_0\hat{B}_{0,c})^{n-1-j}\bar{P}_0\hat{B}_{0,c}^j) \quad \textit{with} \quad |\Psi_n^2| \le n\beta^n. \tag{2.55}$$

Proof. We first claim that for any two operators A, B and $n \ge 0$, we have

$$\bar{P}_0A(\bar{P}_0B)^n = \bar{P}_0AB^n - \sum_{j=0}^{n-1} AB^jP_0B(\bar{P}_0B)^{n-(j+1)} + \sum_{j=0}^{n-1} P_0AB^jP_0B(\bar{P}_0B)^{n-(j+1)}. \tag{2.56}$$

We show this via induction on n. The case $n = 0$ is trivial. For $n = 1$, we have

$$\bar{P}_0A\bar{P}_0B = P_0AP_0B - AP_0B + \bar{P}_0AB. \tag{2.57}$$

For induction step $n-1 \mapsto n$, we have then

$$\bar{P}_0A(\bar{P}_0B)^n = P_0AP_0B(\bar{P}_0B)^{n-1} - AP_0B(\bar{P}_0B)^{n-1} + \bar{P}_0AB(\bar{P}_0B)^{n-1}. \tag{2.58}$$

Applying induction assumption on the last term, we obtain

$$\begin{aligned}
&\bar{P}_0AB(\bar{P}_0B)^{n-1}\\
=&\bar{P}_0ABB^{n-1} - \sum_{j=0}^{n-2} ABB^jP_0B(\bar{P}_0B)^{n-1-(j+1)} + \sum_{j=0}^{n-2} P_0ABB^jP_0B(\bar{P}_0B)^{n-1-(j+1)}\\
=&\bar{P}_0AB^n - \sum_{j=1}^{n-1} AB^jP_0B(\bar{P}_0B)^{n-1-j} + \sum_{j=1}^{n-1} P_0AB^jP_0B(\bar{P}_0B)^{n-1-j}.
\end{aligned} \tag{2.59}$$

Plugging this back to (2.58), we obtain then (2.56). Applying now (2.56) with $A = \tilde{B}_{0,c}e^{L\tilde{\Delta}}$ and $B = \hat{B}_{0,c}$, we then obtain

$$\begin{aligned}
&\operatorname{Tr}(\tilde{\Phi}) = \operatorname{Tr}(\bar{P}_0\tilde{B}_{0,c}e^{L\tilde{\Delta}}(\bar{P}_0\hat{B}_{0,c})^{n-2}\bar{P}_0e^{-L\Delta}B_{0,c})\\
=&\operatorname{Tr}(\bar{P}_0\tilde{B}_{0,c}e^{L\tilde{\Delta}}\hat{B}_{0,c}^{n-2}\bar{P}_0e^{-L\Delta}B_{0,c}) - \sum_{j=0}^{n-3}\operatorname{Tr}(\tilde{B}_{0,c}e^{L\tilde{\Delta}}\hat{B}_{0,c}^jP_0\hat{B}_{0,c}(\bar{P}_0\hat{B}_{0,c})^{n-2-(j+1)}\bar{P}_0e^{-L\Delta}B_{0,c})\\
+&\sum_{j=0}^{n-3}\operatorname{Tr}(P_0\tilde{B}_{0,c}e^{L\tilde{\Delta}}\hat{B}_{0,c}^jP_0\hat{B}_{0,c}(\bar{P}_0\hat{B}_{0,c})^{n-2-(j+1)}\bar{P}_0e^{-L\Delta}B_{0,c})
\end{aligned} \tag{2.60}$$

Using the identity $e^{L\tilde{\Delta}}\hat{B}_{0,c} = \tilde{B}_{0,c}e^{L\tilde{\Delta}}$ (by (A.12)), $e^{-L\Delta}B_{0,c}\tilde{B}_{0,c}e^{L\tilde{\Delta}} = \hat{B}_{0,c}^2$ (by (A.6)) and cyclic property of trace we obtain

$$\begin{aligned}
\operatorname{Tr}(\tilde{\Phi}) =& \operatorname{Tr}(\bar{P}_0\tilde{B}_{0,c}^{n-1}e^{L\tilde{\Delta}}\bar{P}_0e^{-L\Delta}B_{0,c}) - \sum_{j=2}^{n-1}\operatorname{Tr}(P_0\hat{B}_{0,c}(\bar{P}_0\hat{B}_{0,c})^{n-1-j}\bar{P}_0\hat{B}_{0,c}^j)\\
&+\sum_{j=1}^{n-2}\operatorname{Tr}(P_0\tilde{B}_{0,c}^je^{L\tilde{\Delta}}P_0\hat{B}_{0,c}(\bar{P}_0\hat{B}_{0,c})^{n-2-j}\bar{P}_0e^{-L\Delta}B_{0,c}).
\end{aligned} \tag{2.61}$$

By Lemma 2.6, we have

$$|\operatorname{Tr}(\bar{P}_0\tilde{B}_{0,c}^{n-1}e^{L\tilde{\Delta}}\bar{P}_0e^{-L\Delta}B_{0,c}) - 2n^{-2/3}\mathrm{Ai}'(2n^{2/3}c)L| \le \beta^n e^{-\frac{4L^3}{3}}. \tag{2.62}$$

The first sum in (2.61) is just Ψ_n^2 in (2.55), which is independent of L. In order to prove $|\Psi_n^2| \le n\beta^n$, we apply bounds (A.23) for each $j \in \{2,3,\dots,n-1\}$ to deduce

$$\begin{aligned} &|\operatorname{Tr}(P_0\hat{B}_{0,c}(\bar{P}_0\hat{B}_{0,c})^{n-1-j}\bar{P}_0\hat{B}_{0,c}^j)| \\ &\le \|U_rP_0\hat{B}_{0,c}\bar{P}_0U_r^{-1}\|_{\mathrm{HS}}\|U_r\bar{P}_0\hat{B}_{0,c}\bar{P}_0U_r^{-1}\|_{\mathrm{op}}^{n-1-j}\|U_r\bar{P}_0\hat{B}_{0,c}^jP_0U_r^{-1}\|_{\mathrm{HS}} \le \beta^n. \end{aligned} \tag{2.63}$$

The bound on $|\Psi_n^2|$ follows directly from (2.63) and triangle inequality. It remains to bound the last sum in (2.61). Let $j \in \{1,\dots,n-2\}$ and define

$$\begin{aligned} \Phi =& P_0\tilde{B}_{0,c}^je^{L\tilde{\Delta}}P_0\hat{B}_{0,c}(\bar{P}_0\hat{B}_{0,c})^{n-2-j}\bar{P}_0e^{-L\Delta}B_{0,c} \\ =& P_0\tilde{B}_{0,c}^je^{L\tilde{\Delta}}P_0\hat{B}_{0,c}(\bar{P}_0\hat{B}_{0,c})^{n-2-j}e^{-L\Delta}B_{0,c} \\ &- P_0\tilde{B}_{0,c}^je^{L\tilde{\Delta}}P_0\hat{B}_{0,c}(\bar{P}_0\hat{B}_{0,c})^{n-2-j}P_0e^{-L\Delta}B_{0,c}. \end{aligned} \tag{2.64}$$

Now we claim that for any $n \ge 1$, we have

$$(\bar{P}_0\hat{B}_{0,c})^n = \hat{B}_{0,c}^n - \sum_{i=1}^{n}(\bar{P}_0\hat{B}_{0,c})^{n-i}P_0\hat{B}_{0,c}^i. \tag{2.65}$$

We show this via induction on n. For $n=1$, we have $\bar{P}_0\hat{B}_{0,c} = \hat{B}_{0,c} - P_0\hat{B}_{0,c}$. For induction step $n-1 \mapsto n$, we have then

$$\begin{aligned} (\bar{P}_0\hat{B}_{0,c})^n =& (\bar{P}_0\hat{B}_{0,c})^{n-1}\hat{B}_{0,c} - (\bar{P}_0\hat{B}_{0,c})^{n-1}P_0\hat{B}_{0,c} \\ =& \left(\hat{B}_{0,c}^{n-1} - \sum_{i=1}^{n-1}(\bar{P}_0\hat{B}_{0,c})^{n-1-i}P_0\hat{B}_{0,c}^i\right)\hat{B}_{0,c} - (\bar{P}_0\hat{B}_{0,c})^{n-1}P_0\hat{B}_{0,c} \\ =& \hat{B}_{0,c}^n - \sum_{i=1}^{n}(\bar{P}_0\hat{B}_{0,c})^{n-i}P_0\hat{B}_{0,c}^i. \end{aligned} \tag{2.66}$$

Applying now (2.65) on the second line of (2.64), we have then

$$\begin{aligned} \Phi =& P_0\tilde{B}_{0,c}^je^{L\tilde{\Delta}}P_0\hat{B}_{0,c}^{n-1-j}e^{-L\Delta}B_{0,c} \\ &- \sum_{i=1}^{n-2-j}P_0\tilde{B}_{0,c}^je^{L\tilde{\Delta}}P_0\hat{B}_{0,c}(\bar{P}_0\hat{B}_{0,c})^{n-2-j-i}P_0\hat{B}_{0,c}^ie^{-L\Delta}B_{0,c} \\ &- P_0\tilde{B}_{0,c}^je^{L\tilde{\Delta}}P_0\hat{B}_{0,c}(\bar{P}_0\hat{B}_{0,c})^{n-2-j}P_0e^{-L\Delta}B_{0,c} \\ =& P_0\tilde{B}_{0,c}^je^{L\tilde{\Delta}}P_0\hat{B}_{0,c}^{n-1-j}e^{-L\Delta}B_{0,c} \\ &- \sum_{i=0}^{n-2-j}P_0\tilde{B}_{0,c}^je^{L\tilde{\Delta}}P_0\hat{B}_{0,c}(\bar{P}_0\hat{B}_{0,c})^{n-2-j-i}P_0\hat{B}_{0,c}^ie^{-L\Delta}B_{0,c}. \end{aligned} \tag{2.67}$$

Note that for any $p \ge 1, q \ge 0$ we have $\|U_r^{-1}P_0\tilde{B}_{0,c}^pe^{L\tilde{\Delta}}P_0U_r^{-1}\|_{\mathrm{HS}} \le \beta^pe^{Lr^2}$ by (A.41), $\|U_rP_0\hat{B}_{0,c}^qe^{-L\Delta}B_{0,c}U_r\|_{\mathrm{HS}} \le \beta^{q+1}e^{-\frac{4L^3}{3(q+1)^2}}e^{-2Lc}$ by (A.38) and $\|\hat{U}_rB_{0,c}U_r^{-1}\|_{\mathrm{op}} \le \beta$

by (A.23). Applying those upper bounds, Theorem 2.3 and assumption $r^2 \le 2c$ on (2.67), we have then

$$|\operatorname{Tr}(\Phi)| = |\operatorname{Tr}(U_r^{-1}\Phi U_r)| \le \beta^n e^{-\frac{4L^3}{3n^2}} + \sum_{i=0}^{n-2-j} \beta^n e^{-\frac{4L^3}{3n^2}} \le n\beta^n e^{-\frac{4L^3}{3n^2}} \tag{2.68}$$

Plugging this back to (2.61), we obtain the claimed results. □

Hence, we have the following result for $\operatorname{Tr}(K_w^n)$:

Proposition 2.13. *Let $n \in \mathbb{Z}_{\ge 2}, L, r \ge 1$ and $r^2 \le 2c$, it holds*

$$|\operatorname{Tr}(K_w^n) - (\kappa_n(c)L + \Psi_n^1 - \Psi_n^2 - \Psi_n^3)| \le 3^{n+1}\beta^n e^{-\frac{4L^3}{3n^2}}, \tag{2.69}$$

where $\kappa_n(c) = -2n^{-2/3}\operatorname{Ai}'(2n^{2/3}c)$ and

$$\begin{aligned}
\Psi_n^1 &= \sum_{\substack{\sigma_{n-1}\in\{a,b\}^{n-1}\\ b\in\sigma_{n-1}}} \mathcal{S}^b_{\sigma_{n-1}} \operatorname{Tr}\left(\bar{P}_0\tilde{B}_{0,c}\left[\prod_{i=1}^{n-1} K_{\sigma_{n-1}(i)}\right]\right),\\
\Psi_n^2 &= \mathbb{1}_{n\ge 3}\sum_{j=2}^{n-1} \operatorname{Tr}(P_0\hat{B}_{0,c}(\bar{P}_0\hat{B}_{0,c})^{n-1-j}\bar{P}_0\hat{B}_{0,c}^j),\\
\Psi_n^3 &= \operatorname{Tr}(P_0\hat{B}_{0,c}(\bar{P}_0\hat{B}_{0,c})^{n-1}).
\end{aligned} \tag{2.70}$$

In particular,

$$\sum_{n=1}^{\infty} \frac{|\Psi_n^1| + |\Psi_n^2| + |\Psi_n^3|}{n} \le \sum_{n=1}^{\infty} \frac{2^n\beta^n + n\beta^n + \beta^n}{n} < \infty. \tag{2.71}$$

Proof. Combining results in Lemma 2.9, 2.11 and 2.12 we have

$$\begin{aligned}
&|\operatorname{Tr}(K_w^n) - (\kappa_n(c)L + \Psi_n^1 - \Psi_n^2 - \Psi_n^3)|\\
&\le 3^{n-1}\beta^n e^{-\frac{4L^3}{3n^2}} + 2\cdot 3^{n-1}\beta^n e^{-\frac{4L^3}{3n^2}} + n^2\beta^n e^{-\frac{4L^3}{3n^2}} \le 3^{n+1}\beta^n e^{-\frac{4L^3}{3n^2}}.
\end{aligned} \tag{2.72}$$

(2.71) follows from (2.38), (2.45) and (2.55). □

2.3.2 Mixed u, v Terms

Lemma 2.14. *Let $n \in \mathbb{Z}_{\ge 2}, L, r \ge 1$, $r^2 \le 2c$ and $\sigma_n \in \{u,v\}^n$ with $u, v \in \sigma_n$, then*

$$\left|\operatorname{Tr}\left(\prod_{i=1}^{n} K_{\sigma_n(i)}\right)\right| \le \beta^n e^{-\frac{4L^3}{3}}. \tag{2.73}$$

Proof. Choose $\sigma_n \in \{u,v\}^n$ with $u, v \in \sigma_n$. Using the cyclic property, we can assume without loss of generality $\sigma_n(1) = u$. Since $v \in \sigma_n$, it holds $2 \le \ell_v =$

$\max\{i|\sigma_n(i)=v\}\leq n$. Then we have

$$\begin{aligned}\Phi =&P_0B_{0,c}\left[\prod_{i=2}^{n}K_{\sigma_n(i)}\right]P_0\\ =&\underbrace{P_0B_{0,c}\left[\prod_{i=2}^{\ell_v-1}K_{\sigma_n(i)}\right]e^{L\Delta}P_0}_{=:\varphi_1}\cdot\underbrace{P_0e^{-L\Delta}B_{0,c}P_0}_{=:\varphi_2}\underbrace{(P_0B_{0,c})^{n-\ell_v}P_0}_{=:\varphi_3}.\end{aligned}\tag{2.74}$$

Applying $\|U_r^{-1}\varphi_1U_r^{-1}\|_{\mathrm{HS}}\leq\beta^{\ell_v-1}e^{Lr^2}$ (by Lemma B.14), $\|U_r\varphi_2U_r\|_{\mathrm{HS}}\leq\beta e^{-\frac{4L^3}{3}}$ (by (A.38)), $\|U_r^{-1}\varphi_3U_r\|_{\mathrm{HS}}\leq\beta^{n-\ell_v}$ (by (A.20)) and $r^2\leq 2c$, we have

$$|\operatorname{Tr}(\Phi)|\leq\|U_r^{-1}\varphi_1U_r^{-1}\|_{\mathrm{HS}}\|U_r\varphi_2U_r\|_{\mathrm{HS}}\|U_r^{-1}\varphi_3U_r\|_{\mathrm{HS}}\leq\beta^ne^{-\frac{4L^3}{3}}\tag{2.75}$$

□

2.3.3 Mixed u,w Terms

First, we define a new operator $K_{\tilde w}=\bar P_0\tilde B_{0,c}$. For a word $\sigma_n\in\{u,w\}^n$, we define

$$\sigma_n^{w\mapsto\tilde w}(i)=\begin{cases}u, & \text{if } \sigma_n(i)=u,\\ \tilde w, & \text{otherwise.}\end{cases}\tag{2.76}$$

Lemma 2.15. *Let $n\in\mathbb{Z}_{\geq2}$, $L,r\geq1$, $r^2\leq 2c$ and $\sigma_n\in\{u,w\}^n$ with $u,w\in\sigma_n$, it holds*

$$\left|\operatorname{Tr}\left(\prod_{i=1}^nK_{\sigma_n^{w\mapsto\tilde w}(i)}\right)\right|\leq\beta^n\tag{2.77}$$

and

$$\left|\operatorname{Tr}\left(\prod_{i=1}^nK_{\sigma_n(i)}\right)-\operatorname{Tr}\left(\prod_{i=1}^nK_{\sigma_n^{w\mapsto\tilde w}(i)}\right)\right|\leq3^n\beta^ne^{-\frac{4L^3}{3n^2}}.\tag{2.78}$$

Proof. We first show (2.77), let $\hat\sigma_n\in\{u,\tilde w\}^n$ with $u,\tilde w\in\hat\sigma_n$, using cyclyc property, we assume without loss of generality that $\hat\sigma_n(1)=u$. We define

$$\Phi_{\hat\sigma_n}=P_0B_{0,c}\left[\prod_{i=2}^nK_{\hat\sigma_n(i)}\right]P_0.\tag{2.79}$$

Instead of (2.77), we will use induction to show $\max\{|\operatorname{Tr}(\Phi_{\hat\sigma_n})|,\|U_r^{-1}\Phi_{\hat\sigma_n}U_r\|_{\mathrm{HS}}\}\leq\beta^n$. For $n=2$, we have then $\Phi_{\hat\sigma_2}=P_0B_{0,c}\bar P_0\tilde B_{0,c}P_0$, we can then apply the bounds (A.20) and (A.22) to deduce

$$\max\{|\operatorname{Tr}(\Phi_{\hat\sigma_2})|,\|U_r^{-1}\Phi_{\hat\sigma_2}U_r\|_{\mathrm{HS}}\}\leq\|U_r^{-1}P_0B_{0,c}\bar P_0U_r\|_{\mathrm{HS}}\|U_r^{-1}\bar P_0\tilde B_{0,c}P_0U_r\|_{\mathrm{HS}}\leq\beta^2.\tag{2.80}$$

For induction step $n-1\mapsto n$. Let us start with the case $\hat\sigma_n(i)=\tilde w$ for all $i\in\{2,\dots,n\}$, then $\Phi_{\hat\sigma_n}=P_0B_{0,c}\bar P_0\cdot(\bar P_0\tilde B_{0,c}\bar P_0)^{n-2}\bar P_0\tilde B_{0,c}P_0$, then we can apply the bounds (A.20), (A.22) respectively. Then we have

$$|\operatorname{Tr}(\Phi_{\hat\sigma_n})|\leq\|U_r^{-1}P_0B_{0,c}\bar P_0U_r\|_{\mathrm{HS}}\|U_r^{-1}\bar P_0\tilde B_{0,c}\bar P_0U_r\|_{\mathrm{op}}^{n-2}\|U_r^{-1}\bar P_0\tilde B_{0,c}P_0U_r\|_{\mathrm{HS}}\leq\beta^n.\tag{2.81}$$

Next suppose now there exists $i \in \{2, \dots, n\}$ such that $\sigma_n(i) = u$. Then we have $2 \le \ell_u = \min\{i | \sigma_n(i) = u\} \le n$ and hence

$$\Phi_{\hat{\sigma}_n} = \underbrace{P_0 B_{0,c}\left[\prod_{i=2}^{\ell_u - 1} K_{\sigma_n(i)}\right] P_0}_{=:\varphi_1} \cdot \underbrace{P_0 B_{0,c}\left[\prod_{i=\ell_u+1}^{n} K_{\sigma_n(i)}\right] P_0}_{=:\varphi_2} . \tag{2.82}$$

By induction assumption, we have $\|U_r^{-1}\varphi_1 U_r\|_{\mathrm{HS}} \le \beta^{\ell_u - 1}$, $\|U_r^{-1}\varphi_2 U_r\|_{\mathrm{HS}} \le \beta^{n-\ell_u}$, the claim follows.

