Universality of the geodesic tree in last passage percolation

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Abstract

In this paper we consider the geodesic tree in exponential last passage percolation. We show that for a large class of initial conditions around the origin, the line-to-point geodesic that terminates in a cylinder located around the point (N, N), and whose width and length are $o(N^{2/3})$ and $o(N)$ respectively, agrees in the cylinder, with the stationary geodesic sharing the same end point. In the case of the point-to-point model where the geodesic starts from the origin, we consider width $\delta N^{2/3}$, length up to $\delta^{3/2}N/(\log(\delta^{-1}))^3$, and provide lower and upper bounds for the probability that the geodesics agree in that cylinder.

1 Introduction

The last passage percolation model (LPP) is one of the most well studied models in the Kardar-Parisi-Zhang (KPZ) universality class of stochastic growth models. In this model, to each site $(i, j) \in \mathbb{Z}^2$, one associates an independent random variable $\omega_{i,j}$ exponentially distributed with parameter one. In the simplest case, the point-to-point LPP model, for a given point (m, n) in the first quadrant, one defines the last passage time as

$$G(m, n) = \max_{\pi:(0,0)\to(m,n)} \sum_{(i,j)\in\pi} \omega_{i,j}, \tag{1.1}$$

where the maximum is taken over all up-right paths, that is, paths whose incremental steps are either $(1, 0)$ or $(0, 1)$. Consider the spatial direction x to be $(1, -1)$ and the time direction t to be $(1, 1)$. Then one defines a height function $h(x, t = N) = G(N+x, N-x)$, see [40] and also [48,50] for a continuous analogue related to the Hammersley process [35]. The height function has been studied extensively: at time N, it has fluctuations of order $N^{1/3}$ and non-trivial correlation over distance $N^{2/3}$, which are the KPZ scaling exponents [16, 43, 44]. Furthermore, both the one-point distributions [1, 5, 6, 22, 38, 48] as well as the limiting processes [2, 17, 18, 40, 50] are known for a few initial conditions, i.e., distributions of $h(x, 0)$ (or geometries in LPP framework). Finally, the correlations

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in time of the interface are non-trivial over macroscopic distances [23, 30] and they have been recently partially studied [4, 9, 12, 33, 41, 42, 45].

Another very interesting but less studied aspect of models in the KPZ universality class is the geometrical properties of the geodesics (also known as maximizers). For the LPP model, geodesics are the paths achieving the maximum in (1.1). In the case of the exponential random LPP described above, for any end-point (m, n), there is a unique geodesic. Geodesics follow characteristic directions and if the end-point is at distance $\mathcal{O}(N)$ from the origin, then it has spatial fluctuations of order $\mathcal{O}(N^{2/3})$ with respect to the line joining the origin with the end-point [15, 39].

Consider two or more end-points. To each of the end-points there is one geodesic from $(0, 0)$ and thus the set of geodesics as seen from the end-points in the direction of the origin, have a non-trivial coalescing structure. Some recent studies of this structure in LPP and related models can be found in [7, 11, 14, 24, 28, 36, 37, 46]. One might expect that on a large scale the coalescing structure is universal and thus not depending on the details of the chosen random variables defining the LPP models (provided, of course, that the model is still in the KPZ class, which rules out, for instance, heavy tailed random variable).

The decorrelation of the height function in space and time is reflected in the geometrical behaviour of the geodesics. *(a) Spatial-decorrelation.* Consider the height function at a given time of $\mathcal{O}(N)$. For two end-points at spatial distance $\mathcal{O}(N^{2/3})$ the height functions are non-trivially correlated. This property of the height function is related with the behaviour of the geodesics. Indeed, the two geodesics ending at those points have a coalescence point at a distance $\mathcal{O}(N)$ from the end-points [34], with a non-trivial distribution over the full macroscopic scale, as already noticed in some numerical studies in [29]. More refined recent results are also available [51, 53]. This is necessary since the fluctuations over a time-distance t grow as $t^{1/3}$ and therefore, if the coalescence point is at distance $o(N)$ from the end-points, the fluctuations coming from the disjoint part of the two geodesics after the coalescing time will be $o(N^{1/3})$, leading to a trivial correlation (since the fluctuation scale is $N^{1/3}$). *(b) Decorrelation in time.* Consider the height function at two times $t_1 < t_2$ both of $\mathcal{O}(N)$ along a given characteristic line. Let P_1, P_2 be the end-points in the LPP setting. If $t_2 - t_1 = o(N)$, then the geodesic ending at P_2 will be at time t_1 at some point A with $|A - P_1| = o(N^{2/3})$. But this implies that the height functions at A and P_1 are trivially correlated. Moreover, since $|A - P_2| = o(N)$, the height functions at A and P_2 are also trivially correlated. Thus to have non-trivial time correlations, two points along a characteristic must be at a macroscopic distance $\mathcal{O}(N)$ from each other.

This aspect has been used in the study of the covariance of the time-time correlations [33], where it was proven that taking one end-point as (N, N) and the second $(\tau N, \tau N)$, then as $\tau \to 1$, the first order correction to the covariance of the LPP is $\mathcal{O}((1-\tau)^{2/3} N^{2/3})$ and is completely independent of the geometry of the LPP, i.e., it is the same whether one considers the point-to-point LPP as in (1.1) or the line-to-point LPP, for which the geodesics start from a point on the antidiagonal crossing the origin. This suggests that the coalescing structure of the end-points in $\{(N + k, N - k), |k| \leq \delta N^{2/3}\}$, for a small $\delta > 0$, should be independent of the LPP geometry over a time-span $o(N)$ from the end-points. In particular, the coalescing structure should be locally the same as the one from the stationary model, introduced in [8]. In [7] a result in this direction has been proven. Among other results, they showed that the tree of point-to-point geodesics starting from every vertex in a box of side length $\delta N^{2/3}$ going to a point at distance N

agree inside the box with the tree of stationary geodesics.

The goal of this work is to improve on previous results in the following points:

1. In the case of point-to-point LPP, we extend previous results by showing (Theorem 2.2) that the coalescence to the stationary geodesics holds with high probability for any geodesic starting in a large box around the origin and terminating in a cylinder whose width is of order $N^{2/3}$ and its length is of order N (see Figure 2.1). In other words, we obtain the correct dimensions of the cylinder around the point (N, N).

2. In the case of point-to-point LPP, we improve the lower bound of the coalescence result from exponent $3/8$ (see [7, Theorem 2.4]) to the correct exponent $1/2$ (Theorem 2.2). In the process of proving it, we provide a simple probabilistic proof (Theorem 2.8) for the concentration of geodesics around their characteristics with the optimal exponential decay.

3. In the case of point-to-point LPP, we obtain an upper bound on the coalescence event (Theorem 2.6) that differs from the lower bound only by a logarithmic factor, i.e., we have indeed obtained the correct exponent.

4. We also consider the case of line to point LPP with general initial conditions. In that case, we obtain (Theorem 2.5) a lower bound on the probability that the geodesic tree agree with that of stationary one in a cylinder of width $N^{2/3}$ and length N. The order of the lower bound depends on the concentration of the exit point of the geodesics around the origin.

Another problem that is closely related to the coalescence of the point-to-point geodesic with the stationary one is the question of coalescence of point-to-point geodesics. More precisely, consider the probability that two infinite geodesics starting $k^{2/3}$ away from each other will coalesce after Rk steps. A lower bound of the order CR^{-c} was obtained in [46]. Matching upper bound together with the identification of the constant $c = -2/3$ was found in [14]. The analogue result for the point-to-point coalescence was completed more recently in [53]. Finally, in [7] it is proven that the infinite geodesics in fact coalesce with their point-to-point counterparts and identified the polynomial decay obtained in [53].