It remains to show (2.78), let now $\sigma_n \in \{u, w\}^n$, using cyclic property, we assume $\sigma_n(1) = u$. Then there exists $m \ge 1$, $q_m \ge 1$ such that σ_n is given as following:

$$\underbrace{u, \dots, u}_{p_1 \text{times}}, \underbrace{w, \dots, w}_{q_1 \text{times}}, \cdots, \underbrace{u, \dots, u}_{p_m \text{times}}, \underbrace{w, \dots, w}_{q_m \text{times}} . \tag{2.83}$$

Denote

$$\Phi = P_0 B_{0,c}\left[\prod_{i=2}^{n} K_{\sigma_n(i)}\right] P_0 = \prod_{i=1}^{m}\left[(P_0 B_{0,c})^{p_i}(\bar{P}_0 \tilde{B}_{0,c} - e^{L\tilde{\Delta}} P_0 e^{-L\Delta} B_{0,c})^{q_i}\right] . \tag{2.84}$$

Recall that $K_c = e^{L\tilde{\Delta}} P_0 e^{-L\Delta} B_{0,c}$, together with (2.84) and cyclic property, we have then

$$\begin{aligned} &\left| \operatorname{Tr}(\Phi) - \operatorname{Tr}\left(\prod_{i=1}^{m} (P_0 B_{0,c})^{p_i} (\bar{P}_0 \tilde{B}_{0,c})^{q_i} P_0\right) \right| \\ &\le \sum_{\sigma_{q_1} \in \{a,b,c\}^{q_1}} \cdots \sum_{\sigma_{q_m} \in \{a,b,c\}^{q_m}} \mathbb{1}_{\{\exists\ i \text{ s.t. } c \in \sigma_{q_i}\}} |\operatorname{Tr}(\Phi_m)| , \end{aligned} \tag{2.85}$$

where

$$\Phi_m = \prod_{i=1}^{m}\left((P_0 B_{0,c})^{p_i}\left[\prod_{j=1}^{q_i} K_{\sigma_{q_i}(j)}\right] P_0\right). \tag{2.86}$$

Since there exists i such that $c \in \sigma_{q_i}$, we can rewrite

$$\Phi_m = (P_0 B_{0,c})^{p_1 - 1} \underbrace{P_0 B_{0,c}\left[\prod_{i=1}^{\ell_c - 1} K_i\right] e^{L\tilde{\Delta}} P_0}_{=:\varphi_1} \cdot \underbrace{P_0 e^{-L\Delta} B_{0,c}\left[\prod_{i=\ell_c+1}^{n-p_1} K_i\right] P_0}_{=:\varphi_2} \tag{2.87}$$

with $K_i \in \{K_u, K_a, K_b, K_c\}$ and

$$\ell_c = q_1 + \cdots + p_i + \min\{j | \sigma_{q_i}(j) = c\}. \tag{2.88}$$

Applying the bounds $\|U_r^{-1} P_0 B_{0,c} U_r\|_{\mathrm{HS}} \le \beta$ (by (A.20)), $\|U_r^{-1}\varphi_1 U_r^{-1}\|_{\mathrm{HS}} \le \beta^{\ell_c} e^{Lr^2}$ (Corollary B.15), $\|U_r \varphi_2 U_r\|_{\mathrm{HS}} \le \beta^{n-p_1-\ell_c+1} e^{-\frac{4L^3}{3(n-p_1-\ell_c+1)^2}} e^{-2Lc}$ (Corollary B.13) and $r^2 \le 2c$, we then have

$$|\operatorname{Tr}(U_r^{-1}\Phi_m U_r)| \le \|U_r^{-1} P_0 B_{0,c} U_r\|_{\mathrm{HS}}^{p_1 - 1} \|U_r^{-1}\varphi_1 U_r^{-1}\|_{\mathrm{HS}} \|U_r \varphi_2 U_r\|_{\mathrm{HS}} \le \beta^n e^{-\frac{4L^3}{3n^2}}, \tag{2.89}$$

Plugging this back to (2.85) and using the fact there are in total $3^{q_1+\cdots+q_m} - 2^{q_1+\cdots+q_m} \le 3^n$ summations, the proof is completed. ☐

2.3.4 Mixed v, w Terms

Let $\sigma_n \in \{v,w\}^n$ and without loss of generality we can set $\sigma_n(1) = v$. Denote $K_\alpha = P_0\hat{B}_{0,c}$, $K_{\tilde{\beta}} = \bar{P}_0\hat{B}_{0,c}$ and $K_\gamma = \bar{P}_0 B_{0,c}$. For a fixed $\sigma_n \in \{v,w\}^n$, we define its transformed word as

$$\sigma_n^{w\to\tilde{\beta}}(i) = \begin{cases} u, & \text{if } \sigma_n(i) = \sigma_n(i+1) = v, \\ \tilde{\beta}, & \text{if } \sigma_n(i) = \sigma_n(i+1) = w, \\ \gamma, & \text{if } \sigma_n(i) = w, \sigma_n(i+1) = v, \\ \alpha, & \text{if } \sigma_n(i) = v, \sigma_n(i+1) = w, \end{cases} \tag{2.90}$$

where we set $\sigma_n(n+1) = \sigma_n(1)$.

Lemma 2.16. *Let $n \in \mathbb{Z}_{\geq 2}, L, r \geq 1$, $r^2 \leq 2c$ and $\sigma_n \in \{v,w\}^n$ with $v, w \in \sigma_n$, it holds*

$$\left| \operatorname{Tr}\left(\prod_{i=1}^n K_{\sigma_n^{w\mapsto\tilde{\beta}}(i)} \right) \right| \leq \beta^n \tag{2.91}$$

and

$$\left| \operatorname{Tr}\left(\prod_{i=1}^n K_{\sigma_n(i)} \right) - \operatorname{Tr}\left(\prod_{i=1}^n K_{\sigma_n^{w\mapsto\tilde{\beta}}(i)} \right) \right| \leq 3^n \beta^n e^{-\frac{4L^3}{3n^2}}. \tag{2.92}$$

Proof. One can obtain (2.91) using the same method for (2.77), so we omit the proof here. Let $\sigma_n \in \{v,w\}^n$ with $v, w \in \sigma_n$. By cyclic property, we can assume $\sigma_n(1) = v$. In particular, there exists $m \geq 1, q_m \geq 1$ such that σ_n is given as

$$\underbrace{v,\ldots,v}_{p_1\ \textit{many}}, \underbrace{w,\ldots,w}_{q_1\ \textit{many}}, \cdots, \underbrace{v,\ldots,v}_{p_m\ \textit{many}}, \underbrace{w,\ldots,w}_{q_m\ \textit{many}} \tag{2.93}$$

In this case we have

$$\begin{aligned} &\operatorname{Tr}\left(\prod_{i=1}^m \left[e^{L\Delta}(P_0 B_{0,c})^{p_i-1} P_0 e^{-L\Delta} B_{0,c} (\tilde{B}_{0,c} - P_0\tilde{B}_{0,c} - e^{L\tilde{\Delta}} P_0 e^{-L\Delta} B_{0,c})^{q_i} \right] \right) \\ &= \operatorname{Tr}\left(\prod_{i=1}^m \left[(P_0 B_{0,c})^{p_i-1} P_0 e^{-L\Delta} B_{0,c} (e^{L\tilde{\Delta}} \bar{P}_0 e^{-L\Delta} B_{0,c} - P_0 \tilde{B}_{0,c})^{q_i} e^{L\Delta} P_0 \right] \right), \end{aligned} \tag{2.94}$$

where we use $e^{-L\Delta} B_{0,c} e^{L\Delta} = B_{0,c}$ to deduce $K_v^{p_i} = e^{L\Delta}(P_0 B_{0,c})^{p_i-1} P_0 e^{-L\Delta} B_{0,c}$. Applying the identity $e^{-L\Delta} B_{0,c} e^{L\tilde{\Delta}} = \hat{B}_{0,c}$ (see (A.10)), we have then

$$\begin{aligned} &\left| \operatorname{Tr}\left(\prod_{i=1}^n K_{\sigma_n(i)} \right) - \operatorname{Tr}\left(\prod_{i=1}^n K_{\sigma_n^{w\mapsto\tilde{\beta}}(i)} \right) \right| \\ &\leq \sum_{\sigma_{q_1} \in \{a,b,c\}^{q_1}} \cdots \sum_{\sigma_{q_m} \in \{a,b,c\}^{q_m}} \mathbb{1}_{\{\exists\ j \text{ s.t. } b \in \sigma_{q_j}\}} |\operatorname{Tr}(\Phi)| \end{aligned} \tag{2.95}$$

where

$$\Phi = \prod_{i=1}^m \left((P_0 B_{0,c})^{p_i-1} P_0 e^{-L\Delta} B_{0,c} \left[\prod_{k=1}^{q_i} K_{\sigma_{q_i}(k)} \right] e^{L\Delta} P_0 \right). \tag{2.96}$$

Let $j \in \{1, \dots, m\}$ such that $b \in \sigma_{q_j}$ and define

$$\ell_b = q_1 + \dots + p_j + \min\{i | \sigma_{q_j(i)} = b\}. \tag{2.97}$$

Then we have

$$\Phi = \underbrace{(P_0 B_{0,c})^{p_1 - 1}}_{=:\varphi_1} \underbrace{P_0 e^{-L\Delta} B_{0,c} \left[\prod_{i=1}^{\ell_b - 1} K_{\sigma_i(k)} \right] P_0}_{=:\varphi_2} \cdot \underbrace{P_0 \tilde{B}_{0,c} \left[\prod_{i=\ell_b+1}^{n-p_1} K_i \right] e^{L\Delta} P_0}_{=:\varphi_3} \tag{2.98}$$

with $K_i \in \{K_a, K_b, K_c, K_u\}$. Applying the bounds $\|U_r \varphi_1 U_r^{-1}\|_{\mathrm{HS}} \le \beta^{p_1 - 1}$ (by (A.20)), $\|U_r \varphi_2 U_r\|_{\mathrm{HS}} \le \beta^{\ell_b} e^{-\frac{4L^3}{3\ell_b^2}} e^{-2Lc}$ (by Corollary B.13), $\|U_r^{-1} \varphi_3 U_r^{-1}\|_{\mathrm{HS}} \le \beta^{n - p_1 - \ell_b + 1} e^{Lr^2}$ (by Corollary B.17) and $r^2 \le 2c$, we have

$$|\operatorname{Tr}(\Phi)| \le \|U_r \varphi_1 U_r^{-1}\|_{\mathrm{HS}} \|U_r \varphi_2 U_r\|_{\mathrm{HS}} \|U_r^{-1} \varphi_3 U_r^{-1}\|_{\mathrm{HS}} \le \beta^n e^{-\frac{4L^3}{3n^2}}. \tag{2.99}$$

Plugging this back to (2.95) and noticing that the number of summations is smaller than 3^n, we obtain the result. □

2.3.5 Mixed u, v, w terms

Lemma 2.17. *Let $n \in \mathbb{Z}_{\ge 3}$, $L, r \ge 1$, $r^2 \le 2c$ and $\sigma_n \in \{u, v, w\}^n$ with $u, v, w \in \sigma_n$. Then*

$$\left| \operatorname{Tr} \left(\prod_{i=1}^{n} K_{\sigma_n}(i) \right) \right| \le 5^n \beta^n e^{-\frac{4L^3}{3n^2}}. \tag{2.100}$$

Proof. Without loos of generality we assume $\sigma_n(1) = u$. Then we have

$$\begin{aligned} \left| \operatorname{Tr} \left(\prod_{i=1}^{n} K_{\sigma_n}(i) \right) \right| &= \left| \operatorname{Tr} \left(P_0 B_{0,c} \prod_{i=2}^{n} K_{\sigma_n}(i) P_0 \right) \right| \\ &\le \sum_{\sigma_n \in \Sigma_n} \left| \operatorname{Tr} \left(P_0 B_{0,c} \prod_{i=2}^{n} K_{\sigma_n}(i) P_0 \right) \right|, \end{aligned} \tag{2.101}$$

where

$$\Sigma_n = \{\sigma_n \in \{a, b, c, u, v\}^n | \sigma_n \notin \{a, b, c\}^n, \sigma_n \notin \{a, b, c, u\}^n, \sigma_n \notin \{a, b, c, v\}^n\}. \tag{2.102}$$

For $\sigma_n \in \Sigma_n$, we define

$$\begin{aligned} \Phi &= P_0 B_{0,c} \prod_{i=2}^{n} K_{\sigma_n}(i) P_0 \\ &= \underbrace{P_0 B_{0,c} \left[\prod_{i=2}^{\ell_v - 1} K_{\sigma_n}(i) \right] e^{L\Delta} P_0}_{=:\varphi_1} \cdot \underbrace{P_0 e^{-L\Delta} B_{0,c} \left[\prod_{i=\ell_v+1}^{n} K_{\sigma_n}(i) \right] P_0}_{=:\varphi_2}, \end{aligned} \tag{2.103}$$

where $2 \le \ell_v = \min\{i \mid \sigma_n(i) = v\} \le n$. Applying the bounds $\|U_r^{-1}\varphi_1 U_r^{-1}\|_{\mathrm{HS}} \le \beta^{\ell_v - 1} e^{Lr^2}$ (by Corollary B.14), $\|U_r \varphi_2 U_r\|_{\mathrm{HS}} \le \beta^{n-\ell_v+1} e^{-\frac{4L^3}{3(n-\ell_v+1)^2}} e^{-2Lc}$ (by Corollary B.13) and $r^2 \le 2c$, we have

$$|\operatorname{Tr}(\Phi)| \le \|U_r^{-1}\varphi_1 U_r^{-1}\|_{\mathrm{HS}} \|U_r \varphi_2 U_r\|_{\mathrm{HS}} \le \beta^n e^{-\frac{4L^3}{3n^2}}. \tag{2.104}$$

The result follows by the fact that $|\Sigma_n| \le 5^n$. □

2.4 Proof of Theorem 1.1

Now we are able to prove Theorem 1.1. Before going to the main part, we need to control the upper bound of error terms, that is,

$$\beta = \max\{2e^{r^3/3 - 2rc}, e^{(r-1/7)^3/3 - 2(r-1/7)c}\} \tag{2.105}$$

with $r^2 \le 2c, r > 1$. For fixed c, we set $r = \sqrt{2c}$, then $\beta \le 2e^{-\frac{4\sqrt{2}}{3} c^{3/2}}$. In particular, for $c \ge 3/2$, $\beta < 1/7$. Recall that

$$\ln(\mathbb{P}(\mathcal{A}_1(s) \le c, \ s \in [0, L])) = \ln(\det(\mathbb{1} - K_{L,c})) = -\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Tr}(K_{L,c}^n). \tag{2.106}$$

In Proposition 2.5 we have obtained

$$|\operatorname{Tr}(K_{L,c}) - 2\operatorname{Tr}(P_0 B_{0,c}) + 2L\mathrm{Ai}'(2c)| \le e^{-\frac{4L^3}{3}}. \tag{2.107}$$

For $n \ge 2$, applying Lemma 2.7 to (2.26) we get

$$\left| \operatorname{Tr}(K_{L,c}^n) - \sum_{\sigma_n \in \{u,v,w\}^n} \operatorname{Tr}\left(\prod_{i=1}^n K_{\sigma_n(i)}\right) \right| \le 7^n e^{-\frac{4L^3}{3n^2}} \beta^n. \tag{2.108}$$

Furthermore, by Lemma 2.8, Proposition 2.13, Lemma 2.14, Lemma 2.15, Lemma 2.16 and Lemma 2.17 we get

$$\sum_{\sigma_n \in \{u,v,w\}^n} \operatorname{Tr}\left(\prod_{i=1}^n K_{\sigma_n(i)}\right) = \kappa_n(c) L + \Psi_n + R_n \tag{2.109}$$

where

$$\begin{aligned} \Psi_n =& 2\operatorname{Tr}((P_0 B_{0,c})^n) + \mathbb{1}_{n\ge 2}\Bigg[\Psi_n^1 - \Psi_n^2 - \Psi_n^3 \\ &+ \sum_{\substack{\sigma_n \in \{u,w\}^n \\ u,w \in \sigma_n}} \operatorname{Tr}\left(\prod_{i=1}^n K_{\sigma_n^{w\mapsto\tilde{w}}(i)}\right) + \sum_{\substack{\sigma_n \in \{v,w\}^n \\ v,w \in \sigma_n}} \operatorname{Tr}\left(\prod_{i=1}^n K_{\sigma_n^{w\mapsto\tilde{\beta}}(i)}\right)\Bigg], \end{aligned} \tag{2.110}$$

Ψ_n^1, Ψ_n^2 and Ψ_n^3 are defined in Proposition 2.13, $\kappa_n(c) = -2n^{-2/3}\mathrm{Ai}'(2n^{2/3}c)$ and

$$|R_n| \le (5^n + 4 \cdot 3^n)\beta^n e^{-\frac{4L^3}{3n^2}} + \beta^n e^{-\frac{4L^3}{3}} \le 4 \cdot 5^n \beta^n e^{-\frac{4L^3}{3n^2}}. \tag{2.111}$$

Together with (2.108) we have

$$\mathrm{Tr}(K_{L,c}^n) = \kappa_n(c)L + \Psi_n + \tilde{R}_n \tag{2.112}$$

where $|\tilde{R}_n| \leq (7^n + 4\cdot 5^n)\beta^n e^{-\frac{4L^3}{3n^2}} \leq 7^{n+1}\beta^n e^{-\frac{4L^3}{3n^2}}$. By (2.33), (2.71), (2.77) and (2.91) we have the following bound for the L-independent terms

$$\left|\sum_{n\geq 1}\frac{\Psi_n}{n}\right| \leq 6\sum_{n=1}^{\infty}\frac{2^n\beta^n}{n} = -6\ln(1-2\beta) < \infty, \tag{2.113}$$

where we use $\beta < 1/7$. Next, we want to show $\sum_{n\geq 1}|\tilde{R}_n| \to 0$ as $L\to\infty$. Set $\alpha = 7\beta < 1$ and notice that

$$\sum_{n=1}^{\infty}\frac{1}{n}\alpha^n e^{-\frac{4L^3}{3n^2}} \leq \sum_{n=1}^{\infty}\alpha^n e^{-\frac{4L^3}{3n^2}}. \tag{2.114}$$

The function

$$n \mapsto \alpha^n e^{-\frac{4L^3}{3n^2}} \tag{2.115}$$

is increasing on $(1, n_0)$ and decreasing on $[n_0, \infty)$ with

$$n_0 = 2\cdot 3^{-1/3}\ln(\alpha^{-1})^{-1/3}L. \tag{2.116}$$

In particular, $f'_{L,\alpha}(n) > 0$ for all $n < n_0$ and $f'_{L,\alpha}(n) < 0$ for all $n > n_0$. Together with Riemann approximation, we have then

$$\begin{aligned}\sum_{n=1}^{\infty}\alpha^n e^{-\frac{4L^3}{3n^2}} &= \sum_{n=1}^{\lceil n_0\rceil}\alpha^n e^{-\frac{4L^3}{3n^2}} + \sum_{n=\lceil n_0\rceil+1}^{\infty}\alpha^n e^{-\frac{4L^3}{3n^2}}\\ \leq \lceil n_0\rceil f_{L,\alpha}(n_0) + \int_{\lceil n_0\rceil}^{\infty} dn f_{L,\alpha}(n) &\leq \lceil n_0\rceil f_{L,\alpha}(n_0) + \int_1^{\infty} dn f_{L,\alpha}(n).\end{aligned} \tag{2.117}$$

We define $C_\alpha = \left(-\frac{1}{\ln(\alpha)}\right)^{1/3} > 0$ for $\alpha\in(0,1)$. Then we have

$$\lceil n_0\rceil f_{L,\alpha}(n_0) \leq (n_0+1) f_{L,\alpha}(n_0) = \alpha^{3^{2/3}LC_\alpha}\left(\frac{2LC_\alpha}{3^{1/3}}+1\right) \tag{2.118}$$

Since $C_\alpha > 0$ and $\alpha < 1$, for L large, we have then

$$\lceil n_0\rceil f_{L,\alpha}(\lceil n_0\rceil) \leq \tilde{C}_\alpha e^{-\delta L}, \tag{2.119}$$

where $\delta, \tilde{C}_\alpha > 0$ independent of L. As for the integral term in (2.117) note that

$$\int_1^{\infty} dx\alpha^x e^{-\frac{4L^3}{3x^2}} = \int_1^{L^\gamma} dx\alpha^x e^{-\frac{4L^3}{3x^2}} + \int_{L^\gamma}^{\infty} dx\alpha^x e^{-\frac{4L^3}{3x^2}} \tag{2.120}$$

with arbitrary $\gamma\geq 1$. Since $\alpha < 1$, the first integral is bounded by $(L^\gamma - 1)e^{-\frac{4L^3}{3L^{2\gamma}}}$. On the other hand, the second integral is bounded by

$$\int_{L^\gamma}^{\infty} dx\alpha^x = \ln(\alpha^{-1})^{-1}\alpha^{L^\gamma} \tag{2.121}$$

Choosing now $\gamma = 1$, we then see that

$$0 \leq \sum_{n=1}^{\infty} \frac{1}{n} \alpha^n e^{-\frac{4L^3}{3n^2}} \leq L e^{-\frac{4L}{3}} + \ln(\alpha^{-1})^{-1} e^{L \ln \alpha} \leq e^{-\delta L} \tag{2.122}$$

with some $\delta > 0$, which then finishes the proof.

3 Proof of Proposition 1.2

In Theorem 1.1, we get the persistence exponent for $c \geq \frac{3}{2}$, in this section, we try to extend our result to the whole real line via analytic continuation. To this end, we need first show the existence of the following quantities

$$\lim_{L\to\infty} \frac{\ln(\mathbb{P}\left(A_1(t) \leq c,\ \forall t \in [0, L]\right))}{L}. \tag{3.1}$$

Recall that a function is called super-additive if

$$f(x+y) \geq f(x) + f(y), \quad \forall x, y \in \mathbb{R}. \tag{3.2}$$

A nice property of supperadditive function is given by Fekete's Lemma.

Theorem 3.1 (Theorem 16.2.9 of [14]). *Let $f : (0, \infty) \to \mathbb{R}$ be a super-additive function. Then $\lim_{t\to\infty} f(t)/t$ exists.*

To apply Fekete's lemma in our case, we define

$$f(L) = \ln(\mathbb{P}\left(\mathcal{A}_1(t) \leq c,\ \forall t \in [0, L]\right)). \tag{3.3}$$

We have

$$\begin{aligned}
\mathbb{P}\left(\max_{0\leq u\leq L_1+L_2} \mathcal{A}_1(u) \leq c\right) &= \mathbb{P}\left(\max_{0\leq u\leq L_1} \mathcal{A}_1(u) \leq c, \max_{L_1\leq u\leq L_2} \mathcal{A}_1(u) \leq c\right) \\
&\geq \mathbb{P}\left(\max_{0\leq u\leq L_1} \mathcal{A}_1(u) \leq c\right) \mathbb{P}\left(\max_{L_1\leq u\leq L_1+L_2} \mathcal{A}_1(u) \leq c\right) \\
&= \mathbb{P}\left(\max_{0\leq u\leq L_1} \mathcal{A}_1(u) \leq c\right) \mathbb{P}\left(\max_{0\leq u\leq L_2} \mathcal{A}_1(u) \leq c\right)
\end{aligned} \tag{3.4}$$

for all $c \in \mathbb{R}$, $L_1, L_2 > 0$.

To prove (3.4) we follow the of Lemma 3.2 of [3]: we start with the line-to-point last passage percolation (LPP) model. For this model, it is known that under appropriate scaling limit the last passage time at different positions converges weakly to the Airy$_1$ process (in TASEP finite-dimensional distribution is proven in [9,35] and by slow-decorrelation [10,15] the result is translated to the LPP setting; tightness is shown in [25]). On the LPP model, we can apply the FKG inequality since the events which converges to $\{\max_{0\leq u\leq L_1} \mathcal{A}_1(u) \leq c\}$ and to $\{\max_{L_1\leq u\leq L_1+L_2} \mathcal{A}_1(u) \leq c\}$ are both decreasing in the randomness. Finally, for the last step we use translation-invariance of the law of the Airy$_1$ process.

To show that the Airy$_1$ process is positively correlated (also called associated in the language of [21]), a similar argument, but using more involved results as input (the convergence of the KPZ equation to the KPZ fixed point [32]), was presented in [27].

Taking the logarithms in (3.4) we get that f is super-additive and by Theorem 3.1, the proof of Proposition 1.2 is completed.

4 Proof of Proposition 1.3

In Theorem 1.1, we have already showed that

$$\kappa(c) = -2\sum_{n=1}^{\infty} n^{-5/3}\mathrm{Ai}'(2n^{2/3}c), \quad \forall c \geq \frac{3}{2}. \tag{4.1}$$

This function is analytic for all $c > 0$ as well. For instance, applying Fubini's theorem and integral representation of Airy function, we obtain

$$\begin{aligned}\kappa(c) &= -2\sum_{n=1}^{\infty} n^{-5/3}\frac{1}{2\pi\mathrm{i}}\int_{e^{-\pi\mathrm{i}/3}\infty}^{e^{\pi\mathrm{i}/3}\infty} dw\, we^{\frac{w^3}{3}-2n^{2/3}wc} \\ &\overset{w\mapsto n^{1/3}w}{=} \frac{-1}{\pi\mathrm{i}}\int_{e^{-\pi\mathrm{i}/3}\infty}^{e^{\pi\mathrm{i}/3}\infty} dw\, w\sum_{n=1}^{\infty}\frac{(e^{w^3/3-2wc})^n}{n} = \frac{-1}{\pi\mathrm{i}}\int_{e^{-\pi\mathrm{i}/3}\infty}^{e^{\pi\mathrm{i}/3}\infty} dw\, w\ln\left(1-e^{w^3/3-2wc}\right).\end{aligned} \tag{4.2}$$

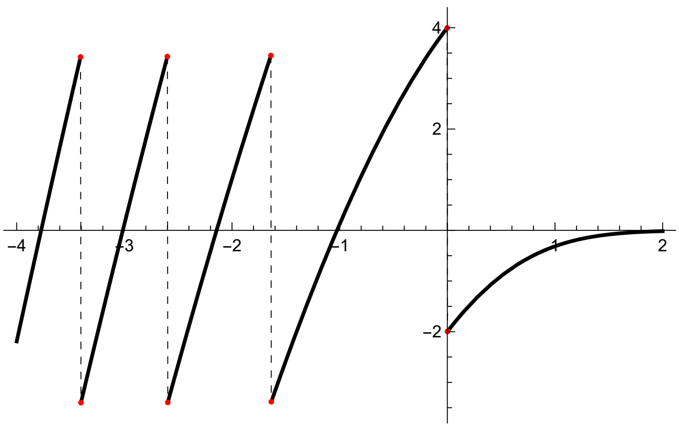

Figure 3: The black solid line is the graph of $(c, \kappa'(c))$. The red point is the jump points in $\mathcal{J}$, that is, $c(0), c(1), \ldots$ defined in Lemma 4.1. In particular, function $\kappa'(c)$ is analytic on each interval $(c(i+1), c(i))$. The analytic continuation of $\kappa'(c)|_{(0,\infty)}$ is obtained by gluing $\kappa'(c)|_{(c(i+1),c(i))}$ together in a smooth way.

To avoid dealing the branch cut of ln function, we consider $\kappa'(c)$ instead of $\kappa(c)$,

$$\kappa'(c) = \frac{2}{\pi\mathrm{i}}\int_{\Gamma} dw \frac{w^2 e^{\frac{w^3}{3}-2wc}}{1-e^{\frac{w^3}{3}-2wc}}, \tag{4.3}$$

where from now on we *fix* the integration contour as follows

$$\Gamma = \{|r|e^{\mathrm{sgn}(r)\pi \mathrm{i}/3} \text{ s.t. } r \in \mathbb{R}\} \tag{4.4}$$

oriented by increasing imaginary part. Different choices of Γ gives rise to different (equivalent) formulas for the analytic continuation. The reason is that when decreasing c, there are zeroes of the denominator crossing the contour Γ.

Define

$$f(w,c) = 1 - e^{\frac{w^3}{3}-2wc} \quad \text{and} \quad g(w,c) = w^2 e^{\frac{w^3}{3}-2wc}. \tag{4.5}$$

First we determine the values of c where the denominator of (4.3) vanishes (the poles). It turns out that these values are exactly the discontinuity points appearing in Figure 3 of function $\kappa'(c)$. Define the set

$$\mathcal{J} = \{c \in \mathbb{R}_- | f(w,c) = 0 \text{ for some } w \in \Gamma\}. \tag{4.6}$$

Lemma 4.1. *We have*

$$\mathcal{J} = \left\{-(2n\pi/3)^{2/3} | n \in \mathbb{Z}_{\geq 0}\right\} \tag{4.7}$$

Moreover, for $c(n) = -(2n\pi/3)^{2/3}$, $f(c(n), w(n)) = f(c(n), \bar{w}(n)) = 0$ *for* $w(n) = 3^{1/2}(2\pi n/3)^{1/3}e^{\pi \mathrm{i}/3}$.

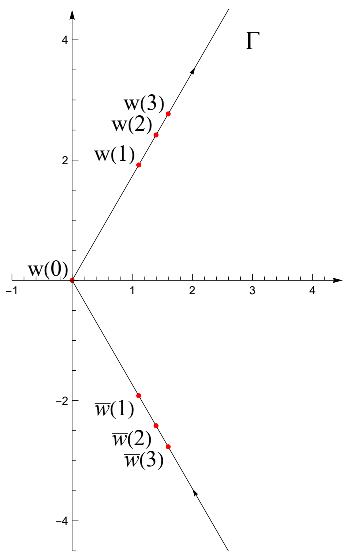

Figure 4: The black solid line is the integral contour Γ, the red points are $w(n)$ and $\bar{w}(n)$, $n \geq 0$.

Proof. By symmetry with respect to the real axis, it is clear that there are complex-conjugated zeroes of the denominator. So parameterize Γ on the upper half plane by $w = re^{\pi i/3}$ with $r \geq 0$. We have $f(w,c) = 0$ if and only if both the real and the imaginary parts are zero. This happens if, for some $n \in \mathbb{Z}$, $w^3/3 - 2wc = 2\pi \mathrm{i} n$, that is,

$$\frac{r^3}{3} + cr = 0 \quad \text{and} \quad \sqrt{3}cr = 2\pi n. \tag{4.8}$$

We can restrict to $n \geq 0$ since the other gives the complex conjugate solutions. The solution of (4.8) are precisely the pairs given by $(c(n), w(n))$ of the lemma. $\square$

For $c \in \mathbb{R}$ and $n \in \mathbb{Z}_{\geq 0}$, we denote by $w_{n,c} \in \mathbb{C}$ with $\Re(w) \geq 0$ such that $w_{n,c}^3/3 - 2w_{n,c}c = 2\pi \mathrm{i} n$. Let $W_0 = \{w_{0,c} | 0 \leq c \leq 3/2\}$ and $W_n = \{w_{n,c} | c(n) \leq c \leq 0\}$ for $n \geq 1$. We also define $\overline{W}_n$ as the conjugate set of W_n. Furthermore, we denote L_Γ (resp. R_Γ) as the set of points that are to the left (resp. right) of contour Γ, that is, $L_\Gamma = \{z \in \mathbb{C} | \arg(z) \in (\pi/3, 5\pi/3)\}$.

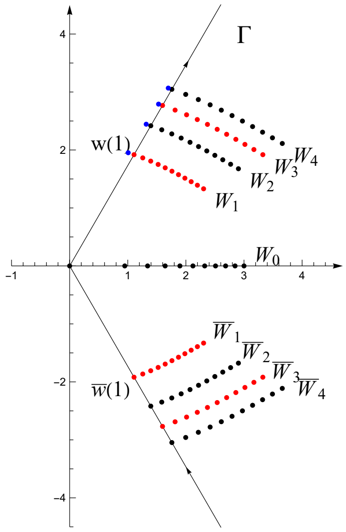

Figure 5: An Illustration for set W_n and $\overline{W}_n$. The black solid line is the original integral contour Γ. The black points for W_0 is $w_{0,c}$ for $c = 3/2 - 3m/20$ with $m \in \{0, 1, \ldots, 10\}$. The red and black points are the set $W_n, \overline{W}_n$ with $n \in \{1, 2, 3, 4\}$ are $w_{n,c}$ for $c = c(n) - mc(n)/10$ with $m \in \{1, 2, \ldots, 10\}$. The four blue points from bottom left to top right is $w_{n,c(n)-1/10}$ with $n = 1, 2, 3, 4$. In particular, note that $w_{n,c(n)-1/10} \in L_\Gamma$ and $d(W_m, W_n) = d(\overline{W}_m, \overline{W}_n) > 0$.

Lemma 4.2.
(a) For $n \in \mathbb{N}$, $w_{n,c} \in L_\Gamma$ for any $c < c(n)$ and $w_{n,c} \in R_\Gamma$ for any $0 \geq c > c(n)$.
(b) For any $m, n \in \mathbb{N}$ with $m \neq n$, it holds $W_n \cap W_m = \varnothing$. See Figure 5 for an illustration.

Proof. Let us prove (a). Parameterize $w = re^{\mathrm{i}\phi}$ with $(r, \phi) \in \mathbb{R}_+ \times (0, \pi/2)$. The condition $\frac{w^3}{3} - 2wc = 2\pi \mathrm{i} n$ for some $c < 0$ and $n \in \mathbb{Z}_{\geq 0}$ is then equivalent to

$$c = h(r, \phi) = \tfrac{1}{6} r^2 \cos(2\phi) - \frac{n\pi \sin(\phi)}{r} + \tfrac{1}{6} \mathrm{i} r^2 \sin(2\phi) - \frac{\mathrm{i} n\pi \cos(\phi)}{r}. \tag{4.9}$$

Since $c \in \mathbb{R}$, we need to have $\mathrm{Im}(h(r, \phi)) = 0$. This and the condition $r > 0$ leads to the following relation between r and ϕ:

$$r_{n,\phi} = \left(6n\pi \cos(\phi) \sin(2\phi)^{-1}\right)^{1/3}, \tag{4.10}$$

now we define

$$k(\phi) = h(r_{n,\phi}, \phi) = 3^{-1/3} n^{2/3} \pi^{2/3} \sin(\phi)^{-4/3} (\cot(\phi) \cot(2\phi) - 1). \tag{4.11}$$

Its derivative is given by

$$k'(\phi) = 3^{-4/3} n^{2/3} \pi^{2/3} \cos(\phi) \sin(\phi)^{-5/3} (4\cos(2\phi) - 5) < 0, \tag{4.12}$$

where we use the fact $\cos(\phi), \sin(\phi) > 0$ for $\phi \in (0, \pi/2)$ and $\cos(2x) \leq 1$ for all x. In particular, this implies that $k(\phi)$ is monotone decreasing. On the other hand, we have $k(\pi/3) = c(n)$, this implies that for any $c < c(n)$, $\arg(w_{n,c}) > \frac{\pi}{3}$ for any $w_{n,c} \in W_n$, which shows the first claim.

Next we show (b). Choose now $m, n \in \mathbb{N}$ with $m \neq n$, it is enough to show that the trajectory of W_n and W_m will not intersect with each other. Note that by definition, $w_{n,0}$ is the solution of $w^3/3 = 2\pi i n$ for $n \in \mathbb{N}$, this implies that $\arg(w_{n,0}) = \pi/6$, together with (4.12), we have

$$W_n = \left\{ r_{n,\phi} e^{i\phi} | \phi \in (\pi/6, \pi/3) \right\}, \quad W_m = \left\{ r_{m,\phi} e^{i\phi} | \phi \in (\pi/6, \pi/3) \right\}, \tag{4.13}$$

where $r_{n,\phi}$ is defined as in (4.10).

For a fixed $\phi \in [\pi/6, \pi/3]$, it is clear that $r_{m,\phi} \neq r_{n,\phi}$ when $m \neq n$, which then implies $W_n \cap W_m = \varnothing$ for $m, n \in \mathbb{Z}_{\geq 1}$ with $m \neq n$. Note that $W_0 = \{\sqrt{6c} | 0 \leq c \leq 3/2\}$ and hence $W_0 \cap W_n = \varnothing$ for any $n \geq 1$, since $r_{n,\phi} > 0$ and $\pi/6 \leq \phi \leq \pi/3$. This completes then the proof. ∎

With this in hand, we are able to do the analytic continuation of $\kappa'(c)$ from $c > 0$ to the whole real line. Denote now $\tilde{\kappa}'(c)$ as the function obtained by extending $\kappa'(c)$ analytically from $(0, \infty)$ to all $\mathbb{R}$. We first consider the extension from $(0, \infty)$ to

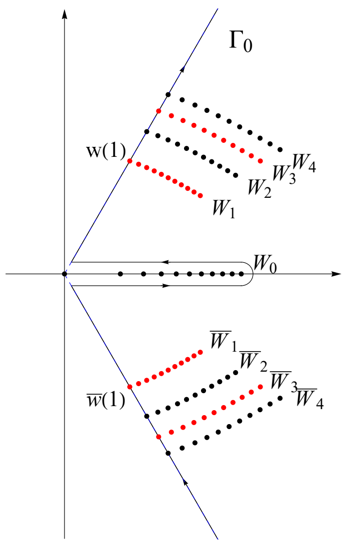

Figure 6: The blue dashed line is the original contour Γ and the black line is the deformed contour Γ_0. In particular, the region between Γ and Γ_0 contain only W_0 but not W_n, $\overline{W}_n$ for any $n \geq 1$.

$(c(1), \infty)$. Now we deform the contour Γ to the contour Γ_0 (see also Figure 6 for an illustration) such that the region between Γ and Γ_0 contains only W_0 but not $W_n, \overline{W}_n$ for any $n \geq 1$, this is possible by Lemma 4.2. Denote the integrand in (4.3) as

$$Q(w, c) = \frac{g(w, c)}{f(w, c)}. \tag{4.14}$$

By Cauchy residue theorem, we know that

$$\begin{aligned}\kappa'(c) &= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma} dwQ(w,c)\\ &=\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_0} dwQ(w,c) - \operatorname{Res}\left(Q(w,c)|w=\sqrt{6c}\right), \quad \forall c\in(0,1).\end{aligned} \tag{4.15}$$

For $c>0$, $\sqrt{6c}$ is a simple pole for $Q(w,c)$, we then have

$$\operatorname{Res}\left(Q(w,c)|w=\sqrt{6c}\right) = \lim_{w\to\sqrt{6c}} \frac{(w-\sqrt{6c})g(w,c)}{f(w,c)} = 6. \tag{4.16}$$

By the definition of Γ_0, there exists no $w\in\Gamma_0$ such that $f(w,c)=0$ for some $c\in(c(1),0)$, moreover, $Q(w,c)$ is bounded on $(w,c)\in\Gamma_0\times(c(1),0)$, hence (see for instance Lemma B.2 of [6]), the function

$$c\mapsto \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_0} dwQ(w,c) \tag{4.17}$$

is analytic on $(c(1),1)$. On the other hand, by the choice of Γ_0, the region between Γ and Γ_0 does not contain any $w_{n,c}$ for $n\in\mathbb{N},\ c\in(c(1),0)$, we have

$$\kappa'(c) = \frac{1}{2\pi\mathrm{i}}\int_{\Gamma} dwQ(w,c) = \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_0} dwQ(w,c), \quad \forall c\in(c(1),0). \tag{4.18}$$

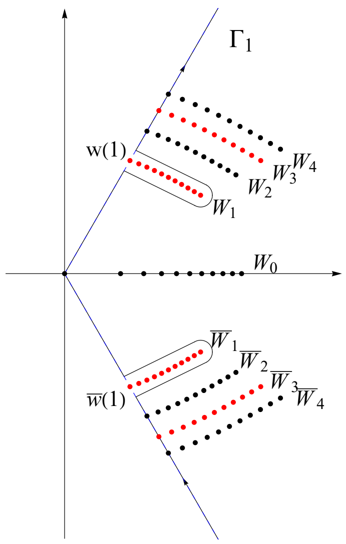

Figure 7: Illustration of Γ_1. The blue dashed line is the original integral contour Γ, while the black solid line is the new integral contour Γ_1. In particular, the region between Γ and Γ_1 should only contain W_1 and $\overline{W}_1$ but not $W_n\cup\overline{W}_n$ for $n\geq 2$.