In a second type of coalescence results, one considers the probability that geodesics leaving from two points that are located at distance of order $N^{2/3}$ away from each other and terminate at or around (N, N) coalesce. In the setup of Brownian LPP, one takes k geodesics leaving from a small interval of order $\epsilon N^{2/3}$ and terminating at time N in an interval of the same order. Then, the probability that they are disjoint is of order $\epsilon^{(k^2-1)/2}$ with a subpolynomial correction, see [37, Theorem 1.1]. It was conjectured there that the lower bound should have the same exponent. For $k = 2$ this is proven in [10, Theorem 2.4]. See also [13, Theorem 2] for a weaker upper bound. Furthermore, an upper bound of order $\tau^{2/9}$ on the probability that two geodesics starting from the points $(0, 0)$ and $(0, N^{2/3})$ and terminating at (N, N) do not coalesce by time $(1 - \tau)N$ is obtained in [7, Theorem 2.8].

Our Theorem 2.7 gives the exact exponent $1/2$, for the probability that any two geodesics starting from a large box of dimensions of order $N \times N^{2/3}$ and terminating at a common point in a small box of size $N \times N^{2/3}$ coalesce (see Figure 2.1 for more accurate dimensions). Theorem 2.7 can be compared with rarity of disjoint geodesics that was considered in [37] although for geodesics starting from a big box rather than a small one.

What is then the reason for the discrepancy in the different exponents (exponent 3/2 in the results in [37] and the 1/2 exponent in this paper)? Clearly, the geometry is different as in this paper we consider geodesics starting from a large box around the origin, as opposed to a small box of size $\delta N^{2/3}$ in [37]. Let us try to give a heuristic argument for a possible settlement of this discrepancy. Let us divide the interval $I := \{(0,i)\}_{0\leq i\leq N^{2/3}}$ into δ^{-1} sub-intervals $\{I_k\}_{0\leq k\leq \delta^{-1}}$ of size $\delta N^{2/3}$. Let A_k (resp. A) denote the event that two geodesics start from I_k (resp. I) and do not meet by the time they reach the small interval around the point (N,N). If the events $\{A_k\}_{1\leq k\leq \delta^{-1}}$ decorrelate fast enough, then by [37, Theorem 1.1] we have roughly δ^{-1} decorrelated events of probability (up to logarithmic correction) $\delta^{3/2}$. This implies that $\mathbb{P}(A)$ is (up to logarithmic correction) $\delta^{-1}\delta^{3/2} = \delta^{1/2}$.

Concerning the methods used in this paper, one input we use is a control over the lateral fluctuations of the geodesics in the LPP. In Theorem 2.8 we show that the probability that the geodesic of the point-to-point LPP is not localized around a distance $MN^{2/3}$ from the characteristic line decay like e^{-cM^3}, which is the optimal power of the decay. This is proven using the approach of [15], see Theorem 4.4, once the mid-point analogue estimate is derived, see Theorem 2.2. The novelty here is a simple and short proof of this latter by using only comparison with stationary models. This probabilistic method is much simpler than previous ones.

To prove Theorem 2.2, the first step is to prove that with high probability the spatial trajectories of both the geodesic of the point-to-point LPP as well as the one of the stationary model with density 1/2 are sandwiched between the geodesics for the stationary models with some densities $\rho_+ > 1/2$ and $\rho_- < 1/2$ respectively. This then reduces the problem to finding bounds on the coalescing probability only for the two geodesics of the stationary models. This is done using the coupling between different stationary models introduced in [26] and techniques introduced in [7]. Using techniques from [15] and exact tail decay of the fluctuations of geodesics we are able to improve the results in [7]. The main ingredient in proving Theorem 2.6 is to show that the geodesics of the stationary models with different densities did not coalesce too early with some positive probability. Here, the application of the queueing representation of the coupling in [26] is more delicate than the one needed for the lower bound, as we now have to force the geodesic away from each other. This application of the queuing techniques developed in [7] is new. The idea is to look at the behaviour of the coupled stationary initial conditions as two coupled random walks, and analyse their behaviour on two disjoint interval. Finally, we extend the local universality result of the geodesic tree for a class of initial conditions without restricting to the point-to-point case as in [15].

Outline of the paper. In Section 2 we define the model and state the main results. We recall in Section 3 some recurrent notations and basic results on stationary LPP. In Section 4 we prove first Theorem 2.8 on the localization of the point-to-point geodesics and then show that the geodesics can be sandwiched between two version of the stationary model, see Lemma 4.7. This allows us to prove Theorems 2.2 and 2.5 in Section 5. Section 6 deals with the proof of Theorem 2.6 and Theorem 2.7.

1.1 Some general notation

We mention here some of the notations which will be used throughout the paper. We denote $\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \dots\}$ and $\mathbb{Z}_{>0} = \{1, 2, 3, \dots\}$. We use four standard vectors in $\mathbb{R}^2$, namely $\mathrm{e}_1 = (1, 0)$ and $\mathrm{e}_2 = (0, 1)$. We also denote $\mathrm{e}_3 = (1, -1)$ and $\mathrm{e}_4 = (1, 1)$. The direction e_3 represents the *spatial direction*, while e_4 represents the *temporal direction*. Furthermore, for a point $x = (x_1, x_2) \in \mathbb{R}^2$ the ℓ^1-norm is $|x| = |x_1| + |x_2|$. We also use the partial ordering on $\mathbb{R}^2$: for $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$, we write $x \leq y$ if $x_1 \leq y_1$ and $x_2 \leq y_2$. Given two points $x, y \in \mathbb{Z}^2$ with $x \leq y$, we define the box $[x, y] = \{z \in \mathbb{Z}^2 | x \leq z \leq y\}$. For $u \in \mathbb{Z}^2$ we denote $\mathbb{Z}^2_{\geq u} = \{x : x \geq u\}$.

Finally, for $\lambda > 0$, $X \sim \mathrm{Exp}(\lambda)$ denotes a random variable X which has exponential distribution with rate λ, in other words $\mathbb{P}(X > t) = e^{-\lambda t}$ for $t \geq 0$, and thus the mean is $\mathbb{E}(X) = \lambda^{-1}$ and variance $\mathrm{Var}(X) = \lambda^{-2}$. To lighten notation, we do not write explicitly the integer parts, as our results are insensitive to shifting points by order 1. For instance, for a $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, ξN means $(\lfloor \xi_1 N \rfloor, \lfloor \xi_2 N \rfloor)$.

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2 Main results

Let $\omega = \{\omega_x\}_{x \in \mathbb{Z}^2}$ be i.i.d. Exp(1)-distributed random weights on the vertices of $\mathbb{Z}^2$. For $o \in \mathbb{Z}^2$, define the last-passage percolation (LPP) process on $o + \mathbb{Z}^2_{\geq 0}$ by

$$G_{o,y} = \max_{x_\bullet \in \Pi_{o,y}} \sum_{k=0}^{|y-o|_1} \omega_{x_k} \quad \text{for } y \in o + \mathbb{Z}^2_{\geq 0}. \tag{2.1}$$

$\Pi_{o,y}$ is the set of paths $x_\bullet = (x_k)_{k=0}^n$ that start at $x_0 = o$, end at $x_n = y$ with $n = |y - o|_1$, and have increments $x_{k+1} - x_k \in \{\mathrm{e}_1, \mathrm{e}_2\}$. The a.s. unique path $\pi_{o,y} \in \Pi_{o,y}$ that attains the maximum in (2.1) is the *geodesic* from o to y.