The value at $\tilde{\kappa}'(c)$ is then given by $\lim_{c\downarrow 0}\kappa'(c)$. In conclusion, the analytic extension $\tilde{\kappa}'(c)$ on $(c(1),\infty)$ is given by

$$\tilde{\kappa}'(c) = \begin{cases}\kappa'(c), & \text{for } c>0,\\ \lim_{c\downarrow 0}\kappa'(c), & \text{for } c=0,\\ \kappa'(c)-6, & \text{for } c\in(c(1),0).\end{cases} \tag{4.19}$$

Using the same method, we can also extend the result to $(c(2), \infty)$, namely, we choose the contour Γ_1 such that the region between Γ_1 and Γ contains only W_1 and $\overline{W}_1$ but not $W_n, \overline{W}_n$ for any $n \neq 1$ (see Figure 7). Similar as above, the function

$$c \mapsto \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_1} dw Q(w, c) \tag{4.20}$$

is analytic on $(c(2), 0)$. And for $c \in (c(1), 0)$, we have

$$\begin{aligned} \tilde{\kappa}'(c) &= \int_{\Gamma} dw Q(w, c) - 6 \\ &= \int_{\Gamma_1} dw Q(w, c) - 6 - \operatorname{Res}\left(Q(w, c) | w = w_{1,c}\right) - \operatorname{Res}\left(Q(w, c) | w = \bar{w}_{1,c}\right). \end{aligned} \tag{4.21}$$

Note that $w_{1,c}$ and $\bar{w}_{1,c}$ are poles of c of order 1, hence, we have

$$\begin{aligned} &\lim_{c \downarrow c(1)} \left(\operatorname{Res}\left(Q(w, c) | w = w_{1,c}\right) + \operatorname{Res}\left(Q(w, c) | w = \bar{w}_{1,c}\right)\right) \\ =& \lim_{c \downarrow c(1)} \left(\frac{[(w - w_{1,c}) g(w, c)]' |_{w = w_{1,c}}}{f'(w, c)|_{w = w_{1,c}}} + \frac{[(w - w_{1,c}) g(w, c)] |_{w = \bar{w}_{1,c}}}{f'(w, c)|_{w = \bar{w}_{1,c}}} \right) \\ =& \operatorname{Res}\left(Q(w, c(1)) | w = w(1)\right) + \operatorname{Res}\left(Q(w, c(1)) | w = \bar{w}(1)\right) = \frac{48}{7}. \end{aligned} \tag{4.22}$$

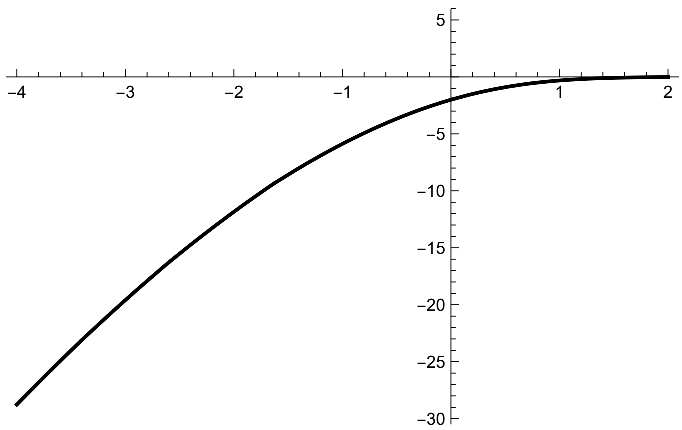

Figure 8: The black line is the graph $(c, \tilde{\kappa}'(c))$, where $\tilde{\kappa}'(c)$ is defined in (4.25). Comparing to the graph in Figure 3, we notice that we really glue the analytical part together.

Hence, we can then extend $\kappa'(c)$ to $(c(2), c(1))$ as the following function:

$$\tilde{\kappa}'(c) = \begin{cases} \kappa'(c), & \text{for } c > 0, \\ \lim_{c \downarrow 0} \kappa'(c) - 6, & \text{for } c = 0, \\ \kappa'(c) - 6, & \text{for } c \in (c(1), 0), \\ \lim_{c \downarrow c(1)} \kappa'(c) - 6 - 48/7, & \text{for } c = c(1), \\ \kappa'(c) - 6 - 48/7, & \text{for } c \in (c(2), c(1)). \end{cases} \tag{4.23}$$

Using the similar method, we can also extend the above function to the interval $(c(3), c(2))$ and eventually to the whole real line. It turns out that for $n \geq 1$, we have

$$\begin{aligned}&\lim_{c \downarrow c(n)} \left(\operatorname{Res}\left(Q(w,c)|w = w_{1,c}\right) + \operatorname{Res}\left(Q(w,c)|w = \bar{w}_{1,c}\right)\right)\\ =&\operatorname{Res}\left(Q(w,c(n))|w = w(n)\right) + \operatorname{Res}\left(Q(w,c)|w = \bar{w}(n)\right) = \frac{48}{7}, \quad \forall n \geq 1.\end{aligned} \tag{4.24}$$

Hence, we obtain the following analytical continuation of $\kappa'(c)$ (see also Figure 8)

$$\tilde{\kappa}'(c) = \begin{cases} \kappa'(c) - 6 \cdot \mathbb{1}_{c<0} - \frac{48}{7} \sum_{n=1}^{\infty} \mathbb{1}_{c<c(n)}, & \text{for } c \notin \mathcal{J}, \\ \lim_{\epsilon \downarrow 0} \tilde{\kappa}'(c+\epsilon), & \text{for } c \in \mathcal{J}. \end{cases} \tag{4.25}$$

As a consequence, we then obtain the full solution of the exponent:

Lemma 4.3. *The analytic continuation of $\kappa|_{(0,\infty)}$ is given by*

$$\tilde{\kappa}(c) = \begin{cases} \kappa(c), & \text{for } c \geq 0, \\ \kappa(0) - \int_c^0 dx \kappa'(x) - 6c - \frac{48}{7} \sum_{n \geq 1} (c - c(n)) \mathbb{1}_{c<c(n)}, & \text{for } c < 0, \end{cases} \tag{4.26}$$

where $\kappa'(c)$ is defined in (4.3) and $c(n)$ is defined as in Lemma 4.1.

A Preliminary Upper Bounds

In this section, we deduce some preliminary upper bounds for later use. To this end, we first deduce some identities regarding to Airy function. Recall the definitions

$$\begin{aligned} B_{0,c}(x,y) &= \operatorname{Ai}(x+y+2c), \\ \tilde{B}_{0,c}(x,y) &= \operatorname{Ai}(y-x+2c), \\ \hat{B}_{0,c}(x,y) &= \operatorname{Ai}(x-y+2c), \end{aligned} \tag{A.1}$$

the heat kernel $e^{L\Delta}(x,y) = \frac{1}{\sqrt{4\pi L}} e^{-\frac{(x-y)^2}{4L}}$, a variant of it, that is, $e^{L\tilde{\Delta}}(x,y) = e^{L\Delta}(-x,y)$. For $L > 0$, $e^{-L\Delta} B_{0,c}$ is still well-defined with

$$e^{-L\Delta} B_{0,c}(x,y) = e^{-2L^3/3 - L(x+y+2c)} \operatorname{Ai}(L^2 + x + y + 2c). \tag{A.2}$$

We have the following identities.

Lemma A.1. *Let $n \in \mathbb{Z}_{\geq 1}$, $L > 0$ and $c \in \mathbb{R}$, then*

$$\hat{B}^n_{0,c}(x,y) = n^{-1/3}\mathrm{Ai}(n^{-1/3}(x-y+2nc)) \tag{A.3}$$

$$\tilde{B}^n_{0,c}(x,y) = n^{-1/3}\mathrm{Ai}(n^{-1/3}(y-x+2nc)). \tag{A.4}$$

$$\hat{B}^{n-1}_{0,c}B_{0,c}(x,y) = B_{0,c}\tilde{B}^{n-1}_{0,c}(x,y) = n^{-1/3}\mathrm{Ai}(n^{-1/3}(x+y+2nc)) \tag{A.5}$$

$$e^{-L\Delta}B_{0,c}\tilde{B}^{n-1}_{0,c}e^{L\tilde{\Delta}} = \hat{B}^n_{0,c}, \tag{A.6}$$

$$e^{-L\Delta}B_{0,c}\tilde{B}_{0,c} = \hat{B}_{0,c}e^{-L\Delta}B_{0,c},, \tag{A.7}$$

$$e^{L\tilde{\Delta}}e^{-L\Delta}B_{0,c} = \tilde{B}_{0,c}, \tag{A.8}$$

$$e^{L\Delta}B_{0,c}(x,y) = e^{2L^3/3+L(x+y+2c)}\mathrm{Ai}(L^2+x+y+2c),, \tag{A.9}$$

$$e^{L\Delta}\hat{B}_{0,c}(x,y) = B_{0,c}e^{L\tilde{\Delta}}(x,y) = e^{\frac{2L^3}{3}+L(x-y+2c)}\mathrm{Ai}\left(L^2+x-y+2c\right), \tag{A.10}$$

$$\tilde{B}^n_{0,c}e^{L\Delta}(x,y) = e^{L\tilde{\Delta}}B^n_{0,c}(x,y) = \frac{e^{2Lc+\frac{2L^3}{3n^2}}}{n^{1/3}}e^{\frac{L(y-x)}{n}}\mathrm{Ai}\left(\frac{L^2}{n^{4/3}}+\frac{y-x+2nc}{n^{1/3}}\right), \tag{A.11}$$

$$e^{L\tilde{\Delta}}\hat{B}^n_{0,c}(x,y) = \tilde{B}^n_{0,c}e^{L\tilde{\Delta}}(x,y) = \frac{e^{2Lc+\frac{2L^3}{3n^2}}}{n^{1/3}}e^{-\frac{L(x+y)}{n}}\mathrm{Ai}\left(\frac{L^2}{n^{4/3}}-\frac{x+y+2nc}{n^{1/3}}\right), \tag{A.12}$$

$$\hat{B}^{n-1}_{0,c}e^{-L\Delta}B_0(x,y) = \frac{e^{-2Lc-\frac{2L^3}{3n^2}}}{n^{1/3}}e^{-\frac{L(x+y)}{n}}\mathrm{Ai}\left(\frac{L^2}{n^{4/3}}+\frac{x+y+2nc}{n^{1/3}}\right), \tag{A.13}$$

$$B_{0,c}\tilde{B}^{n-1}_{0,c}e^{L\tilde{\Delta}}(x,y) = \frac{e^{2Lc+\frac{2L^3}{3n^2}}}{n^{1/3}}e^{\frac{L(x-y)}{n}}\mathrm{Ai}\left(\frac{L^2}{n^{4/3}}+\frac{x-y+2nc}{n^{1/3}}\right). \tag{A.14}$$

Proof. Let us show in detail how to derive (A.3) and (A.12) only, since the others follows using similar computations. Recall that

$$\mathrm{Ai}(x) = \frac{1}{2\pi \mathrm{i}}\int_{\mathrm{i}\mathbb{R}+\varepsilon} dw e^{\frac{w^3}{3}-wx}, \quad \varepsilon > 0. \tag{A.15}$$

Then we have

$$\begin{aligned}\hat{B}^n_{0,c}(x,y) = \int_{\mathbb{R}^{n-1}} dz_1\cdots dz_{n-1}\int_{\mathrm{i}\mathbb{R}+\varepsilon_1} dw_1\cdots\int_{\mathrm{i}\mathbb{R}+\varepsilon_n} dw_n\frac{1}{(2\pi\mathrm{i})^n}\left(\prod_{k=1}^n e^{-w_k^3/3-2cw_k}\right)\\ \times e^{-w_1(x-z_1)}\left(\prod_{\ell=2}^{n-1}e^{-w\ell(z_{\ell-1}-z_\ell)}\right)e^{-z_{n-1}(w_n-w_{n-1})}.\end{aligned} \tag{A.16}$$

We can take the integral over z_{n-1} separately for $z_{n-1}\in\mathbb{R}_+$ and $z_{n-1}\in\mathbb{R}_-$. In the first case we need to assume $\varepsilon_n > \varepsilon_{n-1}$, while in the second case $\varepsilon_n < \varepsilon_{n-1}$. Then the integral over $z_{n-1}\in\mathbb{R}_+$ gives a factor $\frac{1}{w_n-w_{n-1}}$ while the integral over $z_{n-1}\in\mathbb{R}_-$ gives a factor $-\frac{1}{w_n-w_{n-1}}$. For fixed w_{n-1}, putting the two integrals together we get that the integral over w_n is a simple anticlockwise oriented path enclosing w_{n-1}, which has a simple pole at w_{n-1}. Doing the same for the integrals over z_{n-2} until z_1 we obtain

$$\hat{B}^n_{0,c}(x,y) = \frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}+\varepsilon_1} e^{nw_1^3/3-w_1(x-y+2nc)} = n^{-1/3}\mathrm{Ai}(n^{-1/3}(x-y+2nc)). \tag{A.17}$$

Calculating the Gaussian integral, we then obtain

$$
\begin{aligned}
e^{L\tilde{\Delta}}\hat{B}^n_{0,c}(x,y) &= \frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}+\epsilon} dw \int_{\mathbb{R}} dz \frac{1}{\sqrt{4\pi L}} e^{-\frac{(x+z)^2}{4L}} e^{\frac{w^3}{3}-wn^{-1/3}(z-y+2nc)} \\
&= \frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}+\epsilon} dw e^{\frac{w^3}{3}+wn^{-1/3}(x+y)+n^{-2/3}Lw^2-2n^{2/3}c}, \quad \forall \epsilon>0.
\end{aligned}
\tag{A.18}
$$

Clearly we have $\tilde{B}^n_{0,c}(x,y)=\hat{B}^n_{0,c}(y,x)=n^{-1/3}\mathrm{Ai}(n^{-1/3}(y-x+2nc))$. Using this we get $\tilde{B}^n_{0,c}e^{L\tilde{\Delta}}(x,y)=$ r.h.s. of (A.18). By the change of variable $w\mapsto w-n^{-2/3}L$ we get the claimed expression (A.12). $\square$

For $r>0$, let $U_r: L^2(\mathbb{R})\to L^2(\mathbb{R})$ be as $U_rf(x)=e^{rx}f(x)$ and

$$
\beta=\max\{2e^{r^3/3-2rc}, e^{(r-1/7)^3/3-2(r-1/7)c}\}. \tag{A.19}
$$

Lemma A.2. *Let $n\in\mathbb{Z}_{\geq 1}$ and $1\leq r^2\leq 2c$ with $r>0$, then*

$$
\max\{\left\|U_r^{-1}P_0B_{0,c}U_r\right\|_{\mathrm{HS}}, \left\|U_rP_0B_{0,c}P_0U_r^{-1}\right\|_{\mathrm{HS}}\}\leq\beta, \tag{A.20}
$$

$$
\|U_rP_0\hat{B}^n_{0,c}B_{0,c}P_0U_r^{-1}\|_{\mathrm{HS}}=\|U_r^{-1}P_0B_{0,c}\tilde{B}^n_{0,c}P_0U_r\|_{\mathrm{HS}}\leq\beta^{n+1}, \tag{A.21}
$$

$$
\max\{\|U_r^{-1}\tilde{B}^n_{0,c}U_r\|_{\mathrm{op}}, \|U_r^{-1}\bar{P}_0\tilde{B}^n_{0,c}P_0U_r\|_{\mathrm{HS}}, \|U_r^{-1}P_0\tilde{B}^n_{0,c}\bar{P}_0U_r\|_{\mathrm{HS}}\}\leq\beta^n, \tag{A.22}
$$

$$
\max\{\|U_r\hat{B}^n_{0,c}U_r^{-1}\|_{\mathrm{op}}, \|U_rP_0\hat{B}^n_{0,c}\bar{P}_0U_r^{-1}\|_{\mathrm{HS}}, \|U_r\bar{P}_0\hat{B}^n_{0,c}P_0U_r^{-1}\|_{\mathrm{HS}}\}\leq\beta^n, \tag{A.23}
$$

$$
\max\{\|U_r^{-1}P_0e^{L\tilde{\Delta}}P_0U_r^{-1}\|_{\mathrm{HS}}, \|U_r^{-1}P_0e^{L\Delta}P_0U_r^{-1}\|_{\mathrm{HS}}\}\leq\tfrac{1}{\sqrt{L}}. \tag{A.24}
$$

Proof. Let $f\in L^2(\mathbb{R})$. For (A.20), by symmetry of $B_{0,c}$, we have

$$
\left\|U_rP_0B_{0,c}P_0U_r^{-1}\right\|_{\mathrm{HS}}=\left\|U_r^{-1}P_0B_{0,c}P_0U_r\right\|_{\mathrm{HS}}\leq\left\|U_r^{-1}P_0B_{0,c}U_r\right\|_{\mathrm{HS}}. \tag{A.25}
$$

It is enough to show $\|U_r^{-1}P_0B_{0,c}U_r\|_{\mathrm{HS}}\leq\beta$. By definition of Hilbert-Schmidt norm, we have

$$
\|U_r^{-1}P_0B_{0,c}U_r\|^2_{\mathrm{HS}}=\int_0^\infty dx\int_{-\infty}^\infty dy e^{-2r(x-y)}\mathrm{Ai}(x+y+2c)^2=\frac{e^{\frac{2r^3}{3}-4cr}}{8\sqrt{2\pi}r^{3/2}}, \tag{A.26}
$$

where in the last step we use (Lemma 2.6 of [24])

$$
\int_{\mathbb{R}} dy e^{Ly}\mathrm{Ai}(y)^2=\frac{e^{L^3/12}}{\sqrt{4L\pi}}, \quad \forall L>0. \tag{A.27}
$$

(A.20) follows then from $r\geq 1$. The first equality of (A.21) follows from (A.5) and symmetry. Similar as (A.26), we have

$$
|U_r\hat{B}^n_{0,c}B_{0,c}P_0U_r^{-1}\|^2_{\mathrm{HS}}=\frac{e^{(n+1)(2r^3/3-4cr)}}{8\sqrt{2\pi(n+1)}r^{3/2}}\leq\beta^{2(n+1)}. \tag{A.28}
$$

(A.21) follows from $\|U_rP_0\hat{B}^n_{0,c}B_{0,c}P_0U_r^{-1}\|_{\mathrm{HS}}\leq|U_r\hat{B}^n_{0,c}B_{0,c}P_0U_r^{-1}\|_{\mathrm{HS}}$. For (A.22), we first show $\|U_r^{-1}\tilde{B}^n_{0,c}U_r\|_{\mathrm{op}}\leq\beta$. Define $h(x)=e^{-rx}\mathrm{Ai}(-x+2c)$, then

$$
\|U_r^{-1}\tilde{B}_{0,c}U_rf\|_{L^2(\mathbb{R})}=\|h*f\|_{L^2(\mathbb{R})}\leq\|f\|_{L^2(\mathbb{R})}\|h\|_{L^1(\mathbb{R})}, \tag{A.29}
$$

where we apply Young's inequality for convolution in the last step. It is enough to bound $\|h\|_{L^1(\mathbb{R})}$. Note that

$$\|h\|_{L^1(\mathbb{R})} = e^{-2rc}\int_{\mathbb{R}} dx|e^{rx}\mathrm{Ai}(x)| \leq e^{-2rc}\left(\int_{\mathbb{R}} dxe^{rx}\mathrm{Ai}(x) + 2\int_{-\infty}^{0} dxe^{rx}|\mathrm{Ai}(x)|\right). \tag{A.30}$$

By $\max|\mathrm{Ai}(x)| \leq 3/5$, we have

$$2\int_{-\infty}^{0} dxe^{rx}|\mathrm{Ai}(x)| \leq \frac{6}{5r} \leq e^{r^3/3}, \quad \forall r \geq 1. \tag{A.31}$$

Together with the identity (see (9.10.13) in [13])

$$\int_{\mathbb{R}} dxe^{rx}\mathrm{Ai}(x) = e^{r^3/3}, \quad \forall r > 0. \tag{A.32}$$

we have $\|h\|_{L^1(\mathbb{R})} \leq 2e^{r^3/3-2rc} \leq \beta$. For the rest two quantities in (A.22), it is enough to show $\|U_r^{-1}\bar{P}_0\tilde{B}_{0,c}^n P_0 U_r\|_{\mathrm{HS}} \leq \beta$. Using (A.4) and the definition of Hilbert-Schmidt norm, we have

$$\begin{aligned}\|U_r^{-1}\bar{P}_0\tilde{B}_{0,c}^n P_0 U_r\|_{\mathrm{HS}}^2 &= \int_{-\infty}^{0} dxn^{-2/3}\int_0^{\infty} dye^{2r(y-x)}\mathrm{Ai}\left(\tfrac{y-x}{n^{1/3}} + 2n^{2/3}c\right)^2 \\ &= \int_0^{\infty} due^{2rn^{1/3}u}\mathrm{Ai}(u+2n^{2/3}c)^2 u,\end{aligned} \tag{A.33}$$

where we made the change of variable $u = (y-x)/n^{1/3}$ and $v = (y+x)/n^{1/3}$ and integrated over v. Using the identity ((2.4) of [33])

$$\mathrm{Ai}(y)^2 = \frac{1}{4\pi^{3/2}\mathrm{i}}\int_{\mathrm{i}\mathbb{R}+\varepsilon} dww^{-1/2}e^{\frac{1}{12}w^3-wy}, \quad \varepsilon > 0, y \in \mathbb{R}, \tag{A.34}$$

the last integral in (A.33) is equal to

$$\frac{1}{4\pi^{3/2}\mathrm{i}}\int_{\mathrm{i}\mathbb{R}+\varepsilon} dw\frac{e^{\frac{w^3}{12}-2wn^{2/3}c}}{\sqrt{w}(w-2n^{1/3}r)^2} \tag{A.35}$$

for any $\varepsilon > 2n^{1/3}r$. Choosing $\varepsilon = 2n^{1/3}r + 1/(r^{1/2}n^{1/6})$ we get that the absolute value of the last integral is, for $1 \leq r^2 \leq 2c$, bounded by

$$\frac{e^{2nr^3/3-4nc}}{\sqrt{2}\pi r^{1/2}} \leq \beta^{2n}. \tag{A.36}$$

Applying (A.3) and same method as the one for (A.22), we can obtain (A.23). For (A.24), by definition, we have

$$\|U_r^{-1}P_0e^{L\tilde{\Delta}}P_0U_r^{-1}\|_{\mathrm{HS}}^2 = \frac{1}{4\pi L}\int_{\mathbb{R}_+^2} dxdye^{-2rx-2ry}e^{-\frac{(x+y)^2}{4L}} \leq \frac{1}{16\pi Lr^2} \leq \frac{1}{L}, \tag{A.37}$$

since $r \geq 1$. Similarly, we can also show the case for $e^{L\Delta}$. □

Next, we deduce upper bounds for operators involving heat kernel.