Let $\mathcal{L} = \{x \in \mathbb{Z}^2 | x_1 + x_2 = 0\}$ be the antidiagonal crossing through the origin. Given some random variables (in general non independent) $\{h_0(x)\}_{x \in \mathcal{L}}$ on $\mathcal{L}$, independent from ω, define the last passage time with initial condition h_0 by

$$G^{h_0}_{\mathcal{L},y} = \max_{x_\bullet \in \Pi_{\mathcal{L},y}} \left(h_0(x_0) + \sum_{k=1}^{|y-o|_1} \omega_{x_k} \right) \quad \text{for } y > \mathcal{L}, \tag{2.2}$$

where $y > \mathcal{L}$ is meant in the sense of the order on the lattice, that is, y is located above and to the right of $\mathcal{L}$ (see Subsection 3.1 for exact definition). Consider the point $x_0 \in \mathcal{L}$ from where the geodesic from $\mathcal{L}$ to y leaves the line $\mathcal{L}$. Then $x_0 = (Z^{h_0}_{\mathcal{L},y}, -Z^{h_0}_{\mathcal{L},y})$ for some

$Z^{h_0}_{\mathcal{L},y} \in \mathbb{Z}$ and we refer to $Z^{h_0}_{\mathcal{L},y}$ as the *exit point*[1] of the last passage percolation with initial condition h_0.

One can define stationary models parameterized by a density $\rho \in (0,1)$, both for the LPP on the positive quadrant as for the LPP on the north-east of $\mathcal{L}$, see Section 3.2 for detailed explanations. In that case we denote the stationary LPP by $G^\rho_{o,y}$ or $G^\rho_{\mathcal{L},y}$ respectively.

For $\sigma \in \mathbb{R}_+$ and $0 < \tau < 1$ we define the cylinder with principal axis in the e_4 direction ending at (N,N) of width $\sigma N^{2/3}$ and length τN by

$$\mathcal{C}^{\sigma,\tau} = \{i\mathrm{e}_4 + j\mathrm{e}_3 \in \mathbb{Z}^2 : (1-\tau)N \le i \le N, -\tfrac{\sigma}{2}N^{2/3} \le j \le \tfrac{\sigma}{2}N^{2/3}\}. \tag{2.3}$$

As the points in $\mathcal{C}^{\sigma,\tau}$ are a subset of $\mathbb{Z}^2$, one necessarily have that $i,j \in \mathbb{Z}/2$ with $i+j \in \mathbb{Z}$. Similarly, for $\sigma \in \mathbb{R}_+$ and $0 < \tau < 1$ we define a set of width $\sigma N^{2/3}$ and length τN

$$\mathcal{R}^{\sigma,\tau} = \{i\mathrm{e}_4 + j\mathrm{e}_3 \in \mathbb{Z}^2 : 0 \le i \le \tau N, -\tfrac{\sigma}{2}N^{2/3} \le j \le \tfrac{\sigma}{2}N^{2/3}, |j| < i\}. \tag{2.4}$$

Remark 2.1. Note that the shape of $\mathcal{R}^{\sigma,\tau}$ in (2.4) is somewhat different than the blue cylinder in Figure 2.1. The reason for that is that in this paper we use exit points with respect to the vertical and horizontal axis so that the shape defined in (2.4) is easier to work with. We stress that similar results can be obtained for a box as in Figure 2.1 by using exit points with respect to the antidiagonal $\mathcal{L}$.

Due to the correspondence to stochastic growth models in the KPZ universality class, we denote the time direction by $(1,1)$ and the spatial direction by $(1,-1)$. In particular, for any $0 < \tau < 1$ define the time horizon

$$L_\tau = \{\tau N\mathrm{e}_4 + i\mathrm{e}_3 : -\infty < i < \infty\}, \tag{2.5}$$

Let $x,y,z \in \mathbb{Z}^2$ be such that $x,y \le z$. For the geodesics $\pi_{x,z}$ and $\pi_{y,z}$ we define the coalescence point

$$C_p(\pi_{x,z}, \pi_{y,z}) = \inf\{u \in \mathbb{Z}^2 : u \in \pi_{x,z} \cap \pi_{y,z}\}, \tag{2.6}$$

where the infimum is with respect to the order $\le$ on the lattice.

Upper and lower bounds on the coalescing point

The main results of this paper can be thought of as a local universality of the geodesic trees in a cylinder of spatial width $o(N^{2/3})$ and time width $o(N)$ around the end-point (N,N). For a class of initial conditions, we show that with high probability the geodesics starting from any point of $\mathcal{C}^{\delta,\tau}$ are indistinguishable from the stationary ones beyond the time horizon L_τ. In other words, the geodesic tree generated by the end-points in $\mathcal{C}^{\delta,\tau}$ have universal character beyond L_τ.

The first result is for the point-to-point geometry: with probability going to 1 as $\delta \to 0$, the set of geodesics ending at any point in the cylinder $\mathcal{C}^{\delta,\tau}$ of the stationary LPP with density $1/2$ is indistinguishable from the geodesics of the point-to-point LPP from the origin for any $\tau \le \delta^{3/2}/(\log(\delta^{-1}))^3$.

Theorem 2.2. *Let $o = (0,0)$. There exist $C, \delta_0 > 0$ such that for any $\delta \in (0,\delta_0)$ and $\tau \le \delta^{3/2}/(\log(\delta^{-1}))^3$,*

$$\mathbb{P}\Big(C_p(\pi^{1/2}_{o,x}, \pi_{y,x}) \le L_{1-\tau} \quad \forall x \in \mathcal{C}^{\delta,\tau}, y \in \mathcal{R}^{\frac{1}{8}\log\delta^{-1},1/4}\Big) \ge 1 - C\delta^{1/2}\log(\delta^{-1}) \tag{2.7}$$

for all N large enough.

[1] Equivalently, one can think of the point x_0 as the exit point, which is one-to-one with the value $Z^{h_0}_{\mathcal{L},y}$.

As a direct corollary we have that in a cylinder of spatial width $o(N^{2/3})$ and time width $o(N)$ around the end-point (N,N), the geodesics are indistinguishable from the stationary ones.

Corollary 2.3. *For any $\varepsilon > 0$,*

$$\lim_{N\to\infty} \mathbb{P}\Big(C_p(\pi^{1/2}_{o,x}, \pi_{o,x}) \le L_{1-N^{-\varepsilon}} \quad \forall x \in \mathcal{C}^{N^{-\varepsilon},N^{-\varepsilon}}\Big) = 1. \tag{2.8}$$

Theorem 2.2 generalizes to LPP with a large class of initial conditions, i.e., consider LPP from $\mathcal{L}$ with initial condition h_0, like the ones considered in [22, 33]. We make the following assumption on h_0. Recall the exit point $Z^{h_0}_{\mathcal{L},x}$ mentioned right after (2.2).

Assumption 2.4. *Let $x^1 = N\mathrm{e}_4 + \frac{3}{4}\delta N^{2/3}\mathrm{e}_3$ and $x^2 = N\mathrm{e}_4 - \frac{3}{4}\delta N^{2/3}\mathrm{e}_3$. Assume that*

$$\mathbb{P}\left(Z^{h_0}_{\mathcal{L},x^1} \le \log(\delta^{-1})N^{2/3}\right) \ge 1 - Q(\delta) \tag{2.9}$$

and

$$\mathbb{P}\left(Z^{h_0}_{\mathcal{L},x^2} \ge -\log(\delta^{-1})N^{2/3}\right) \ge 1 - Q(\delta) \tag{2.10}$$

for all N large enough, with a function $Q(\delta)$ satisfying $\lim_{\delta\to 0} Q(\delta) = 0$.

Under Assumption 2.4 the analogue of Theorem 2.2 (and thus of Corollary 2.3) holds true.