Lemma A.3. *Let $n \in \mathbb{Z}_{\geq 1}, L, r \geq 1$ and $r^2 \leq 2c$, then*

$$\|U_r P_0 \hat{B}_{0,c}^{n-1} e^{-L\Delta} B_0 P_0 U_r\|_{\mathrm{HS}} \leq \beta^n e^{-\frac{4L^3}{3n^2}} e^{-2Lc}, , \tag{A.38}$$

$$\|U_r^{-1} P_0 e^{L\Delta} B_{0,c} P_0 U_r^{-1}\|_{\mathrm{HS}} \leq \beta e^{Lr^2}, \tag{A.39}$$

$$\|U_r^{-1} P_0 B_{0,c} e^{L\tilde{\Delta}} P_0 U_r^{-1}\|_{\mathrm{HS}} \leq \beta e^{Lr^2}, \tag{A.40}$$

$$\|U_r^{-1} P_0 e^{L\tilde{\Delta}} \hat{B}_{0,c}^n P_0 U_r^{-1}\|_{\mathrm{HS}} = \|U_r^{-1} P_0 \tilde{B}_{0,c}^n e^{L\tilde{\Delta}} P_0 U_r^{-1}\|_{\mathrm{HS}} \leq \beta^n e^{Lr^2}, \tag{A.41}$$

$$\|U_r^{-1} P_0 \tilde{B}_{0,c}^n e^{L\Delta} P_0 U_r^{-1}\|_{\mathrm{HS}}^2 \leq \beta^n e^{Lr^2}, \tag{A.42}$$

$$\|U_r^{-1} P_0 B_{0,c} \tilde{B}_{0,c}^{n-1} e^{L\tilde{\Delta}} P_0 U_r^{-1}\|_{\mathrm{HS}} \leq \beta^n e^{Lr^2}. \tag{A.43}$$

Proof. For (A.38): applying (A.13) and definition of Hilbert-Schmidt norm, we have

$$\begin{aligned}
&\|U_r P_0 \hat{B}_{0,c}^{n-1} e^{-L\Delta} B_0 P_0 U_r\|_{\mathrm{HS}}^2 \\
=&\frac{e^{-4Lc} e^{-\frac{4L^3}{3n^2}}}{n^{2/3}} \int_{\mathbb{R}_+^2} dx dy e^{2rx+2ry-\frac{2L(x+y)}{n}} \mathrm{Ai}\left(\tfrac{L^2}{n^{4/3}} + \tfrac{x+y}{n^{1/3}} + 2n^{2/3} c\right)^2 \\
=&e^{-4Lc-\frac{4L^3}{3n^2}} \int_0^\infty du u e^{-u\left(\frac{2L}{n^{2/3}} - 2n^{1/3} r\right)} \mathrm{Ai}\left(\tfrac{L^2}{n^{4/3}} + u + 2n^{2/3} c\right)^2 \\
\overset{(A.34)}{=}&\frac{e^{-4Lc} e^{-\frac{4L^3}{3n^2}}}{4\pi^{3/2} \mathrm{i}} \int_{\mathrm{i}\mathbb{R}+\alpha} dw \frac{e^{\frac{w^3}{12} - w\left(\frac{L^2}{n^{4/3}} + 2n^{2/3} c\right)}}{\left(w - \frac{2(-L+nr)}{n^{2/3}}\right)^2 w^{-1/2}}
\end{aligned} \tag{A.44}$$

with $\alpha = \frac{2(-L+nr)}{n^{2/3}} + \varepsilon$ for arbitrary $\varepsilon > 0$. With the choice $\alpha = \frac{2(L+nr)}{n^{2/3}}$ we get

$$|(A.44)| \leq \frac{n^{5/3}}{32\sqrt{2} L^2 \pi \sqrt{L+nr}} e^{2Lr^2 - 8cL} e^{-\frac{8L^3}{3n^2}} (e^{2r^3/3 - 4cr})^n, \tag{A.45}$$

which, for $1 \leq r^2 \leq 2c$, will be dominated by the claimed bound (we used that $(e^{2r^3/3-4cr})^n \leq (\beta/2)^{2n}$ and $n^{5/3} \leq 4^n$ for $n \geq 1$). Applying (A.9) and similar method as above, we will get (A.39). For (A.40), applying (A.10), definition of Hilbert-Schmidt norm, (A.27) and $L, r \geq 1$, we have

$$\|U_r^{-1} P_0 B_{0,c} e^{L\tilde{\Delta}} P_0 U_r^{-1}\|_{\mathrm{HS}}^2 \leq \|U_r^{-1} P_0 B_{0,c} e^{L\tilde{\Delta}} U_r^{-1}\|_{\mathrm{HS}}^2 = \frac{e^{\frac{2r^3}{3} - 4rc + 2Lr^2}}{8\sqrt{2\pi} r \sqrt{L+r}} \leq \beta^2 e^{2Lr^2}. \tag{A.46}$$

For (A.41), the first equality follows from (A.12). For the inequality, since $r \geq 1$, we have

$$\|U_r^{-1} P_0 \tilde{B}_{0,c}^n e^{L\tilde{\Delta}} P_0 U_r^{-1}\|_{\mathrm{HS}} \leq \|U_r^{-1} P_0 \tilde{B}_{0,c}^n e^{L\tilde{\Delta}} P_0 U_{r-1/7}^{-1}\|_{\mathrm{HS}} \leq \|U_r^{-1} P_0 \tilde{B}_{0,c}^n e^{L\tilde{\Delta}} U_{r-1/7}^{-1}\|_{\mathrm{HS}}. \tag{A.47}$$

Similarly as (A.46), applying (A.27), we have

$$\|U_r^{-1} P_0 \tilde{B}_{0,c}^n e^{L\tilde{\Delta}} U_s^{-1}\|_{\mathrm{HS}}^2 = \frac{e^{(2s^3/3 - 4cs)n + 2Ls^2}}{4\sqrt{2\pi}(r-s)\sqrt{L+ns}}, \quad \forall r > s > 0. \tag{A.48}$$

Applying this with $s = r - 1/7$, $7 \leq 4\sqrt{2\pi}$ and plugging back to (A.47), we obtain (A.41). For (A.42), applying (A.11), $r \geq 1$ and definition of Hilbert-Schmidt norm, we have

$$\begin{aligned}
&\|U_r^{-1} P_0 \tilde{B}_{0,c}^n e^{L\Delta} P_0 U_r^{-1}\|_{\mathrm{HS}}^2 \\
&= n^{-2/3} e^{4Lc + \frac{4L^3}{3n^2}} \int_{\mathbb{R}_+^2} dx dy e^{\frac{2L(y-x)}{n} - 2r(x+y)} \mathrm{Ai}\left(\tfrac{L^2}{n^{4/3}} + \tfrac{y-x}{n^{1/3}} + 2n^{2/3} c\right)^2 \\
&= \frac{e^{4Lc + \frac{4L^3}{3n^2}}}{4n^{1/3} r} \left(\int_{-\infty}^0 du e^{\frac{2Lu}{n^{2/3}} + 2n^{1/3} ru} \mathrm{Ai}(L^2 n^{-4/3} + u + 2n^{2/3} c) \right. \\
&\qquad\qquad \left. + \int_0^\infty du e^{\frac{2Lu}{n^{2/3}} - 2n^{1/3} ru} \mathrm{Ai}(L^2 n^{-4/3} + u + 2n^{2/3} c) \right) \\
&\leq \frac{e^{4Lc + \frac{4L^3}{3n^2}}}{4n^{1/3} r} \int_{\mathbb{R}} du e^{\frac{2Lu}{n^{2/3}} + 2n^{1/3} ru} \mathrm{Ai}(L^2 n^{-4/3} + u + 2n^{2/3} c) \\
&\overset{(A.27)}{=} \frac{e^{\frac{2nr^3}{3} - 4cnr + 2Lr^2}}{8\sqrt{2\pi} r \sqrt{L + nr}} \leq \beta^{2n} e^{2Lr^2}
\end{aligned} \tag{A.49}$$

For (A.43), applying (A.14), definition of Hilbert-Schmidt norm, (A.27) and $r \geq 1$, we have 6. By A.14 in Lemma A.1 and

$$\|U_r^{-1} P_0 B_{0,c} \tilde{B}_{0,c}^n e^{L\tilde{\Delta}} P_0 U_r^{-1}\|_{\mathrm{HS}}^2 \leq \|U_r^{-1} P_0 B_{0,c} \tilde{B}_{0,c}^n e^{L\tilde{\Delta}} U_r^{-1}\|_{\mathrm{HS}}^2 \leq \beta^{2(n+1)} e^{2Lr^2}. \tag{A.50}$$

□

B Upper bounds

In this section, we provide some useful upper bounds. Let's first recall that

$$\begin{aligned}
&K_a = \tilde{B}_{0,c}, \quad K_b = P_0 \tilde{B}_{0,c}, \quad K_c = e^{L\tilde{\Delta}} P_0 e^{-L\Delta} B_{0,c}, \quad K_d = P_0 e^{L\tilde{\Delta}} P_0 e^{-L\Delta} B_{0,c}, \\
&K_e = P_0 e^{L\Delta} P_0 e^{-L\Delta} B_{0,c}, \quad K_u = P_0 B_{0,c}, \quad K_v = e^{L\Delta} P_0 e^{-L\Delta} B_{0,c}.
\end{aligned} \tag{B.1}$$

From here, we will use the convention

$$\prod_{i=1}^{0} A_i = 1, \quad \text{for any operators } A_1, \ldots, A_n. \tag{B.2}$$

Also, we will always assume $L, r \geq 1$ and $2c \geq r^2$ so that we can apply the bounds of Section A.

Lemma B.1. *Let $n \in \mathbb{Z}_{\geq 0}$, $\sigma_n \in \{a, b\}^n$, then $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{op}} \leq \beta^{n+1}$, where*

$$\hat{\Phi}_{\sigma_n} = U_r^{-1} P_0 B_{0,c} \left[\prod_{i=1}^{n} K_{\sigma_n(i)} \right] P_0 U_r. \tag{B.3}$$

Proof. We show this via induction on n. The case $n = 0$ follows from (A.20). Consider now the induction step $n-1 \mapsto n$.
Case I: $\sigma_n \equiv a$. Then $\hat{\Phi}_{\sigma_n} = U_r^{-1} P_0 B_{0,c} \tilde{B}_{0,c}^n P_0 U_r$, the result follows from (A.21).
Case II: $\sigma_n \in \{a,b\}^n$ with $b \in \sigma_n$. Define $\ell_b = \max\{i | \sigma_n(i) = b\} \geq 1$, then

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_r^{-1} P_0 B_{0,c} \left[\prod_{i=1}^{\ell_b - 1} K_{\sigma_n(i)} \right] P_0 U_r}_{=:\hat{\varphi}_1} \underbrace{U_r^{-1} P_0 \tilde{B}_{0,c}^{n-\ell_b+1} P_0 U_r}_{=:\hat{\varphi}_2} . \tag{B.4}$$

Applying $\|\hat{\varphi}_1\|_{\mathrm{op}} \leq \beta^{\ell_b}$ (induction assumption) and $\|\hat{\varphi}_2\|_{\mathrm{op}} \leq \beta^{n-\ell_b+1}$ (by (A.22)), we have $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{op}} \leq \|\hat{\varphi}_1\|_{\mathrm{op}} \|\hat{\varphi}_2\|_{\mathrm{op}} \leq \beta^{n+1}$. ☐

Lemma B.2. *Let $n \in \mathbb{Z}_{\geq 0}$, $\sigma_n \in \{a,b\}^n$, then $\|\Phi_{\sigma_n}\|_{\mathrm{HS}} \leq \beta^{n+1}$, where*

$$\Phi_{\sigma_n} = U_r^{-1} \bar{P}_0 \tilde{B}_{0,c} \left[\prod_{i=1}^{n} K_{\sigma_n(i)} \right] P_0 U_r. \tag{B.5}$$

Proof. For $n = 0$, $\Phi_{\sigma_0} = U_r^{-1} \bar{P}_0 \tilde{B}_{0,c} P_0 U_r$, the result follows from (A.22). For $n = 1$, there are only two possible Φ_{σ_1}, namely $\sigma_1(1) \in \{a,b\}$. For the first case, applying (A.22), we have $\|\Phi_{\sigma_1}\|_{HS} = \|U_r^{-1} \bar{P}_0 \tilde{B}_{0,c}^2 U_r\|_{HS} \leq \beta^2$. Suppose $\sigma_1(1) = b$. Applying (A.22), we get

$$\|\Phi_{\sigma_1}\|_{\mathrm{HS}} \leq \|U_r^{-1} \bar{P}_0 \tilde{B}_{0,c} P_0 U_r\|_{\mathrm{HS}} \|U_r^{-1} P_0 \tilde{B}_{0,c} P_0 U_r\|_{\mathrm{op}} \leq \beta^2. \tag{B.6}$$

For $n \geq 2$ we prove it by induction. For the induction step form $n-1$ to n we need to consider the following cases:
Case I: $\sigma_n(i) = a$ for all $i = 1, 2, \ldots, n$. We have $\Phi_{\sigma_n} = U_r^{-1} \bar{P}_0 \tilde{B}_{0,c}^{n+1} P_0 U_r$, the results follows from (A.22).
Case II: $b \in \sigma_n$. In this case, denote by $1 \leq \ell_b = \max\{i | \sigma_n(i) = b\} \leq n$. Then

$$\Phi_{\sigma_n} = \underbrace{U_r^{-1} \bar{P}_0 \tilde{B}_{0,c} \left[\prod_{i=1}^{\ell_b - 1} K_{\sigma_n(i)} \right] P_0 U_r}_{=:\varphi_1} \cdot \underbrace{U_r^{-1} P_0 \tilde{B}_{0,c}^{n-\ell_b+1} P_0 U_r}_{=:\varphi_2} . \tag{B.7}$$

Applying now $\|\varphi_1\|_{\mathrm{HS}} \leq \beta^{\ell_b}$ (induction assumption) and $\|\varphi_2\|_{\mathrm{op}} \leq \beta^{n-\ell_b+1}$ (by (A.22)), we have $\|\Phi_{\sigma_n}\|_{\mathrm{HS}} \leq \|\varphi_1\|_{\mathrm{HS}} \|\varphi_2\|_{\mathrm{op}} \leq \beta^{n+1}$. ☐

Lemma B.3. *Let $n \in \mathbb{Z}_{\geq 0}$, $\sigma_n \in \{a,b,c\}^n$, then $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \leq \beta^{n+1} e^{Lr^2}$ where*

$$\hat{\Phi}_{\sigma_n} = U_r^{-1} P_0 \tilde{B}_{0,c} \left[\prod_{i=1}^{n} K_{\sigma(i)} \right] e^{L\tilde{\Delta}} P_0 U_r^{-1}. \tag{B.8}$$

Proof. The case $n = 0$ follows from (A.41). We show this via induction on $n \geq 1$. For $n = 1$, we have the following cases. If $\sigma_1(1) = a$, we have $\hat{\Phi}_{\sigma_1} = U_r^{-1} P_0 \tilde{B}_{0,c}^2 e^{L\tilde{\Delta}} P_0 U_r^{-1}$, the result follows from (A.41). If $\sigma_1(1) = b$, then we have

$\hat{\Phi}_{\sigma_1} = U_r^{-1}P_0\tilde{B}_{0,c}P_0\tilde{B}_{0,c}e^{L\tilde{\Delta}}P_0U_r^{-1}$. Applying the bounds $\|U_r^{-1}P_0\tilde{B}_{0,c}P_0U_r\|_{\mathrm{op}} \leq \beta$ by (A.22), $\|U_r^{-1}P_0\tilde{B}_{0,c}e^{L\tilde{\Delta}}P_0U_r^{-1}\|_{\mathrm{HS}} \leq \beta e^{Lr^2}$ (by (A.41)) and Theorem 2.3, we have

$$\|\hat{\Phi}_{\sigma_1}\|_{\mathrm{HS}} \leq \|U_r^{-1}P_0\tilde{B}_{0,c}P_0U_r\|_{\mathrm{op}}\|U_r^{-1}P_0\tilde{B}_{0,c}e^{L\tilde{\Delta}}P_0U_r^{-1}\|_{\mathrm{HS}} \leq \beta^2 e^{Lr^2}, \tag{B.9}$$

If $\sigma_1(1) = c$, using $e^{-L\Delta}B_{0,c}e^{L\tilde{\Delta}} = \hat{B}_{0,c}$ (by (A.10)), we have

$$\hat{\Phi}_{\sigma_1} = U_r^{-1}P_0\tilde{B}_{0,c}e^{L\tilde{\Delta}}P_0e^{-L\Delta}B_{0,c}e^{L\tilde{\Delta}}P_0U_r^{-1} = U_r^{-1}P_0\tilde{B}_{0,c}e^{L\tilde{\Delta}}P_0\hat{B}_{0,c}P_0U_r^{-1}. \tag{B.10}$$

Applying $\|U_r^{-1}P_0\tilde{B}_{0,c}e^{L\tilde{\Delta}}P_0U_r^{-1}\|_{\mathrm{HS}} \leq \beta e^{Lr^2}$ (by (A.41)) and $\|U_rP_0\hat{B}_{0,c}P_0U_r^{-1}\|_{\mathrm{op}} \leq \beta$ (by (A.23)), we have

$$\|\hat{\Phi}_{\sigma_1}\|_{\mathrm{HS}} \leq \|U_r^{-1}P_0\tilde{B}_{0,c}e^{L\tilde{\Delta}}P_0U_r^{-1}\|_{\mathrm{HS}}\|U_rP_0\hat{B}_{0,c}P_0U_r^{-1}\|_{\mathrm{op}} \leq \beta^2 e^{Lr^2}. \tag{B.11}$$

Now we consider the induction step $n-1 \mapsto n$:
Case I: $\sigma_n \equiv a$, then $\hat{\Phi}_{\sigma_n} = U_r^{-1}P_0\tilde{B}_{0,c}^{n+1}e^{L\tilde{\Delta}}P_0U_r^{-1}$, the result follows from (A.41).
Case II: $\sigma_n \in \{a,b\}^n \setminus \{a\}^n$. We then have

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_r^{-1}P_0\tilde{B}_{0,c}^{f_b}P_0U_r}_{=:\hat{\varphi}_1}\underbrace{U_r^{-1}P_0\tilde{B}_{0,c}\left[\prod_{i=f_b+1}^{n}K_{\sigma_n(i)}\right]e^{L\tilde{\Delta}}P_0U_r^{-1}}_{=:\hat{\varphi}_2}, \tag{B.12}$$

where $f_b = \min\{i|\sigma_n(i) = b\} \geq 1$. Applying $\|\hat{\varphi}_1\|_{\mathrm{op}} \leq \beta^{f_b}$ (by (A.22)) and $\|\hat{\varphi}_2\|_{\mathrm{HS}} \leq \beta^{n-f_b+1}e^{Lr^2}$ (induction assumption), we have

$$\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \leq \|\hat{\varphi}_1\|_{\mathrm{op}}\|\hat{\varphi}_2\|_{\mathrm{HS}} \leq \beta^{n+1}e^{Lr^2}. \tag{B.13}$$