Theorem 2.5. *Under Assumption 2.4, there exist $C, \delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$ and $\tau \le \delta^{3/2}/(\log(\delta^{-1}))^3$,*

$$\mathbb{P}\Big(C_p(\pi^{1/2}_{\mathcal{L},x}, \pi^{h_0}_{\mathcal{L},x}) \le L_{1-\tau} \quad \forall x \in \mathcal{C}^{\delta,\tau}\Big) \ge 1 - C\delta^{1/2}\log(\delta^{-1}) - Q(\delta) \tag{2.11}$$

for all N large enough.

The exponent 1/2 in Theorem 2.2 is optimal as our next result shows.

Theorem 2.6. *Let $o = (0,0)$. There exist $C, \delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$ and $\tau \le \delta^{3/2}/(\log(\delta^{-1}))^3$,*

$$\mathbb{P}\Big(C_p(\pi^{1/2}_{o,x}, \pi_{y,x}) \le L_{1-\tau} \quad \forall x \in \mathcal{C}^{\delta,\tau}, y \in \mathcal{R}^{(\frac{1}{8}\log\delta^{-1}),\tau}\Big) \le 1 - C\delta^{1/2} \tag{2.12}$$

for all N large enough.

The following result is closely related to Theorem 2.2 and Theorem 2.6, it considers the question of coalescence of point-to-point geodesics.

Theorem 2.7. *There exist $C, \delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$ and $\tau \le \delta^{3/2}/(\log(\delta^{-1}))^3$,*

$$1 - C\delta^{1/2}\log(\delta^{-1}) \le \mathbb{P}\Big(C_p(\pi_{w,x}, \pi_{y,x}) \le L_{1-\tau} \quad \forall x \in \mathcal{C}^{\delta,\tau}, w, y \in \mathcal{R}^{(\frac{1}{8}\log\delta^{-1}),\tau}\Big) \le 1 - C\delta^{1/2} \tag{2.13}$$

for all N large enough.

Cubic decay of localization

In order to prove the main theorems we will need some control on the spatial fluctuations of the geodesics for the point-to-point problem. As this estimate has its own interest, we state it below as Theorem 4.2. For a an up-right path γ we denote

$$\begin{aligned} \Gamma_k^u(\gamma) &= \max\{l : (k,l) \in \gamma\}, \\ \Gamma_k^l(\gamma) &= \min\{l : (k,l) \in \gamma\}. \end{aligned} \tag{2.14}$$

When γ is a geodesic associated with a direction, we denote the direction by $\xi = (\xi_1, \xi_2)$ with $\xi_1 + \xi_2 = 1$, and we set

$$\Gamma_k(\gamma) = \max\{|\Gamma_k^u(\gamma) - \tfrac{\xi_2}{\xi_1} k|, |\Gamma_k^l(\gamma) - \tfrac{\xi_2}{\xi_1} k|\}. \tag{2.15}$$

Theorem 2.8. *Let $\varepsilon \in (0,1]$. Then there exists $N_0(\varepsilon)$ and $c_1(\varepsilon)$ such that for ξ satisfying $\varepsilon \leq \xi_2/\xi_1 \leq 1/\varepsilon$,*

$$\mathbb{P}\big(\Gamma_k(\pi_{o,\xi N}) > M(\tau N)^{2/3} \text{ for all } k \in [0, \tau \xi_1 N]\big) \leq e^{-c_1 M^3} \tag{2.16}$$

for all $\tau N \geq N_0$ and all $M \leq (\tau N)^{1/3}/\log(N)$.

Statements similar to Theorem 2.8 with Gaussian bound can be found in [9, Proposition 2.1], [14, Theorem 3] and [15, Theorem 11.1]. The authors employed Theorems 10.1 and 10.5 of [15]. A cubic decay can be found in [12, Proposition 4.7]. We provide a short and self-contained proof of the localization result using comparison with stationarity only.

Remark 2.9. Theorem 2.8 states the optimal localization scale for small τ. By symmetry of the point-to-point problem, the same statement holds with τ replaced by $1-\tau$ and and gives the optimal localization scale for τ close to 1.

A family of random initial conditions.

Now let us consider the family of initial conditions, interpolating between flat initial condition (i.e., point-to-line LPP) and the stationary initial condition, for which the time-time covariance was studied in [33]. For $\sigma \geq 0$, let us define

$$h_0(k,-k) = \sigma \times \begin{cases} \sum_{\ell=1}^{k} (X_\ell - Y_\ell), & \text{for } k \geq 1, \\ 0, & \text{for } k = 0, \\ -\sum_{\ell=k+1}^{0} (X_\ell - Y_\ell), & \text{for } k \leq -1. \end{cases} \tag{2.17}$$

where $\{X_k, Y_k\}_{k \in \mathbb{Z}}$ are independent random variables $X_k, Y_k \sim \text{Exp}(1/2)$. For $\sigma = 0$ it corresponds to the point-to-line LPP, while for $\sigma = 1$ it is the stationary case with density 1/2.

Proposition 2.10. *For LPP with initial condition* (2.17)*, Assumption 2.4 holds with*

$$Q(\delta) = Ce^{-c(\log(\delta^{-1}))^2} \tag{2.18}$$

for some constants $C, c > 0$.

This result is essentially contained in the proof of part (b) of Lemma 5.2 of [33]. For completeness we present a sketch of the proof in Appendix B.

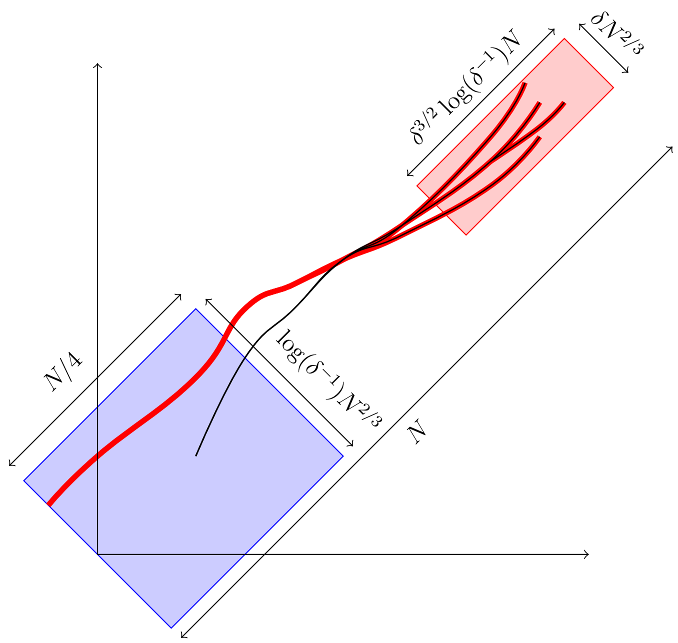

Figure 2.1: Illustration of Theorems 2.2 and 2.6. With high probability, any geodesic tree (black curve) consisting of all geodesics starting from a fixed point in the blue cylinder and terminating at any point in the red cylinder, will agree in the red box with the stationary tree (red curve) of intensity 1/2.

3 Preliminaries

3.1 Ordering of paths

We construct two partial orders on directed paths in $\mathbb{Z}^2$.

$\leq$: For $x, y \in \mathbb{Z}^2$ we write $x \leq y$ if y is above and to the right of x, i.e.

$$x_1 \leq y_1 \quad \text{and} \quad x_2 \leq y_2. \tag{3.1}$$

We also write $x < y$ if

$$x \leq y \quad \text{and} \quad x \neq y \tag{3.2}$$

An up-right path is a (finite or infinite) sequence $\mathcal{Y} = (y_k)_{k\in\mathbb{Z}}$ in $\mathbb{Z}^2$ such that $y_k - y_{k-1} \in \{\mathrm{e}_1, \mathrm{e}_2\}$ for all k. Let $\mathcal{UR}$ be the set of up-right paths in $\mathbb{Z}^2$. If $A, B \subset \mathbb{Z}^2$, we write $A \leq B$ if

$$x \leq y \quad \forall x \in A \cap \mathcal{Y}, y \in B \cap \mathcal{Y} \quad \forall \mathcal{Y} \in \mathcal{UR}. \tag{3.3}$$

where we take the inequality to be vacuously true if one of the intersections in (3.3) is empty.