Case III: $\sigma_n \in \{a,b,c\}^n \setminus \{a,b\}^n$. Then we have $\ell_c = \max\{i|\sigma_n(i) = c\} \geq 1$ and hence

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_r^{-1}P_0\tilde{B}_{0,c}\left[\prod_{i=1}^{\ell_c-1}K_{\sigma_n(i)}\right]e^{L\tilde{\Delta}}P_0U_r^{-1}}_{=:\hat{\varphi}_1}\cdot\underbrace{U_rP_0e^{-L\Delta}B_{0,c}\left[\prod_{i=\ell_c+1}^{n}K_{\sigma_n(i)}\right]e^{L\tilde{\Delta}}P_0U_r^{-1}}_{=:\hat{\varphi}_2}. \tag{B.14}$$

By induction assumption, $\|\hat{\varphi}_1\|_{\mathrm{HS}} \leq \beta^{\ell_c}e^{Lr^2}$. As for $\hat{\varphi}_2$, if $\sigma_n(i) = a$ for all $i \in \{\ell_c+1,\ldots,n\}$, then by (A.6), we have

$$\hat{\varphi}_2 = U_rP_0e^{-L\Delta}B_{0,c}\tilde{B}_{0,c}^{n-\ell_c}e^{L\tilde{\Delta}}P_0U_r^{-1} = U_rP_0\hat{B}_{0,c}^{n-\ell_c+1}P_0U_r^{-1}. \tag{B.15}$$

Applying $\|\hat{\varphi}_2\|_{\mathrm{op}} \leq \beta^{n-\ell_c+1}$ (by (A.23)), we have $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \leq \|\hat{\varphi}_1\|_{\mathrm{HS}}\|\hat{\varphi}_2\|_{\mathrm{op}} \leq \beta^{n+1}e^{Lr^2}$. If there exists $j \geq \ell_c+1$ such that $\sigma_n(j) = b$, we set $f_b = \min\{j \geq \ell_c+1|\sigma_n(j) = b\}$. Then we have

$$\hat{\varphi}_2 = \underbrace{U_rP_0\hat{B}_{0,c}^{f_b-\ell_c-1}e^{-L\Delta}B_{0,c}P_0U_r}_{=:\hat{\varphi}_2^1}\cdot\underbrace{U_r^{-1}P_0\tilde{B}_{0,c}\left[\prod_{i=f_b+1}^{n}K_{\sigma_n(i)}\right]e^{L\tilde{\Delta}}P_0U_r^{-1}}_{=:\hat{\varphi}_2^2}, \tag{B.16}$$

where we use the definition of f_b, ℓ_c and the identity $e^{-L\Delta}B_{0,c}\tilde{B}_{0,c}^{f_b-\ell_c-1} = \hat{B}_{0,c}^{f_b-\ell_c-1}e^{-L\Delta}B_{0,c}$ (by (A.7)). Applying now $\|\hat{\varphi}_2^2\|_{\mathrm{HS}} \le \beta^{n-f_b+1}e^{Lr^2}$ (induction assumption), $\|\hat{\varphi}_2^1\|_{\mathrm{HS}} \le \beta^{f_b-\ell_c}e^{-\frac{4L^3}{3(f_b-\ell_c)^2}}e^{-2Lc}$ (by (A.38)) and $r^2 \le 2c$, we have

$$\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \le \|\hat{\varphi}_1\|_{\mathrm{HS}}\|\hat{\varphi}_2^1\|_{\mathrm{HS}}\|\hat{\varphi}_2^2\|_{\mathrm{HS}} \le \beta^{n+1}e^{-\frac{4L^3}{3n^2}}. \tag{B.17}$$

□

Lemma B.4. *For $\sigma_n \in \{a,b,c,d,e\}^n$ with $n \in \mathbb{Z}_{\ge 0}$, it holds $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \le \beta^{n+1}e^{-\frac{4L^3}{3(n+1)^2}}e^{-2Lc}$, where*

$$\hat{\Phi}_{\sigma_n} = U_rP_0e^{-L\Delta}B_{0,c}\left[\prod_{i=1}^{n}K_{\sigma_n(i)}\right]P_0U_r. \tag{B.18}$$

Proof. We show this via induction on n. The case $n=0$ follows from (A.38). For $n=1$, we have the following cases:

1. $\sigma_1(1)=a$: then $\hat{\Phi}_{\sigma_n} = U_rP_0\hat{B}_{0,c}e^{-L\Delta}B_{0,c}P_0U_r$ by (A.7). The results follows from (A.38).

2. $\sigma_1(1)=b$: then $\hat{\Phi}_{\sigma_n} = U_rP_0e^{-L\Delta}B_{0,c}P_0\tilde{B}_{0,c}P_0U_r$. Applying (A.38) and (A.22), we have

$$\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \le \|U_rP_0e^{-L\Delta}B_{0,c}P_0U_r\|_{\mathrm{HS}}\|U_r^{-1}P_0\tilde{B}_{0,c}P_0U_r\|_{\mathrm{op}} \le \beta^2e^{-\frac{4L^3}{3}}e^{-2Lc}, \tag{B.19}$$

3. $\sigma_1(1)=c$: then $\hat{\Phi}_{\sigma_n} = U_rP_0\hat{B}_{0,c}P_0e^{-L\Delta}B_{0,c}P_0U_r$ (by (A.10)). Applying $\|U_rP_0\hat{B}_{0,c}P_0U_r^{-1}\|_{\mathrm{op}} \le \beta$ (by (A.23)) and (A.38), we have

$$\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \le \|U_rP_0\hat{B}_{0,c}P_0U_r^{-1}\|_{\mathrm{op}}\|U_rP_0e^{-L\Delta}B_{0,c}P_0U_r\|_{\mathrm{HS}} \le \beta^2e^{-\frac{4L^3}{3}}e^{-2Lc}. \tag{B.20}$$

4. $\sigma_1(1)=d$: then

$$\hat{\Phi}_{\sigma_n} = U_rP_0e^{-L\Delta}B_{0,c}P_0U_r \cdot U_r^{-1}P_0e^{L\tilde{\Delta}}P_0U_r^{-1} \cdot U_rP_0e^{-L\Delta}B_{0,c}P_0U_r. \tag{B.21}$$

   Applying $\|U_r^{-1}P_0e^{L\tilde{\Delta}}P_0U_r^{-1}\|_{\mathrm{HS}} \le \frac{1}{\sqrt{L}}$ (by (A.24)), (A.38) and $L \ge 1$, we have $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \le \beta^2e^{-\frac{8L^3}{3}}e^{-4Lc}$.

5. Similarly, we can also show the case for $\sigma_1(1)=e$..

Now let's consider the induction step: $n-1 \mapsto n$.
Case I: $\sigma_n \equiv a$, then $\hat{\Phi}_{\sigma_n} = U_rP_0\hat{B}_{0,c}^ne^{-L\Delta}B_{0,c}P_0U_r$ (by (A.7)). The result follows from (A.38).
Case II: $\sigma_n \in \{a,c\}^n \setminus \{a\}^n$. Define $f_c = \min\{j|\sigma_{n+1}(j)=c\} \ge 1$, then

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_rP_0e^{-L\Delta}B_{0,c}\tilde{B}_{0,c}^{f_c-1}e^{L\tilde{\Delta}}P_0U_r^{-1}}_{=:\hat{\varphi}_1}\underbrace{U_rP_0e^{-L\Delta}B_{0,c}\left[\prod_{j=f_c+1}^{n}K_{\sigma_n(i)}\right]P_0U_r}_{=:\hat{\varphi}_2}. \tag{B.22}$$

Using (A.7) and (A.10), we have $\hat{\varphi}_1 = U_r P_0 \hat{B}_{0,c}^{f_c} P_0 U_r^{-1}$. Applying $\|\hat{\varphi}_1\|_{\mathrm{op}} \le \beta^{f_c}$ by (A.23) and $\|\hat{\varphi}_2\|_{\mathrm{HS}} \le \beta^{n-f_c+1} e^{-\frac{4L^3}{3(n+1-f_c)^2}} e^{-2Lc}$ (induction hypothesis), we have $\|\hat{\Phi}_{\sigma_n}\| \le \|\hat{\varphi}_1\|_{\mathrm{op}} \|\hat{\varphi}_2\|_{\mathrm{HS}} \le \beta^{n+1} e^{-\frac{4L^3}{3(n+1)^2}} e^{-2Lc}$.

Case III: $\sigma_n \in \{a,b,c\}^n \setminus \{a,c\}^n$**.** Define $\ell_b = \max\{j | \sigma_n(j) = b\} \ge 1$, then

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_r P_0 e^{-L\Delta} B_{0,c} \left[\prod_{i=1}^{\ell_b - 1} K_{\sigma_n(i)} \right] P_0 U_r}_{\hat{\varphi}_1} \underbrace{U_r^{-1} P_0 \tilde{B}_{0,c} \left[\prod_{j=\ell_b+1}^{n} K_{\sigma_n(i)} \right] P_0 U_r}_{=:\hat{\varphi}_2} . \tag{B.23}$$

By induction we have $\|\hat{\varphi}_1\|_{\mathrm{HS}} \le \beta^{\ell_b} e^{-\frac{4L^3}{3\ell_b^2}} e^{-2Lc}$. For $\hat{\varphi}_2$, if $\sigma_n(i) = a$ for all $i > \ell_b$, then we have $\hat{\varphi}_2 = U_r^{-1} P_0 \tilde{B}_{0,c}^{n+1-\ell_b} P_0 U_r$. Applying $\|\hat{\varphi}_2\|_{\mathrm{op}} \le \beta^{n+1-\ell_b}$ by (A.22), we have

$$\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \le \|\hat{\varphi}_1\|_{\mathrm{HS}} \|\hat{\varphi}_2\|_{\mathrm{op}} \le \beta^{n+1} e^{-\frac{4L^3}{3(n+1)^2}} e^{-2Lc}. \tag{B.24}$$

If it exists $j \in \{\ell_b + 1, \dots, n\}$ with $\sigma_n(j) = c$, define $f_c = \min\{j > \ell_b | \sigma_n(j) = c\}$, then

$$\hat{\varphi}_2 = \underbrace{U_r^{-1} P_0 \tilde{B}_{0,c}^{f_c - \ell_b} e^{L\tilde{\Delta}} P_0 U_r^{-1}}_{=\hat{\varphi}_2^1} \underbrace{U_r P_0 e^{-L\Delta} B_{0,c} \left[\prod_{j=f_c+1}^{n} K_{\sigma_n(i)} \right] P_0 U_r}_{=:\hat{\varphi}_2^2} . \tag{B.25}$$

Applying $\|\hat{\varphi}_2^2\|_{\mathrm{HS}} \le \beta^{n+1-f_c} e^{-\frac{4L^3}{3(n+1-f_c)^2}} e^{-2Lc}$ (induction assumption), $\|\hat{\varphi}_2^1\|_{\mathrm{HS}} \le \beta^{f_c - \ell_b} e^{Lr^2}$ (by (A.41)) and $r^2 \le 2c$, we obtain

$$\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \le \|\hat{\varphi}_1\|_{\mathrm{HS}} \|\hat{\varphi}_2^1\|_{\mathrm{HS}} \|\hat{\varphi}_2^2\|_{\mathrm{HS}} \le \beta^{n+1} e^{-\frac{4L^3}{3(n+1)^2}} e^{-2Lc}, \tag{B.26}$$

Case IV: $\sigma_n \in \{a,b,c,d\}^n \setminus \{a,b,c\}^n$**.** Set $f_d = \min\{i | \sigma_n(i) = d\} \ge 1$. Then

$$\begin{aligned} \hat{\Phi}_{\sigma_n} &= \underbrace{U_r P_0 e^{-L\Delta} B_{0,c} \left[\prod_{i=1}^{f_d - 1} K_{\sigma_n(i)} \right] P_0 U_r}_{=:\hat{\varphi}_1} \underbrace{U_r^{-1} P_0 e^{L\tilde{\Delta}} P_0 U_r^{-1}}_{=:\hat{\varphi}_2} \\ &\quad \times \underbrace{U_r P_0 e^{-L\Delta} B_{0,c} \left[\prod_{i=f_d+1}^{n} K_{\sigma_n(i)} \right] P_0 U_r}_{=:\hat{\varphi}_3} . \end{aligned} \tag{B.27}$$

Applying $\|\hat{\varphi}_1\|_{\mathrm{HS}} \le \beta^{f_d} e^{-\frac{4L^3}{3f_d^2}} e^{-2Lc}$, $\|\hat{\varphi}_3\|_{\mathrm{HS}} \le \beta^{n+1-f_d} e^{-\frac{4L^3}{3(n-f_d+1)^2}} e^{-2Lc}$ (induction assumption) and $\|\hat{\varphi}_2\|_{\mathrm{HS}} \le \frac{1}{\sqrt{L}}$ (by (A.24)), we have

$$\|\hat{\Phi}_{\sigma_n}^{u,w}\|_{\mathrm{HS}} \le \|\hat{\varphi}_1\|_{\mathrm{HS}} \|\hat{\varphi}_2\|_{\mathrm{HS}} \|\hat{\varphi}_3\|_{\mathrm{HS}} \le \beta^{n+1} e^{-\frac{4L^3}{3(n+1)^2}} e^{-2Lc}. \tag{B.28}$$

Case V: $\sigma_n \in \{a,b,c,d,e\}^n \setminus \{a,b,c,d\}^n$. Same as **Case IV**. ∎

Lemma B.5. *For $n \in \mathbb{Z}_{\geq 0}$ and $\sigma_n \in \{a, c\}^n$, it holds $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{op}} \leq \beta^{n+1}$, where*

$$\hat{\Phi}_{\sigma_n} = U_r P_0 e^{-L\Delta} B_{0,c} \left[\prod_{i=1}^{n} K_{\sigma_n(i)} \right] e^{L\Delta} P_0 U_r^{-1} \tag{B.29}$$

Proof. The case $n = 0$ follows from (A.20) and $\|\cdot\|_{\mathrm{op}} \leq \|\cdot\|_{\mathrm{HS}}$. We show this via induction on n. For $n \geq 1$, we have a few cases. If $\sigma_1 = a$, then $\hat{\Phi}_{\sigma_n} = U_r P_0 \hat{B}_{0,c} B_{0,c} P_0 U_r^{-1}$ by (A.7), the result follows from (A.21). If $\sigma_1 = c$, then $\hat{\Phi}_{\sigma_1} = U_r P_0 \hat{B}_{0,c} P_0 B_{0,c} P_0 U_r^{-1}$ by (A.10). Applying $\|U_r P_0 \hat{B}_{0,c} P_0 U_r^{-1}\|_{\mathrm{op}} \leq \beta$ (by (A.23)) and $\|U_r P_0 B_{0,c} P_0 U_r^{-1}\|_{\mathrm{op}} \leq \|U_r P_0 B_{0,c} P_0 U_r^{-1}\|_{\mathrm{HS}} \leq \beta$ (by (A.20)), we then have

$$\|\hat{\Phi}_{\sigma_1}\|_{\mathrm{op}} \leq \|U_r P_0 \hat{B}_{0,c} P_0 U_r^{-1}\|_{\mathrm{op}} \|U_r P_0 B_{0,c} P_0 U_r^{-1}\|_{\mathrm{op}} \leq \beta^2 \tag{B.30}$$

For induction step $n - 1 \mapsto n$, we consider the following cases.
Case I: $\sigma_n \equiv a$: then $\hat{\Phi}_{\sigma_n} = U_r P_0 \hat{B}_{0,c}^n B_{0,c} P_0 U_r^{-1}$ by (A.7), the result is true by (A.21).
Case II: $\sigma_n \in \{a, c\}^n \setminus \{a\}^n$: Defining $f_c = \min\{j | \sigma_n(j) = c\} \geq 1$ we have

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_r P_0 e^{-L\Delta} B_{0,c} \tilde{B}_{0,c}^{f_c - 1} e^{L\tilde{\Delta}} P_0 U_r^{-1}}_{=:\hat{\varphi}_1} \underbrace{U_r P_0 e^{-L\Delta} B_{0,c} \left[\prod_{i=f_c+1}^{n} K_{\sigma_n(i)} \right] e^{L\Delta} P_0 U_r^{-1}}_{=:\hat{\varphi}_2} . \tag{B.31}$$

By (A.6), we have $\hat{\varphi}_1 = U_r P_0 \hat{B}_{0,c}^{f_c} P_0 U_r^{-1}$. Applying $\|\hat{\varphi}_1\|_{\mathrm{op}} \leq \beta^{f_c}$ (by (A.23)) and $\|\hat{\varphi}_2\|_{\mathrm{op}} \leq \beta^{n - f_c + 1}$ (induction assumption), we obtain $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{op}} \leq \|\hat{\varphi}_1\|_{\mathrm{op}} \|\hat{\varphi}_2\|_{\mathrm{op}} \leq \beta^{n+1}$.

□

Lemma B.6. *Let $n \in \mathbb{Z}_{\geq 0}$, $\sigma_n \in \{a, b, c\}^n$, then $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \leq \beta^{n+1} e^{Lr^2}$, where*

$$\hat{\Phi}_{\sigma_n} = U_r^{-1} P_0 \tilde{B}_{0,c} \left[\prod_{j=1}^{n} K_{\sigma_n(j)} \right] e^{L\Delta} P_0 U_r^{-1} \tag{B.32}$$

Proof. We show this via induction on n. The case $n = 0$ follows from (A.42). The case $n = 1$ can be handled similarly as before, so we omit the proof here. Now we consider the induction step $n - 1 \mapsto n$.
Case I: $\sigma_n \equiv a$: then $\hat{\Phi}_{\sigma_n} = U_r^{-1} P_0 \tilde{B}_{0,c}^{n+1} e^{L\Delta} P_0 U_r^{-1}$, the result follows from (A.42)
Case II: $\sigma_n \in \{a, c\}^n \setminus \{a\}^n$. We set $\ell_c = \min\{j | \sigma_n(j) = c\} \geq 1$. Then

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_r^{-1} P_0 \tilde{B}_{0,c}^{\ell_c} e^{L\tilde{\Delta}} P_0 U_r^{-1}}_{=:\hat{\varphi}_1} \underbrace{U_r P_0 e^{-L\Delta} B_{0,c} \left[\prod_{j=\ell_c+1}^{n} K_{\sigma_n(j)} \right] e^{L\Delta} P_0 U_r^{-1}}_{=:\hat{\varphi}_2} . \tag{B.33}$$

Applying $\|\hat{\varphi}_1\|_{\mathrm{HS}} \leq \beta^{\ell_c} e^{Lr^2}$ (by (A.41)) and $\|\hat{\varphi}_2\|_{\mathrm{op}} \leq \beta^{n - \ell_c + 1}$ (by Lemma B.5), we obtain $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \leq \|\hat{\varphi}_1\|_{\mathrm{HS}} \|\hat{\varphi}_2\|_{\mathrm{op}} \leq \beta^{n+1} e^{Lr^2}$.
Case III: $\sigma_n \in \{a, b, c\}^n \setminus \{a, c\}^n$. Define $f_b = \min\{j | \sigma_n(j) = b\} \geq 1$. Then

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_r^{-1} P_0 \tilde{B}_{0,c} \left[\prod_{j=1}^{f_b - 1} K_{\sigma_n(j)} \right] P_0 U_r}_{=:\hat{\varphi}_1} \underbrace{U_r^{-1} P_0 \tilde{B}_{0,c} \left[\prod_{j=f_b+1}^{n} K_{\sigma_n(j)} \right] e^{L\Delta} P_0 U_r^{-1}}_{=:\hat{\varphi}_2} . \tag{B.34}$$

By induction assumption, $\|\hat{\varphi}_2\|_{\mathrm{HS}} \le \beta^{n+1-f_b} e^{Lr^2}$. Next we deal with $\hat{\varphi}_1$. If $\sigma_n(j) = a$ for all $j \in \{1, \dots, f_b - 1\}$, then $\hat{\varphi}_1 = U_r^{-1} P_0 \tilde{B}_{0,c}^{f_b} P_0 U_r$ and $\|\hat{\varphi}_1\|_{\mathrm{op}} \le \beta^{f_b}$ (by (A.22)). Then $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \le \|\hat{\varphi}_1\|_{\mathrm{op}} \|\hat{\varphi}_2\|_{\mathrm{HS}} \le \beta^{n+1} e^{Lr^2}$. If there exists $j \in \{1, \dots, f_b - 1\}$ such that $\sigma_n(j) = c$, then we define $f_c = \min\{j \le f_b - 1 | \sigma_n(j) = c\}$ and decompose

$$\hat{\varphi}_1 = \underbrace{U_r^{-1} P_0 \tilde{B}_{0,c}^{f_c} e^{L\tilde{\Delta}} P_0 U_r^{-1}}_{=:\hat{\varphi}_1^1} \underbrace{U_r P_0 e^{-L\Delta} B_{0,c} \left[\prod_{f_c+1}^{f_b-1} K_{\sigma_n(j)} \right] P_0 U_r}_{\hat{\varphi}_1^2} . \tag{B.35}$$