$\preceq$: For $x, y \in \mathbb{Z}^2$ we write $x \preceq y$ if y is below and to the right of x, i.e.

$$x_1 \leq y_1 \quad \text{and} \quad x_2 \geq y_2. \tag{3.4}$$

We also write $x \prec y$ if

$$x \preceq y \quad \text{and} \quad x \neq y \tag{3.5}$$

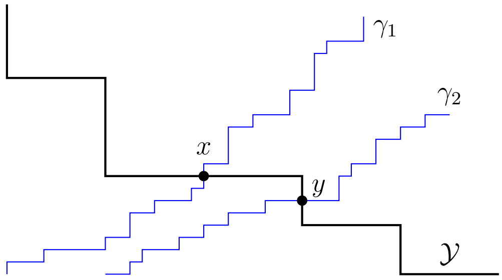

Figure 3.1: The two geodesics γ_1 and γ_2 are ordered i.e., $\gamma_1 \prec \gamma_2$. For any down-right path $\mathcal{Y}$ in $\mathbb{Z}^2$ the set of points $x = \mathcal{Y} \cap \gamma_1$ and $y = \mathcal{Y} \cap \gamma_2$ are ordered, i.e., $x \prec y$.

A down-right path is sequence $\mathcal{Y} = (y_k)_{k\in\mathbb{Z}}$ in $\mathbb{Z}^2$ such that $y_k - y_{k-1} \in \{\mathrm{e}_1, -\mathrm{e}_2\}$ for all $k \in \mathbb{Z}$. Let $\mathcal{DR}$ be the set of infinite down-right paths in $\mathbb{Z}^2$. If $A, B \subset \mathbb{Z}^2$, we write $A \preceq B$ if

$$x \preceq y \quad \forall x \in A \cap \mathcal{Y}, y \in B \cap \mathcal{Y} \quad \forall \mathcal{Y} \in \mathcal{DR}. \tag{3.6}$$

where we take the inequality to be vacuously true if one of the intersections in (3.6) is empty (see Figure 3.1).

3.2 Stationary LPP

Stationary LPP on $\mathbb{Z}^2_{\geq 0}$ has been introduced in [49] by adding boundary terms on the e_1 and e_2 axis. In [8] it was shown that it can be set up by using more general boundary domains. In this paper we are going to use two of them.

Boundary weights on the axis. For a base point $o = (o_1, o_2) \in \mathbb{Z}^2$ and a parameter value $\rho \in (0, 1)$ we introduce the stationary last-passage percolation process $G^\rho_{o,\bullet}$ on $o + \mathbb{Z}^2_{\geq 0}$. This process has boundary conditions given by two independent sequences

$$\{I^\rho_{o+i\mathrm{e}_1}\}_{i=1}^\infty \quad \text{and} \quad \{J^\rho_{o+j\mathrm{e}_2}\}_{j=1}^\infty \tag{3.7}$$

of i.i.d. random variables with $I^\rho_{o+\mathrm{e}_1} \sim \mathrm{Exp}(1-\rho)$ and $J^\rho_{o+\mathrm{e}_2} \sim \mathrm{Exp}(\rho)$. Put $G^\rho_{o,o} = 0$ and on the boundaries

$$G^\rho_{o,\,o+\,k\mathrm{e}_1} = \sum_{i=1}^k I_{o+i\mathrm{e}_1} \quad \text{and} \quad G^\rho_{o,\,o+\,l\mathrm{e}_2} = \sum_{j=1}^l J_{o+j\mathrm{e}_2}. \tag{3.8}$$

Then in the bulk for $x = (x_1, x_2) \in o + \mathbb{Z}^2_{>0}$,

$$G^\rho_{o,\,x} = \max\left\{ \max_{1\leq k\leq x_1-o_1} \left\{ \sum_{i=1}^k I_{o+i\mathrm{e}_1} + G_{o+k\mathrm{e}_1+\mathrm{e}_2,\,x} \right\}, \max_{1\leq \ell\leq x_2-o_2} \left\{ \sum_{j=1}^\ell J_{o+j\mathrm{e}_2} + G_{o+\ell\mathrm{e}_2+\mathrm{e}_1,\,x} \right\} \right\}. \tag{3.9}$$

Boundary weights on antidiagonal. The stationary model with density ρ can be realized by putting boundary weights on $\mathcal{L}$ as follows. Let $\{X_k, k \in \mathbb{Z}\}$ and $\{Y_k, k \in \mathbb{Z}\}$ be independent random variables with $X_k \sim \mathrm{Exp}(1-\rho)$ and $Y_k \sim \mathrm{Exp}(\rho)$. Then, define

$$h_0(k,-k) = \begin{cases} \sum_{\ell=1}^{k}(X_\ell - Y_\ell), & \text{for } k \geq 1, \\ 0, & \text{for } k = 0, \\ -\sum_{\ell=k+1}^{0}(X_\ell - Y_\ell), & \text{for } k \leq -1. \end{cases} \tag{3.10}$$

Then the LPP defined by (2.2) with initial condition h_0 is stationary, that is, the increments $G^{h_0}_{\mathcal{L},x+\mathrm{e}_1} - G^{h_0}_{\mathcal{L},x} \sim \mathrm{Exp}(1-\rho)$ as well as $G^{h_0}_{\mathcal{L},x+\mathrm{e}_2} - G^{h_0}_{\mathcal{L},x} \sim \mathrm{Exp}(\rho)$ for all $x > \mathcal{L}$.

Next we define the *exit points* of geodesics, these will play an important role in our analysis.

Definition 3.1 (Exit points).
(a) For a point $p \in o + \mathbb{Z}^2_{>0}$, let $Z_{o,p}$ be the signed exit point of the geodesic $\pi_{o,p}$ of $G_{o,p}$ from the west and south boundaries of $o + \mathbb{Z}^2_{>0}$. More precisely,

$$Z^\rho_{o,p} = \begin{cases} \underset{k}{\operatorname{argmax}}\left\{ \sum_{i=1}^{k} I_{o+i\mathrm{e}_1} + G_{o+k\mathrm{e}_1+\mathrm{e}_2,\,x}\right\} & \text{if } \pi_{o,p} \cap o + \mathrm{e}_1 \neq \emptyset, \\ -\underset{\ell}{\operatorname{argmax}}\left\{ \sum_{j=1}^{\ell} J_{o+j\mathrm{e}_2} + G_{o+\ell\mathrm{e}_2+\mathrm{e}_1,\,x}\right\} & \text{if } \pi_{o,p} \cap o + \mathrm{e}_2 \neq \emptyset. \end{cases} \tag{3.11}$$

(b) For a point $p > \mathcal{L}$, we denote by $Z^{h_0}_{\mathcal{L},p} \in \mathbb{Z}$ the exit point of the LPP from $\mathcal{L}$ with initial condition h_0, if the starting point of the geodesic from $\mathcal{L}$ to p is given by $(Z^{h_0}_{\mathcal{L},p}, -Z^{h_0}_{\mathcal{L},p})$. In the case of the stationary model with parameter ρ, the exit point is denoted by $Z^\rho_{\mathcal{L},p}$.

As a consequence, the value $G^\rho_{o,x}$ can be determined by (3.8) and the recursion

$$G^\rho_{o,x} = \omega_x + \max\{G^\rho_{o,x-\mathrm{e}_1}, G^\rho_{o,x-\mathrm{e}_2}\}. \tag{3.12}$$

3.3 Backward LPP

Next we consider LPP maximizing down-left paths. For $y \leq o$, define

$$\widehat{G}_{o,y} = G_{y,o}, \tag{3.13}$$

and let the associated geodesic be denoted by $\hat{\pi}_{o,y}$. For each $o = (o_1, o_2) \in \mathbb{Z}^2$ and a parameter value $\rho \in (0,1)$ define a stationary last-passage percolation processes $\widehat{G}^\rho$ on $o + \mathbb{Z}^2_{\leq 0}$, with boundary variables on the north and east, in the following way. Let

$$\{\hat{I}^\rho_{o-i\mathrm{e}_1}\}_{i=1}^\infty \quad \text{and} \quad \{\hat{J}^\rho_{o-j\mathrm{e}_2}\}_{j=1}^\infty \tag{3.14}$$

be two independent sequences of i.i.d. random variables with marginal distributions $\hat{I}^\rho_{o-i\mathrm{e}_1} \sim \mathrm{Exp}(1-\rho)$ and $\hat{J}^\rho_{o-j\mathrm{e}_2} \sim \mathrm{Exp}(\rho)$. The boundary variables in (3.7) and those in (3.14) are taken independent of each other. Put $\widehat{G}^\rho_{o,\,o} = 0$ and on the boundaries

$$\widehat{G}^\rho_{o,\,o-k\mathrm{e}_1} = \sum_{i=1}^{k} \hat{I}_{o-i\mathrm{e}_1} \quad \text{and} \quad \widehat{G}^\rho_{o,\,o-l\mathrm{e}_2} = \sum_{j=1}^{l} \hat{J}_{o-j\mathrm{e}_2}. \tag{3.15}$$

Then in the bulk for $x = (x_1, x_2) \in o + \mathbb{Z}^2_{<0}$,

$$\widehat{G}^\rho_{o,\,x} = \max_{1\leq k\leq o_1-x_1}\left\{ \sum_{i=1}^{k} \hat{I}_{o-i\mathrm{e}_1} + \max\left\{\widehat{G}_{\,o-k\mathrm{e}_1-\mathrm{e}_2,x}\right\}, \max_{1\leq \ell\leq o_2-x_2}\left\{ \sum_{j=1}^{\ell} \hat{J}_{o-j\mathrm{e}_2} + \widehat{G}_{\,o-\ell\mathrm{e}_2-\mathrm{e}_1,x}\right\}\right\}. \tag{3.16}$$

For a southwest endpoint $p \in o + \mathbb{Z}^2_{<0}$, let $\widehat{Z}^\rho_{o,p}$ be the signed exit point of the geodesic $\hat{\pi}_{o,p}$ of $\widehat{G}^\rho_{o,p}$ from the north and east boundaries of $o + \mathbb{Z}^2_{<0}$. Precisely,

$$\widehat{Z}^\rho_{o,\,x} = \begin{cases} \underset{k}{\operatorname{argmax}}\big\{ \sum_{i=1}^{k} \hat{I}_{o-i\mathrm{e}_1} + \widehat{G}_{o-k\mathrm{e}_1-\mathrm{e}_2,x} \big\}, & \text{if } \hat{\pi}_{o,x} \cap o - \mathrm{e}_1 \neq \emptyset, \\ -\underset{\ell}{\operatorname{argmax}}\big\{ \sum_{j=1}^{\ell} \hat{J}_{o-j\mathrm{e}_2} + \widehat{G}_{o-\ell\mathrm{e}_2-\mathrm{e}_1,x} \big\}, & \text{if } \hat{\pi}_{o,x} \cap o - \mathrm{e}_2 \neq \emptyset. \end{cases} \tag{3.17}$$

3.4 The comparison lemma

We are going to use the comparison between point-to-point LPP and stationary LPP using the lemma by Cator and Pimentel.

Lemma 3.2. *Let $o = (0,0)$ and consider two points $p^1 \preceq p^2$.*
If $Z^\rho_{o,p^1} \geq 0$, then

$$G_{o,p^2} - G_{o,p^1} \leq G^\rho_{o,p^2} - G^\rho_{o,p^1}. \tag{3.18}$$

If $Z^\rho_{o,p^2} \leq 0$, then

$$G_{o,p^2} - G_{o,p^1} \geq G^\rho_{o,p^2} - G^\rho_{o,p^1}. \tag{3.19}$$

Clearly by reversion of the space we can use this comparison lemma also for backwards LPP.

Lemma 3.3. *Consider two points $p^1 \preceq p^2$ with $p^1, p^2 > \mathcal{L}$.*
If $Z^\rho_{\mathcal{L},p^1} \geq Z^{h_0}_{\mathcal{L},p^2}$, then

$$G^{h_0}_{\mathcal{L},p^2} - G^{h_0}_{\mathcal{L},p^1} \leq G^\rho_{\mathcal{L},p^2} - G^\rho_{\mathcal{L},p^1}. \tag{3.20}$$

If $Z^\rho_{\mathcal{L},p^2} \leq Z^{h_0}_{\mathcal{L},p^1}$, then

$$G^{h_0}_{\mathcal{L},p^2} - G^{h_0}_{\mathcal{L},p^1} \geq G^\rho_{\mathcal{L},p^2} - G^\rho_{\mathcal{L},p^1}. \tag{3.21}$$

For $p^1_2 = p^2_2$, Lemma 3.2 is proven as Lemma 1 of [21], while Lemma 3.3 is Lemma 2.1 of [47]. The generalization to the case of geodesics starting from $\mathcal{L}$ (or from any down-right paths) is straightforward, see e.g. Lemma 3.5 of [31].

4 Local Stationarity

4.1 Localization over a time-span τN.

In this section we are going to prove Theorem 2.8. We shall need the following estimates on the tail of the exit point of a stationary process. For a density $\rho \in (0,1)$ we associate a direction

$$\xi(\rho) = \left(\frac{(1-\rho)^2}{(1-\rho)^2 + \rho^2}, \frac{\rho^2}{(1-\rho)^2 + \rho^2} \right) \tag{4.1}$$

and, vice versa, to each direction $\xi = (\xi_1, \xi_2)$ corresponds a density

$$\rho(\xi) = \frac{\sqrt{\xi_2}}{\sqrt{\xi_1} + \sqrt{\xi_2}}. \tag{4.2}$$

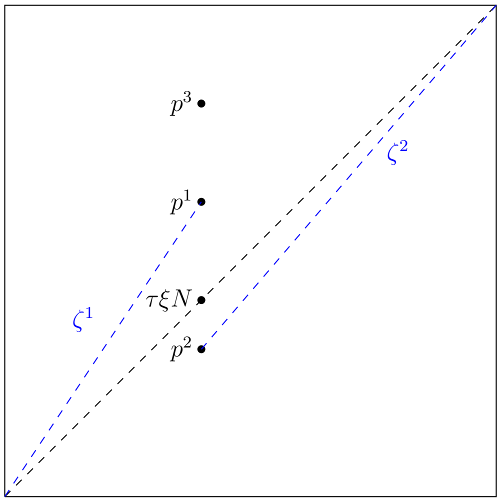

Figure 4.1: Illustration of the geometry of the points and characteristics that appear in the proof of Theorem 4.2.