Applying now $\|\hat{\varphi}_1^1\|_{\mathrm{HS}} \le \beta^{f_c} e^{Lr^2}$ (by (A.41)), $\|\hat{\varphi}_1^2\|_{\mathrm{HS}} \le \beta^{f_b - f_c} e^{-2cL}$ (by Lemma B.4) and $r^2 \le 2c$, we get $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \le \|\hat{\varphi}_1^1\|_{\mathrm{HS}} \|\hat{\varphi}_1^2\|_{\mathrm{HS}} \|\hat{\varphi}_2\|_{\mathrm{HS}} \le \beta^{n+1} e^{Lr^2}$. ☐

Lemma B.7. *For $n \in \mathbb{Z}_{\ge 0}$ and $\sigma_n \in \{a, b, c, d, e, v\}^n$, the operator*

$$\hat{\Phi}_{\sigma_n} = U_r P_0 e^{-L\Delta} B_{0,c} \left[\prod_{i=1}^{n} K_{\sigma_n(i)} \right] e^{L\Delta} P_0 U_r^{-1} \tag{B.36}$$

satisfies the following bounds:

1. *if $\sigma_n \in \{a, c, v\}^n$, then $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{op}} \le \beta^{n+1}$,*
2. *if $\sigma_n \in \{a, b, c, d, e, v\} \setminus \{a, c, v\}^n$, then $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{op}} \le \|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \le \beta^{n+1} e^{-\frac{4L^3}{3n^2}}$.*

Proof. The case $\sigma_n \in \{a, c\}^n$ is solved in Lemma B.5. We show the rest via induction on n and omit the details for $n = 1$. For induction step $n - 1 \mapsto n$.
Case I: $\sigma_n \in \{a, b, c\}^n \setminus \{a, c\}^n$. Define $\ell_b = \max\{j | \sigma_n(j) = b\} \ge 1$, then we have

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_r P_0 e^{-L\Delta} B_{0,c} \left[\prod_{i=1}^{\ell_b - 1} K_{\sigma_n(i)} \right] P_0 U_r}_{=:\hat{\varphi}_1} \underbrace{U_r^{-1} P_0 \tilde{B}_{0,c} \left[\prod_{i=\ell_b+1}^{n} K_{\sigma_n(i)} \right] e^{L\Delta} P_0 U_r^{-1}}_{=:\hat{\varphi}_2} . \tag{B.37}$$

Since $\sigma_n \in \{a, b, c\}^n$, we have $\|\hat{\varphi}_1\|_{\mathrm{HS}} \le \beta^{\ell_b} e^{-\frac{4L^3}{3\ell_b^2}} e^{-2Lc}$ (by Lemma B.4) and $\|\hat{\varphi}_2\|_{\mathrm{HS}} \le \beta^{n-\ell_b+1} e^{Lr^2}$ (by Lemma B.6), which implies $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \le \|\hat{\varphi}_1\|_{\mathrm{HS}} \|\hat{\varphi}_2\|_{\mathrm{HS}} \le \beta^{n+1} e^{-\frac{4L^3}{3\ell_b^2}} e^{Lr^2 - 2Lc} \le \beta^{n+1} e^{-\frac{4L^3}{3n^2}}$, where we use the assumption $r^2 \le 2c$.
Case II: $\sigma_n \in \{a, b, c, d\}^n \setminus \{a, b, c\}^n$. Setting $\ell_d = \min\{i | \sigma_n(i) = d\} \ge 1$, we have

$$\begin{aligned} \hat{\Phi}_{\sigma_n} &= \underbrace{U_r P_0 e^{-L\Delta} B_{0,c} \left[\prod_{i=1}^{\ell_d - 1} K_{\sigma_n(i)} \right] P_0 U_r}_{=:\hat{\varphi}_1} \underbrace{U_r^{-1} P_0 e^{L\Delta} P_0 U_r^{-1}}_{=:\hat{\varphi}_2} \\ &\times \underbrace{U_r P_0 e^{-L\Delta} B_{0,c} \left[\prod_{i=\ell_d+1}^{n} K_{\sigma_n(i)} \right] e^{L\Delta} P_0 U_r^{-1}}_{=:\hat{\varphi}_3} . \end{aligned} \tag{B.38}$$

Applying the bounds $\|\hat{\varphi}_1\|_{\mathrm{HS}} \le \beta^{\ell_d} e^{-\frac{4L^3}{3\ell_d^2}} e^{-2Lc}$ (by Lemma B.4), $\|\hat{\varphi}_2\|_{\mathrm{HS}} \le \frac{1}{\sqrt{L}}$ (by (A.24)) and $\|\hat{\varphi}_3\|_{\mathrm{op}} \le \beta^{n-\ell_d+1}$ (induction assumption), we have $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \le \|\hat{\varphi}_1\|_{\mathrm{HS}}\|\hat{\varphi}_2\|_{\mathrm{HS}}\|\hat{\varphi}_3\|_{\mathrm{op}} \le \beta^{n+1}e^{-\frac{4L^3}{3n^2}}$.

Case III: $\sigma_n \in \{a,b,c,d,e\}^n \setminus \{a,b,c,d\}^n$**.** It is the same decomposition as in the previous case except that in $\hat{\varphi}_2$ there is $e^{L\tilde{\Delta}}$ instead of $e^{L\Delta}$ and we use (A.24).

Case IV: $\sigma_n \in \{a,b,c,d,e,v\}^n \setminus \{a,b,c,d,e\}^n$. Define $\ell_v = \max\{i \mid \sigma_n(i) = v\} \le n$, then

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_r P_0 e^{-L\Delta} B_{0,c}\left[\prod_{i=1}^{\ell_v-1} K_{\sigma_n(i)}\right] e^{L\Delta} P_0 U_r^{-1}}_{=:\hat{\varphi}_1} \underbrace{U_r P_0 e^{-L\Delta} B_{0,c}\left[\prod_{i=\ell_v+1}^{n} K_{\sigma_n(i)}\right] e^{L\Delta} P_0 U_r^{-1}}_{=:\hat{\varphi}_2}. \tag{B.39}$$

The results follows from induction assumption. $\square$

Corollary B.8. *Let $n \in \mathbb{Z}_{\ge 0}$, $\sigma_n \in \{a,b,c,d,e,v\}^n$, then $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \le \beta^{n+1}e^{-\frac{4L^3}{3(n+1)^2}}e^{-2Lc}$, where*

$$\hat{\Phi}_{\sigma_n} = U_r P_0 e^{-L\Delta} B_{0,c}\left[\prod_{i=1}^{n} K_{\sigma_n(i)}\right] P_0 U_r. \tag{B.40}$$

Proof. If $\sigma_n \in \{a,b,c,d,e\}^n$, the result follows from Lemma B.4. Consider $\sigma_n \in \{a,b,c,d,e,v\}^n$ with $v \in \sigma_n$, we have

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_r P_0 e^{-L\Delta} B_{0,c}\left[\prod_{i=1}^{\ell_v-1} K_{\sigma_n(i)}\right] e^{L\Delta} P_0 U_r^{-1}}_{=:\hat{\varphi}_1} \underbrace{U_r P_0 e^{-L\Delta} B_{0,c}\left[\prod_{i=\ell_v+1}^{n} K_{\sigma_n(i)}\right] P_0 U_r}_{=:\hat{\varphi}_2}, \tag{B.41}$$

where $\ell_v = \max\{i|\sigma_n(i) = v\} \ge 1$. Applying $\|\hat{\varphi}_1\|_{\mathrm{op}} \le \beta^{\ell_v}$ (by Lemma B.7) and $\|\hat{\varphi}_2\|_{\mathrm{HS}} \le \beta^{n-\ell_v+1}e^{-\frac{4L^3}{(n-\ell_v+1)^2}}e^{-2Lc}$ (by definition of ℓ_v, we can use Lemma B.4), we then have $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \le \|\hat{\varphi}_1\|_{\mathrm{op}}\|\hat{\varphi}_2\|_{\mathrm{HS}} \le \beta^{n+1}e^{-\frac{4L^3}{3(n+1)^2}}e^{-2Lc}$. $\square$

Lemma B.9. *For $n \in \mathbb{Z}_{\ge 0}$ and $\sigma_n \in \{a,b,c\}^n$, we have $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \le \beta^{n+1}e^{Lr^2}$, where*

$$\hat{\Phi}_{\sigma_n} = U_r^{-1} P_0 B_{0,c}\left[\prod_{i=1}^{n} K_{\sigma_n(i)}\right] e^{L\tilde{\Delta}} P_0 U_r^{-1}. \tag{B.42}$$

Proof. We show this via induction on n the case $n = 0$ follows from (A.40). We omit details for $n = 1$. For induction step $n-1 \mapsto n$, consider the following cases.

Case I: $\sigma_n \equiv a$. Then $\hat{\Phi}_{\sigma_n} = U_r^{-1}P_0 B_{0,c}\tilde{B}_{0,c}^n e^{L\tilde{\Delta}} P_0 U_r^{-1}$, the result is true by (A.43).

Case II: $\sigma_n \in \{a,b\}^n \setminus \{a\}^n$. Define $\ell_b = \max\{j|\sigma_n(j) = b\} \ge 1$, then

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_r^{-1} P_0 B_{0,c}\left[\prod_{i=1}^{\ell_b-1} K_{\sigma_n(i)}\right] P_0 U_r}_{=:\hat{\varphi}_1} \underbrace{U_r^{-1} P_0 \tilde{B}_{0,c}^{n-\ell_b+1} e^{L\tilde{\Delta}} P_0 U_r^{-1}}_{=:\hat{\varphi}_2}. \tag{B.43}$$

Since $\sigma_n \in \{a,b\}^n$, we can apply Lemma B.1 to get $\|\hat{\varphi}_1\|_{\mathrm{op}} \le \beta^{\ell_b}$, together with $\|\hat{\varphi}_2\|_{\mathrm{HS}} \le \beta^{n-\ell_b+1} e^{Lr^2}$ (by (A.41)), we have $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \le \|\hat{\varphi}_1\|_{\mathrm{op}} \|\hat{\varphi}_2\|_{\mathrm{HS}} \le \beta^{n+1} e^{Lr^2}$.
Case III: $\sigma_n \in \{a,b,c\}^n \setminus \{a,b\}^n$. Define $\ell_c = \max\{j | \sigma_n(j) = c\} \ge 1$, then

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_r^{-1} P_0 B_{0,c} \left[\prod_{i=1}^{\ell_c-1} K_{\sigma_n(i)}\right] e^{L\tilde{\Delta}} P_0 U_r^{-1}}_{=:\hat{\varphi}_1} \underbrace{U_r P_0 e^{-L\Delta} B_{0,c} \left[\prod_{i=\ell_c+1}^{n} K_{\sigma_n(i)}\right] e^{L\tilde{\Delta}} P_0 U_r^{-1}}_{=:\hat{\varphi}_2}. \tag{B.44}$$

By induction assumption $\|\hat{\varphi}_1\|_{\mathrm{HS}} \le \beta^{\ell_c} e^{Lr^2}$. Now we need to deal with $\hat{\varphi}_2$. If $\sigma_n(i) = a$ for all $i \in \{\ell_c+1, \dots, n\}$, applying (A.6), we have $\hat{\varphi}_2 = U_r P_0 \hat{B}_{0,c}^{n-\ell_c+1} P_0 U_r^{-1}$ and $\|\hat{\varphi}_2\|_{\mathrm{op}} \le \beta^{n-\ell_c+1}$ (by (A.23)), so that $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \le \|\hat{\varphi}_1\|_{\mathrm{HS}} \|\hat{\varphi}_2\|_{\mathrm{op}} \le \beta^{n+1} e^{Lr^2}$. If it exists $j > \ell_c + 1$ with $\sigma_n(j) = b$, setting $f_b = \min\{j \ge \ell_c + 1 | \sigma_n(j) = b\}$, we get

$$\hat{\varphi}_2 = \underbrace{U_r P_0 e^{-L\Delta} B_{0,c} \left[\prod_{i=\ell_c+1}^{f_b-1} K_{\sigma_n(i)}\right] P_0 U_r}_{=:\hat{\varphi}_2^1} \underbrace{U_r^{-1} P_0 \tilde{B}_{0,c} \left[\prod_{i=f_b+1}^{n} K_{\sigma_n(i)}\right] e^{L\tilde{\Delta}} P_0 U_r^{-1}}_{=:\hat{\varphi}_2^2}. \tag{B.45}$$

Applying $\|\hat{\varphi}_2^1\|_{\mathrm{HS}} \le \beta^{f_b-\ell_c} e^{-\frac{4L^3}{3(f_b-\ell_c)^2}} e^{-2Lc}$ (by Lemma B.8), $\|\hat{\varphi}_2^2\|_{\mathrm{HS}} \le \beta^{n-f_b+1} e^{Lr^2}$ (induction assumption) and $r^2 \le 2c$, we have $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \le \|\hat{\varphi}_1\|_{\mathrm{HS}} \|\hat{\varphi}_2^1\|_{\mathrm{HS}} \|\hat{\varphi}_2^2\|_{\mathrm{HS}} \le \beta^{n+1} e^{Lr^2}$. □

Lemma B.10. *For any $n \in \mathbb{Z}_{\ge 0}$ and $\sigma_n \in \{a,b,c,d,e\}^n$, the operator*

$$\hat{\Phi}_{\sigma_n} = U_r^{-1} P_0 B_{0,c} \left[\prod_{i=1}^{n} K_{\sigma_n(i)}\right] P_0 U_r \tag{B.46}$$

satisfies the following bounds:

1. *if $\sigma_n \in \{a,b\}^n$, then $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{op}} \le \beta^{n+1}$,*
2. *if $\sigma_n \in \{a,b,c,d,e\}^n \setminus \{a,b\}^n$, then $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{op}} \le \|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \le \beta^{n+1} e^{-\frac{4L^3}{3n^2}}$.*

Proof. The case $\sigma_n \in \{a,b\}^n$ is solved in Lemma B.1. We show the rest cases via induction on n and omit the details for $n = 1$. Consider induction step $n-1 \mapsto n$.
Case I: $\sigma_n \in \{a,b,c\}^n \setminus \{a,b\}^n$**.** Define $\ell_c = \min\{i | \sigma_n(i) = c\} \ge 1$, then

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_r^{-1} P_0 B_{0,c} \left[\prod_{i=1}^{\ell_c-1} K_{\sigma_n(i)}\right] e^{L\tilde{\Delta}} P_0 U_r^{-1}}_{=:\hat{\varphi}_1} \underbrace{U_r P_0 e^{-L\Delta} B_{0,c} \left[\prod_{i=\ell_c+1}^{n} K_{\sigma_n(i)}\right] P_0 U_r}_{=:\hat{\varphi}_2}. \tag{B.47}$$

Applying $\|\hat{\varphi}_1\|_{\mathrm{HS}} \le \beta^{\ell_c} e^{Lr^2}$ (by Lemma B.9), $\|\hat{\varphi}_2\|_{\mathrm{HS}} \le \beta^{n-\ell_c+1} e^{-\frac{4L^3}{3(n-\ell_c)^2}} e^{-2Lc}$ (by Lemma B.4) and $r^2 \le 2c$, we get $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \le \|\hat{\varphi}_1\|_{\mathrm{HS}} \|\hat{\varphi}_2\|_{\mathrm{HS}} \le e^{-\frac{4L^3}{3n^2}} \beta^{n+1}$.

Case II: $\sigma_n \in \{a,b,c,d\}^n \setminus \{a,b,c\}^n$**.** Set $\ell_d = \max\{i|\sigma_n(i) = d\} \geq 1$, then

$$\begin{aligned}\hat{\Phi}_{\sigma_n} &= \underbrace{U_r^{-1} P_0 B_{0,c} \left[\prod_{i=1}^{\ell_d - 1} K_{\sigma_n(i)}\right] P_0 U_r}_{=:\hat{\varphi}_1} \underbrace{U_r^{-1} P_0 e^{L\tilde{\Delta}} P_0 U_r^{-1}}_{=:\hat{\varphi}_2} \\ &\quad \times \underbrace{U_r P_0 e^{-L\Delta} B_{0,c} \left[\prod_{i=\ell_d+1}^{n} K_{\sigma_n(i)}\right] P_0 U_r}_{=:\hat{\varphi}_3} . \end{aligned} \tag{B.48}$$

Applying $\|\hat{\varphi}_1\|_{\text{op}} \leq \beta^{\ell_d}$ (induction assumption), $\|\hat{\varphi}_2\|_{\text{HS}} \leq \frac{1}{\sqrt{L}}$ (by (A.24)) and $\|\hat{\varphi}_3\|_{\text{HS}} \leq \beta^{n-\ell_d+1} e^{-\frac{4L^3}{(n-\ell_d+1)^2}} e^{-2Lc}$ (by Lemma B.4), we have $\|\hat{\Phi}_{\sigma_n}\|_{\text{HS}} \leq \|\hat{\varphi}_1\|_{\text{op}} \|\hat{\varphi}_2\|_{\text{HS}} \|\hat{\varphi}_3\|_{\text{HS}} \leq \beta^{n+1} e^{-\frac{4L^3}{3n^2}}$.

Case III: $\sigma_n \in \{a,b,c,d,e\}^n \setminus \{a,b,c,d\}^n$**.** This is the same as the previous case, except that $e^{L\tilde{\Delta}}$ is replaced by $e^{L\Delta}$. ☐

Lemma B.11. *Let* $n \in \mathbb{Z}_{\geq 0}$, $\sigma_n \in \{a,b,c,d,e,v\}^n$, *then* $\|\hat{\Phi}_{\sigma_n}\|_{\text{HS}} \leq \beta^{n+1} e^{Lr^2}$, *where*

$$\hat{\Phi}_{\sigma_n} = U_r^{-1} P_0 B_{0,c} \left[\prod_{i=1}^{n} K_{\sigma_n(i)}\right] e^{L\Delta} P_0 U_r^{-1}. \tag{B.49}$$

Proof. We show this via induction on n. The case $n = 0$ follows from (A.39). We omit the details for $n = 1$. Consider the induction step $n - 1 \mapsto n$.

Case I: $\sigma_n \in \{a,b\}^n$. If $\sigma_n \equiv a$, then $\hat{\Phi}_{\sigma_n} = U_r^{-1} P_0 B_{0,c} \tilde{B}_{0,c}^n e^{L\Delta} P_0 U_r^{-1}$, the result follows from (A.43). If $b \in \sigma_n$, define $\ell_b = \max\{i|\sigma_n(i) = b\} \geq 1$, then

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_r^{-1} P_0 B_{0,c} \left[\prod_{i=1}^{\ell_b - 1} K_{\sigma_n(i)}\right] P_0 U_r}_{=:\hat{\varphi}_1} \underbrace{U_r^{-1} P_0 \tilde{B}_{0,c}^{n-\ell_b+1} e^{L\Delta} P_0 U_r^{-1}}_{=:\hat{\varphi}_2} . \tag{B.50}$$

Applying $\|\hat{\varphi}_1\|_{\text{op}} \leq \beta^{\ell_b}$ (by Lemma B.10) and $\|\hat{\varphi}_2\|_{\text{HS}} \leq \beta^{n-\ell_b+1} e^{Lr^2}$ (by (A.42)), we have $\|\hat{\Phi}_{\sigma_n}\|_{\text{HS}} \leq \|\hat{\varphi}_1\|_{\text{op}} \|\hat{\varphi}_2\|_{\text{HS}} \leq \beta^{n+1} e^{Lr^2}$.