Lemma 4.1 (Theorem 2.5 and Proposition 2.7 of [25])**.** *Let $\varepsilon \in (0,1]$. Then there exists $N_0(\varepsilon)$, $c_0(\varepsilon)$, $r_0(\varepsilon) > 0$ such that for every direction ξ with $\varepsilon \le \xi_2/\xi_1 \le 1/\varepsilon$, $N \ge N_0$ and $r \ge r_0$:*

$$\mathbb{P}(|Z^{\nu}_{o,\xi N}| > rN^{2/3}) \le e^{-c_0 r^3}, \tag{4.3}$$

$$\mathbb{P}(Z^{\nu}_{o,\xi N - rN^{2/3}\mathrm{e}_3} > 0) \le e^{-c_0 r^3}, \tag{4.4}$$

$$\mathbb{P}(Z^{\nu}_{o,\xi N + rN^{2/3}\mathrm{e}_3} < 0) \le e^{-c_0 r^3}, \tag{4.5}$$

for all densities ν satisfying $|\nu - \rho(\xi)| \le N^{-1/3}$.

Using Lemma 4.1 one can control the path of a point-to-point geodesic.

Theorem 4.2. *Let $\varepsilon \in (0,1]$. Then there exist $N_0(\varepsilon)$ and $c_1(\varepsilon)$ such that for ξ satisfying $\varepsilon \le \xi_2/\xi_1 \le 1/\varepsilon$,*

$$\mathbb{P}\big(\Gamma_{\tau\xi_1 N}(\pi_{o,\xi N}) > M(\tau N)^{2/3}\big) \le e^{-c_1 M^3} \tag{4.6}$$

for all $\tau N \ge N_0$ and all $M \le (\tau N)^{1/3}/\log(N)$.

Remark 4.3. A similar result was obtained in [14, Theorem 3] but with weaker Gaussian bounds. Theorem 4.2 gives the optimal power in the exponent.

Proof. We will show in detail that

$$\mathbb{P}\Big(\Gamma^{u}_{\tau\xi_1 N}(\pi_{o^1,o^2}) > \tau\xi_2 N + M(\tau N)^{2/3}\Big) \le e^{-c_1 M^3}, \tag{4.7}$$

where $o^1 = o$ and $o^2 = \xi N$. Similarly one proves

$$\mathbb{P}\Big(\Gamma^{l}_{\tau\xi_1 N}(\pi_{o^1,o^2}) < \tau\xi_2 N - M(\tau N)^{2/3}\Big) \le e^{-c_1 M^3}. \tag{4.8}$$

Then Theorem 4.2 follows directly from the definition of $\Gamma_{\tau\xi_1 N}(\pi_{o,\xi N})$. Set the points (see Figure 4.1)

$$\begin{aligned} p^1 &= \tau N\xi + \tfrac{M}{4}(\tau N)^{2/3}\mathrm{e}_2, \\ p^2 &= \tau N\xi - \tfrac{M}{8}\tfrac{1-\tau}{\tau}(\tau N)^{2/3}\mathrm{e}_2, \\ p^3 &= \tau N\xi + \tfrac{M}{2}(\tau N)^{2/3}\mathrm{e}_2, \end{aligned} \tag{4.9}$$

and the characteristics associated with (o^1,p^1) and (o^2,p^2)

$$\begin{aligned}\zeta^1 &= (\tau\xi_1 N, \tau\xi_2 N + \tfrac{M}{4}(\tau N)^{2/3}),\\ \zeta^2 &= ((1-\tau)\xi_1 N, (1-\tau)\xi_2 N + \tfrac{M}{8}\tfrac{1-\tau}{\tau}(\tau N)^{2/3}).\end{aligned} \tag{4.10}$$

The associated densities are

$$\begin{aligned}\rho_1 &= \frac{\sqrt{\tau\xi_2 N + \frac{M}{4}(\tau N)^{2/3}}}{\sqrt{\tau\xi_1 N} + \sqrt{\tau\xi_2 N + \frac{M}{4}(\tau N)^{2/3}}},\\ \rho_2 &= \frac{\sqrt{(1-\tau)\xi_2 N + \frac{M}{8}\frac{1-\tau}{\tau}(\tau N)^{2/3}}}{\sqrt{(1-\tau)\xi_1 N} + \sqrt{(1-\tau)\xi_2 N + \frac{M}{8}\frac{1-\tau}{\tau}(\tau N)^{2/3}}}.\end{aligned} \tag{4.11}$$

The choice of p^1 and p^2 is due to the following reasons. First of all, they need to be below p^3 so that the exit points associated with the densities have a given sign with high probability, see (4.12). If one would take p^2 to be the same point as p^1, then the second estimate in (4.12) would become τ-dependent (more precisely $e^{-c_0 M^3 \tau^2/(1-\tau)^2}$. Our choice is such that the decay is uniform in τ. Furthermore, they are chosen such that $\rho_1 - \rho_2 > 0$, see (4.16). This is essential since we are going to estimate increments as the difference between the stationary case with density ρ_1 and the one with density ρ_2, see (4.15).

Note that by (4.4)–(4.5) there exists $c_0 > 0$ such that

$$\begin{aligned}\mathbb{P}\big(Z^{\rho_1}_{o^1,p^3} > 0\big) &\le e^{-c_0 M^3},\\ \mathbb{P}\big(\hat{Z}^{\rho_2}_{o^2,p^3} < 0\big) &\le e^{-c_0 M^3}.\end{aligned} \tag{4.12}$$

Define, for $i \ge 0$,

$$\begin{aligned}J_i &= G_{o^1,p^3+(i+1)\mathrm{e}_2} - G_{o^1,p^3+i\mathrm{e}_2},\\ \widehat{J}_i &= G_{o^2,p^3+\mathrm{e}_1+i\mathrm{e}_2} - G_{o^2,p^3+\mathrm{e}_1+(i+1)\mathrm{e}_2},\\ J_i^{\rho_1} &= G^{\rho_1}_{o^1,p^3+(i+1)\mathrm{e}_2} - G^{\rho_1}_{o^1,p^3+i\mathrm{e}_2},\\ \widehat{J}_i^{\rho_2} &= G^{\rho_2}_{o^2,p^3+\mathrm{e}_1+i\mathrm{e}_2} - G^{\rho_2}_{o^2,p^3+\mathrm{e}_1+(i+1)\mathrm{e}_2}.\end{aligned} \tag{4.13}$$

Then, by Lemma 3.2, it follows from (4.12) that with probability $1 - 2e^{-c_0 M^3}$

$$J_i \le J_i^{\rho_1} \text{ and } \widehat{J}_i^{\rho_2} \le \widehat{J}_i \tag{4.14}$$

for all $i \ge 0$, and therefore that

$$J_i - \widehat{J}_i \le J_i^{\rho_1} - \widehat{J}_i^{\rho_2} \tag{4.15}$$

for all $i \ge 0$. Set $\rho = \rho(\xi)$. Using that $M \le (\tau N)^{1/3}/\log(N)$, we get the series expansions

$$\begin{aligned}\rho_1 &= \rho + \kappa(\rho)\frac{M}{8}(\tau N)^{-1/3} + o((\tau N)^{-1/3}),\\ \rho_2 &= \rho + \kappa(\rho)\frac{M}{16}(\tau N)^{-1/3} + o((\tau N)^{-1/3}),\\ \rho_1 - \rho_2 &= \kappa(\rho)\frac{M}{16}(\tau N)^{-1/3} + o((\tau N)^{-1/3}) > 0,\end{aligned} \tag{4.16}$$

with $\kappa(\rho) = (1-\rho)(1-2\rho(1-\rho))/\rho > 0$ for all $\rho \in (0,1)$.