Case II: $\sigma_n \in \{a,b,c\}^n \setminus \{a,b\}^n$**.** Define $\ell_c = \max\{i|\sigma_n(i) = c\} \geq 1$ and

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_r^{-1} P_0 B_{0,c} \left[\prod_{i=1}^{\ell_c - 1} K_{\sigma_n(i)}\right] e^{L\tilde{\Delta}} P_0 U_r^{-1}}_{=:\hat{\varphi}_1} \underbrace{U_r P_0 e^{-L\Delta} B_{0,c} \left[\prod_{i=\ell_c+1}^{n} K_{\sigma_n(i)}\right] e^{L\Delta} P_0 U_r^{-1}}_{=:\hat{\varphi}_2} . \tag{B.51}$$

Applying $\|\hat{\varphi}_1\|_{\text{HS}} \leq \beta^{\ell_c} e^{Lr^2}$ (by Lemma B.9) and $\|\hat{\varphi}_2\|_{\text{op}} \leq \beta^{n-\ell_c+1}$ (by Lemma B.7), we have $\|\hat{\Phi}_{\sigma_n}\|_{\text{HS}} \leq \|\hat{\varphi}_1\|_{\text{HS}} \|\hat{\varphi}_2\|_{\text{op}} \leq \beta^{n+1} e^{Lr^2}$.

Case III: $\sigma_n \in \{a,b,c,d\}^n \setminus \{a,b,c\}^n$**.** Define $\ell_d = \min\{i|\sigma_n(i) = d\} \geq 1$, then

$$\begin{aligned}\hat{\Phi}_{\sigma_n} &= \underbrace{U_r^{-1} P_0 B_{0,c} \left[\prod_{i=1}^{\ell_d - 1} K_{\sigma_n(i)} \right] P_0 U_r}_{=:\hat{\varphi}_1} \underbrace{U_r^{-1} P_0 e^{L\tilde{\Delta}} P_0 U_r^{-1}}_{=:\hat{\varphi}_2} \\ &\quad \times \underbrace{U_r P_0 e^{-L\Delta} B_{0,c} \left[\prod_{i=\ell_d+1}^{n} K_{\sigma_n(i)} \right] e^{L\Delta} P_0 U_r^{-1}}_{=:\hat{\varphi}_3} .\end{aligned} \tag{B.52}$$

Applying $\|\hat{\varphi}_1\|_{\mathrm{op}} \leq \beta^{\ell_d}$ (by Lemma B.10), $\|\hat{\varphi}_2\|_{\mathrm{HS}} \leq \frac{1}{\sqrt{L}}$ (by (A.24)) and $\|\hat{\varphi}_3\|_{\mathrm{op}} \leq \beta^{n-\ell_d+1}$ (by Lemma B.7), we have $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \leq \|\hat{\varphi}_1\|_{\mathrm{op}} \|\hat{\varphi}_2\|_{\mathrm{HS}} \|\hat{\varphi}_3\|_{\mathrm{op}} \leq \beta^{n+1} \leq \beta^{n+1} e^{Lr^2}$.
Case IV: $\sigma_n \in \{a,b,c,d,e\}^n \setminus \{a,b,c,d\}^n$**.** It is the same as the previous case.
Case V: $\sigma_n \in \{a,b,c,d,e,v\}^n \setminus \{a,b,c,d,e\}^n$**.** Define $\ell_v = \min\{i|\sigma_n(i) = v\} \geq 1$, then

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_r^{-1} P_0 B_{0,c} \left[\prod_{i=1}^{\ell_v - 1} K_{\sigma_n(i)} \right] e^{L\Delta} P_0 U_r^{-1}}_{=:\hat{\varphi}_1} \underbrace{U_r P_0 e^{-L\Delta} B_{0,c} \left[\prod_{i=\ell_v+1}^{n} K_{\sigma_n(i)} \right] e^{L\Delta} P_0 U_r^{-1}}_{=:\hat{\varphi}_2} . \tag{B.53}$$

Applying $\|\hat{\varphi}_1\|_{\mathrm{HS}} \leq \beta^{\ell_v} e^{Lr^2}$ (induction assumption) and $\|\hat{\varphi}_2\|_{\mathrm{op}} \leq \beta^{n-\ell_v+1}$ (by Lemma B.7), we have $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \leq \|\hat{\varphi}_1\|_{\mathrm{HS}} \|\hat{\varphi}_2\|_{\mathrm{op}} \leq \beta^{n+1} e^{Lr^2}$. ☐

Corollary B.12. *For any $n \in \mathbb{Z}_{\geq 0}$ and $\sigma_n \in \{a,b,c,d,e,v,u\}^n$, the operator*

$$\hat{\Phi}_{\sigma_n} = U_r^{-1} P_0 B_{0,c} \left[\prod_{i=1}^{n} K_{\sigma_n(i)} \right] P_0 U_r \tag{B.54}$$

satisfies the following bounds:

1. *if $\sigma_n \in \{a,b,u\}^n$, then $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{op}} \leq \beta^{n+1}$,*
2. *if $\sigma_n \in \{a,b,c,d,e,v,u\}^n \setminus \{a,b,u\}^n$, then $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{op}} \leq \|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \leq \beta^{n+1} e^{-\frac{4L^3}{3n^2}}$.*

Proof. The case for $\sigma_n \in \{a,b,c,d,e\}^n$ is proved in Lemma B.10. We show rest cases via induction on n and omit details for $n = 1$. Consider induction step $n - 1 \mapsto n$.
Case I: $\sigma_n \in \{a,b,c,d,e,v\}^n \setminus \{a,b,c,d,e\}^n$**.** Set $\ell_v = \max\{i|\sigma_n(i) = v\} \geq 1$, then

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_r^{-1} P_0 B_{0,c} \left[\prod_{i=1}^{\ell_v - 1} K_{\sigma_n(i)} \right] e^{L\Delta} P_0 U_r^{-1}}_{=:\hat{\varphi}_1} \underbrace{U_r P_0 e^{-L\Delta} B_{0,c} \left[\prod_{i=\ell_v+1}^{n} K_{\sigma_n(i)} \right] P_0 U_r}_{=:\hat{\varphi}_2} . \tag{B.55}$$

Applying $\|\hat{\varphi}_1\|_{\mathrm{HS}} \leq \beta^{\ell_v} e^{Lr^2}$ (by Lemma B.11), $\|\hat{\varphi}_2\|_{\mathrm{HS}} \leq \beta^{n-\ell_v+1} e^{-\frac{4L^3}{3(n-\ell_v+1)^2}} e^{-2Lc}$ (by Corollary B.8) and $r^2 \leq 2c$, we have $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \leq \|\hat{\varphi}_1\|_{\mathrm{HS}} \|\hat{\varphi}_2\|_{\mathrm{HS}} \leq \beta^{n+1} e^{-\frac{4L^3}{3n^2}}$.

Case II: $\sigma_n \in \{a,b,c,d,e,v,u\}^n \setminus \{a,b,c,d,e,v\}^n$. Define $\ell_u = \min\{i|\sigma_n(i) = u\} \geq 1$, then

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_r^{-1}P_0B_{0,c}\left[\prod_{i=1}^{\ell_u-1} K_{\sigma_n(i)}\right]P_0U_r^{-1}}_{=:\hat{\varphi}_1}\underbrace{U_rP_0B_{0,c}\left[\prod_{i=\ell_u+1}^{n} K_{\sigma_n(i)}\right]P_0U_r}_{=:\hat{\varphi}_2}. \tag{B.56}$$

The result follows by induction assumption. □

Corollary B.13. *For any $n \in \mathbb{Z}_{\geq 0}$ and $\sigma_n \in \{a,b,c,d,e,u,v\}^n$, the operator*

$$\hat{\Phi}_{\sigma_n} = U_rP_0e^{-L\Delta}B_{0,c}\left[\prod_{i=1}^{n} K_{\sigma_n(i)}\right]P_0U_r \tag{B.57}$$

satisfies $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \leq \beta^{n+1}e^{-\frac{4L^3}{3(n+1)^2}}e^{-2Lc}$.

Proof. The case for $\sigma_n \in \{a,b,c,d,e,v\}^n$ is already proved in Corollary B.8. We only need to consider the case $u \in \sigma_n$. Define $\ell_u = \min\{i|\sigma_n(i) = u\} \geq 1$, then

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_rP_0e^{-L\Delta}B_{0,c}\left[\prod_{i=1}^{\ell_u-1} K_{\sigma_n(i)}\right]P_0U_r}_{=:\hat{\varphi}_1}\underbrace{U_r^{-1}P_0B_{0,c}\left[\prod_{i=\ell_u+1}^{n} K_{\sigma_n(i)}\right]P_0U_r}_{=:\hat{\varphi}_2}. \tag{B.58}$$

Applying $\|\hat{\varphi}_1\|_{\mathrm{HS}} \leq \beta^{\ell_u}e^{-\frac{4L^3}{3\ell_u^2}}e^{-2Lc}$ (by Corollary B.8) and $\|\hat{\varphi}_2\|_{\mathrm{op}} \leq \beta^{n-\ell_u+1}$ (by Lemma B.12), we have $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \leq \|\hat{\varphi}_1\|_{\mathrm{HS}}\|\hat{\varphi}_2\|_{\mathrm{op}} \leq \beta^{n+1}e^{-\frac{4L^3}{3n^2}}e^{-2Lc}$. □

Corollary B.14. *Let $n \in \mathbb{Z}_{\geq 0}$, $\sigma_n \in \{a,b,c,d,e,v,u\}^n$, then $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \leq \beta^{n+1}e^{Lr^2}$, where*

$$\hat{\Phi}_{\sigma_n} = U_r^{-1}P_0B_{0,c}\left[\prod_{i=1}^{n} K_{\sigma_n(i)}\right]e^{L\Delta}P_0U_r^{-1} \tag{B.59}$$

Proof. The case for $\sigma_n \in \{a,b,c,d,e,v\}^n$ is already proved in Lemma B.11. Now consider the case $u \in \sigma_n$, then we have $1 \leq \ell_u = \max\{i \mid \sigma_n(i) = u\} \leq n$ and

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_r^{-1}P_0B_{0,c}\left[\prod_{i=1}^{\ell_u-1} K_{\sigma_n(i)}\right]P_0U_r}_{=:\hat{\varphi}_1} \cdot \underbrace{U_r^{-1}P_0B_{0,c}\left[\prod_{i=\ell_u+1}^{n} K_{\sigma_n(i)}\right]e^{L\Delta}P_0U_r^{-1}}_{=:\hat{\varphi}_2}. \tag{B.60}$$

By Corollary B.12, we have $\|\hat{\varphi}_1\|_{\mathrm{op}} \leq \beta^{\ell_u}$. By definition of ℓ_u, we have $\sigma_n(i) \neq u$ for all $i \geq \ell_u + 1$, hence, we can apply Lemma B.11 to deduce $\|\hat{\varphi}_2\|_{\mathrm{HS}} \leq \beta^{n-\ell_u+1}e^{Lr^2}$. Combining together, we then have $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \leq \|\hat{\varphi}_1\|_{\mathrm{op}}\|\hat{\varphi}_2\|_{\mathrm{HS}} \leq \beta^{n+1}e^{Lr^2}$. □

Corollary B.15. *Let $n \in \mathbb{Z}_{\geq 0}$, $\sigma_n \in \{a,b,c,u\}^n$, then $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \leq \beta^{1+n}e^{Lr^2}$, where*

$$\hat{\Phi}_{\sigma_n} = U_r^{-1}P_0B_{0,c}\left[\prod_{i=1}^{n} K_{\sigma_n(i)}\right]e^{L\tilde{\Delta}}P_0U_r^{-1}. \tag{B.61}$$

Proof. The case for $\sigma_n \in \{a,b,c\}^n$ is already proved in Lemma B.9. Let now $\sigma_n \in \{a,b,c,u\}^n$ with $u \in \sigma_n$, then

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_r^{-1} P_0 B_{0,c} \left[\prod_{i=1}^{\ell_u - 1} K_{\sigma_n(i)} \right] P_0 U_r}_{=:\hat{\varphi}_1} \underbrace{U_r^{-1} P_0 B_{0,c} \left[\prod_{i=\ell_u+1}^{n} K_{\sigma_n(i)} \right] e^{L\tilde{\Delta}} P_0 U_r^{-1}}_{=:\hat{\varphi}_2}, \tag{B.62}$$

where $\ell_u = \min\{i | \sigma_n(i) = u\} \geq 1$. Since $\sigma_n(i) \neq u$ for $i < \ell_u$, we have $\|\hat{\varphi}_2\|_{\mathrm{HS}} \leq \beta^{n-\ell_u+1} e^{Lr^2}$ (by Lemma B.9), together with $\|\hat{\varphi}_1\|_{\mathrm{op}} \leq \beta^{\ell_u}$ (by Corollary B.12), it holds $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \leq \|\hat{\varphi}_1\|_{\mathrm{op}} \|\hat{\varphi}_2\|_{\mathrm{HS}} \leq \beta^{n+1} e^{Lr^2}$. $\square$

Lemma B.16. *Let $n \in \mathbb{Z}_{\geq 0}$, $\sigma_n \in \{a,b,c,v\}^n$, then $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{op}} \leq \beta^{n+1}$, where*

$$\hat{\Phi}_{\sigma_n} = U_r^{-1} P_0 \tilde{B}_{0,c} \left[\prod_{i=1}^{n} K_{\sigma(i)} \right] P_0 U_r. \tag{B.63}$$

Proof. The case $n = 0$ follows from (A.22). We show this via induction on $n \geq 1$ and omit the proof for $n = 1$. Consider induction step $n - 1 \mapsto n$.
Case I: $\sigma_n \equiv a$. Then $\hat{\Phi}_{\sigma_n} = U_r^{-1} P_0 \tilde{B}_{0,c}^{n+1} P_0 U_r$, the result follows from (A.22).
Case II: $\sigma_n \in \{a,b\}^n \setminus \{a\}^n$. Define $1 \leq \ell_b = \min\{i \mid \sigma_n(i) = b\} \leq n$, then

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_r^{-1} P_0 \tilde{B}_{0,c}^{\ell_b} P_0 U_r}_{=:\hat{\varphi}_1} \cdot \underbrace{U_r^{-1} P_0 \tilde{B}_{0,c} \left[\prod_{i=\ell_b+1}^{n} K_{\sigma(i)} \right] P_0 U_r}_{=:\hat{\varphi}_2}. \tag{B.64}$$

Applying now $\|\hat{\varphi}_2\|_{\mathrm{op}} \leq \beta^{\ell_b}$ (by (A.22)) and $\|\hat{\varphi}_2\|_{\mathrm{op}} \leq \beta^{n-\ell_b}$ (induction assumption), we have $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{op}} \leq \|\hat{\varphi}_1\|_{\mathrm{op}} \|\hat{\varphi}_2\|_{\mathrm{op}} \leq \beta^{n+1}$.
Case III: $\sigma_n \in \{a,b,c\}^n \setminus \{a,b\}^n$. Define $\ell_c = \min\{i \mid \sigma_n(i) = c\} \leq n$, then

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_r^{-1} P_0 \tilde{B}_{0,c} \left[\prod_{i=1}^{\ell_c - 1} K_{\sigma(i)} \right] e^{L\tilde{\Delta}} P_0 U_r^{-1}}_{=:\hat{\varphi}_1} \cdot \underbrace{U_r P_0 e^{-L\Delta} B_{0,c} \left[\prod_{i=\ell_c+1}^{n} K_{\sigma(i)} \right] P_0 U_r}_{=:\hat{\varphi}_2}, \tag{B.65}$$

Applying $\|\hat{\varphi}_1\|_{\mathrm{HS}} \leq \beta^{\ell_c} e^{Lr^2}$ (by Lemma B.3), $\|\hat{\varphi}_2\|_{\mathrm{HS}} \leq \beta^{n-\ell_c+1} e^{-\frac{4L^3}{3(n-\ell_c+1)^2}} e^{-2Lc}$ (by Lemma B.13) and $r^2 \leq 2c$, we have

$$\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{op}} \leq \|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \leq \|\hat{\varphi}_1\|_{\mathrm{HS}} \|\hat{\varphi}_2\|_{\mathrm{HS}} \leq \beta^{n+1}. \tag{B.66}$$

Case IV: $\sigma_n \in \{a,b,c,v\}^n \setminus \{a,b,c\}^n$. Define $\ell_v = \min\{i \mid \sigma_n(i) = v\} \leq n$, then

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_r^{-1} P_0 \tilde{B}_{0,c} \left[\prod_{i=1}^{\ell_v - 1} K_{\sigma_n(i)} \right] e^{L\Delta} P_0 U_r^{-1}}_{=:\hat{\varphi}_1} \cdot \underbrace{U_r P_0 e^{-L\Delta} B_{0,c} \left[\prod_{i=\ell_v+1}^{n} K_{\sigma_n(i)} \right] P_0 U_r}_{=:\hat{\varphi}_2}. \tag{B.67}$$

By definition of ℓ_v, we have $\sigma_n(i) \in \{a, b, c\}$ for any $1 \leq i \leq \ell_v - 1$. Hence, we can apply Lemma B.6 to deduce $\|\hat{\varphi}_1\|_{\mathrm{HS}} \leq \beta^{\ell_v} e^{Lr^2}$. Together with $\|\hat{\varphi}_2\|_{\mathrm{HS}} \leq \beta^{n-\ell_v+1} e^{-2Lc}$. (by Corollary B.13) and $r^2 \leq 2c$, we have

$$\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{op}} \leq \|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \leq \|\hat{\varphi}_1\|_{\mathrm{HS}} \|\hat{\varphi}_2\|_{\mathrm{HS}} \leq \beta^{n+1}. \tag{B.68}$$

□

Corollary B.17. *Let $n \in \mathbb{Z}_{\geq 0}, \sigma_n \in \{a, b, c, u, v\}^n$, then $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \leq \beta^{n+1} e^{Lr^2}$, where*

$$\hat{\Phi}_{\sigma_n} = U_r^{-1} P_0 \tilde{B}_{0,c} \left[\prod_{j=1}^{n} K_{\sigma_n(j)} \right] e^{L\Delta} P_0 U_r^{-1}. \tag{B.69}$$

Proof. The case for $\sigma_n \in \{a, b, c\}^n$ is proved in Lemma B.6. We show the rest cases via induction and omit the details for $n = 1$. Consider induction step $n - 1 \mapsto n$.
Case I: $\sigma_n \in \{a, b, c, v\}^n \setminus \{a, b, c\}^n$. Define $\ell_v = \min\{i | \sigma_n(i) = v\} \geq 1$, then

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_r^{-1} P_0 \tilde{B}_{0,c} \left[\prod_{j=1}^{\ell_v - 1} K_{\sigma_n(j)} \right] e^{L\Delta} P_0 U_r^{-1}}_{=:\hat{\varphi}_1} \underbrace{U_r P_0 e^{-L\Delta} B_{0,c} \left[\prod_{j=\ell_v+1}^{n} K_{\sigma_n(j)} \right] e^{L\Delta} P_0 U_r^{-1}}_{=:\hat{\varphi}_2}. \tag{B.70}$$

Applying $\|\hat{\varphi}_1\|_{\mathrm{HS}} \leq \beta^{\ell_v} e^{Lr^2}$ (induction assumption) and $\|\hat{\varphi}_2\|_{\mathrm{op}} \leq \beta^{n-\ell_v+1}$ (by Lemma B.7), we have $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \leq \|\hat{\varphi}_1\|_{\mathrm{HS}} \|\hat{\varphi}_2\|_{\mathrm{op}} \leq \beta^{n+1} e^{Lr^2}$.
Case II: $\sigma_n \in \{a, b, c, v, u\}^n \setminus \{a, b, c, v\}^n$. Define $\ell_u = \min\{i \mid \sigma_n(i) = u\}$, then

$$\hat{\Phi}_{\sigma_n} = \underbrace{U_r^{-1} P_0 \tilde{B}_{0,c} \left[\prod_{j=1}^{\ell_u - 1} K_{\sigma_n(j)} \right] P_0 U_r}_{=:\hat{\varphi}_1} \cdot \underbrace{U_r^{-1} P_0 B_{0,c} \left[\prod_{j=\ell_u+1}^{n} K_{\sigma_n(j)} \right] e^{L\Delta} P_0 U_r^{-1}}_{=:\hat{\varphi}_2}. \tag{B.71}$$

By definition of ℓ_u, $\sigma_n(j) \in \{a, b, c, v\}$ for any $j \leq \ell_u - 1$, we can then apply Lemma B.16 to deduce $\|\hat{\varphi}_1\|_{\mathrm{op}} \leq \beta^{\ell_u}$. Together with $\|\hat{\varphi}_2\|_{\mathrm{HS}} \leq \beta^{n-\ell_u+1} e^{Lr^2}$ (by Corollary B.14), we have $\|\hat{\Phi}_{\sigma_n}\|_{\mathrm{HS}} \leq \|\hat{\varphi}_1\|_{\mathrm{op}} \|\hat{\varphi}_2\|_{\mathrm{HS}} \leq \beta^{n+1} e^{Lr^2}$.

□

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