Define

$$S_i = \sum_{k=0}^{i} J_k - \widehat{J}_k \quad \text{and} \quad W_i = \sum_{k=0}^{i} J_k^{\rho_1} - \widehat{J}_k^{\rho_2} \tag{4.17}$$

so that by (4.15)

$$S_i \leq W_i \text{ for } i \geq 0. \tag{4.18}$$

Note that

$$\{\Gamma^u_{\tau\xi_1 N}(\pi_{o^1,o^2}) > \tau\xi_2 N + M(\tau N)^{2/3}\} \subseteq \Big\{ \sup_{i\geq \frac{M}{2}(\tau N)^{2/3}} S_i > 0 \Big\} \subseteq \Big\{ \sup_{i\geq \frac{M}{2}(\tau N)^{2/3}} W_i > 0 \Big\}. \tag{4.19}$$

It follows that it is enough to show that there exists $c_1 > 0$ such that

$$\mathbb{P}\Big(\sup_{i\geq \frac{M}{2}(\tau N)^{2/3}} W_i > 0 \Big) \leq e^{-c_1 M^3}. \tag{4.20}$$

Note that

$$\begin{aligned} \mathbb{P}\Big(\sup_{i\geq \frac{M}{2}(\tau N)^{2/3}} W_i > 0 \Big) &\leq \mathbb{P}\Big(W_{\frac{M}{2}(\tau N)^{2/3}} > -\frac{\chi(\rho)M^2}{64}(\tau N)^{1/3} \Big) \\ &+ \mathbb{P}\Big(\sup_{i\geq \frac{M}{2}(\tau N)^{2/3}} W_i - W_{\frac{M}{2}(\tau N)^{2/3}} > \frac{\chi(\rho)M^2}{64}(\tau N)^{1/3} \Big) \end{aligned} \tag{4.21}$$

for $\chi(\rho) = \kappa(\rho)/\rho^2$.

Plugging (4.16) in Lemma A.1

$$\begin{aligned} \mathbb{P}\Big(\sup_{i\geq \frac{M}{2}(\tau N)^{2/3}} W_i - W_{\frac{M}{2}(\tau N)^{2/3}} > \frac{\chi(\rho)M^2}{64}(\tau N)^{1/3} \Big) &\leq \frac{\rho_1}{\rho_2} e^{-(\rho_1-\rho_2)\frac{\chi(\rho)M^2}{64}(\tau N)^{1/3}} \\ &\leq 2e^{-\chi(\rho)\kappa(\rho)M^3/1024} \end{aligned} \tag{4.22}$$

for all τN large enough.

Next, using exponential Tchebishev inequality, we show that

$$\mathbb{P}\Big(W_{\frac{M}{2}(\tau N)^{2/3}} > -\frac{\chi(\rho)M^2}{64}(\tau N)^{1/3} \Big) \leq 2e^{-M^3\chi(\rho)\kappa(\rho)/8192} \tag{4.23}$$

for all τN large enough, which completes the proof. Indeed, using

$$\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) \tfrac{M}{2}(\tau N)^{2/3} = -\frac{M^2\chi(\rho)}{32}(\tau N)^{1/3} + o((\tau N)^{1/3}), \tag{4.24}$$

we get, using also the independence of the J's and $\widehat{J}$'s, that

$$\begin{aligned} (4.23) &= \mathbb{P}\Bigg(\sum_{k=0}^{\frac{M}{2}(\tau N)^{2/3}} (J_k^{\rho_1} - \rho_1^{-1} - \widehat{J}_k^{\rho_2} + \rho_2^{-1}) > \frac{\chi(\rho)M^2}{64}(\tau N)^{1/3} + o((\tau N)^{1/3}) \Bigg) \\ &\leq \inf_{\lambda>0} \frac{\mathbb{E}\Big(e^{\lambda(J_1^{\rho_1} - \rho_1^{-1} - \widehat{J}_1^{\rho_2} + \rho_2^{-1})} \Big)^{\frac{M}{2}(\tau N)^{2/3}}}{e^{\lambda[\frac{\chi(\rho)M^2}{64}(\tau N)^{1/3} + o((\tau N)^{1/3})]}} \\ &= \inf_{\mu>0} e^{-M(M\kappa(\rho)-32\mu)\mu/(64\rho^2)+o(1)} \leq 2e^{-M^3\kappa(\rho)\chi(\rho)/8192}, \end{aligned} \tag{4.25}$$

for all τN large enough, where in the third step we set $\lambda = \mu(\tau N)^{-1/3}$ and performed simple computations.

$\square$

Theorem 4.4. *Let $o = (0,0)$ and $\varepsilon \in (0,1]$. There exists $N_1(\varepsilon)$, $c(\varepsilon)$, $C(\varepsilon)$ such that for every direction ξ with $\varepsilon \le \xi_2/\xi_1 \le 1/\varepsilon$, and $v \le N^{1/3}/\log(N)$, for $N > N_1$*

$$\mathbb{P}\Big(\max_{k\in[0,\xi_1 N]} \Gamma_k(\pi_{o,\xi N}) < vN^{2/3}\Big) \ge 1 - Ce^{-cv^3}. \tag{4.26}$$

Proof. The proof follows the approach of [15], using the pointwise control of the fluctuations of the geodesic around the characteristic from Theorem 4.2. Let $m = \min\{j : 2^{-j}N \le N^{1/2}\}$. Choose $u_1 < u_2 < \dots$ with $u_1 = v/10$ and $u_j - u_{j-1} = u_1 2^{-(j-1)/2}$. We define

$$u(k) = \Gamma^u_k(\pi_{o,\xi N}) - \tfrac{\xi_2}{\xi_1}k, \quad k \in [0, \xi_1 N] \tag{4.27}$$

and the following events

$$\begin{aligned} A_j &= \{u(k2^{-j}N) \le u_j N^{2/3}, 1 \le k \le 2^j - 1\},\\ B_{j,k} &= \{u(k2^{-j}N) > u_j N^{2/3}\}, \quad k = 1,\dots,2^j-1,\\ L &= \{\sup_{x\in[0,1]} |u((k+x)2^{-m}N) - u(k2^{-m}N)| \le \tfrac12 vN^{2/3}, 0 \le k \le 2^m - 1\},\\ G &= \{u(k) \le vN^{2/3} \text{ for all } 0 \le k \le \xi_1 N\}. \end{aligned} \tag{4.28}$$

Notice that $A_j^c = \bigcup_{k=1}^{2^j-1} B_{j,k}$. Also, since $\lim_{j\to\infty} u_j \le v/2$, we have

$$\bigcup_{j=1}^{m}\bigcup_{k=1}^{2^j-1} (B_{j,k}\cap A_{j-1}) \supseteq \{u(k2^{-m}N) \ge \tfrac12 vN^{2/3} \text{ for some } k = 1,\dots,2^m-1\}. \tag{4.29}$$

This implies that

$$G \supseteq \left(\bigcup_{j=1}^{m}\bigcup_{k=1}^{2^j-1} (B_{j,k}\cap A_{j-1}) \right)^c \cap L. \tag{4.30}$$

Thus we have

$$\mathbb{P}(G^c) \le \mathbb{P}(L^c) + \sum_{j=1}^{m}\sum_{k=1}^{2^j-1} \mathbb{P}(B_{j,k}\cap A_{j-1}). \tag{4.31}$$

Since the geodesics have discrete steps, in n time steps a geodesic can wonder off by at most n steps from its characteristic. For all N large enough, $N^{1/2} < \frac12 vN^{2/3}$ and therefore $\mathbb{P}(L) = 1$. Thus we need to bound $\mathbb{P}(B_{j,k}\cap A_{j-1})$ only. As for even k the two events are incompatible, we consider odd k.

If A_{j-1} holds, then the geodesic at $t_1 = (k-1)2^{-j}N$ and $t_2 = (k+1)2^{-j}N$ satisfies

$$u(t_1) \le u_{j-1}N^{2/3} \quad \text{and} \quad u(t_2) \le u_{j-1}N^{2/3}. \tag{4.32}$$

Consider the point-to-point LPP from $\hat o^1$ to $\hat o^2$ with

$$\hat o^1 = (t_1, t_1\tfrac{\xi_2}{\xi_1} + u_{j-1}) \quad \text{and} \quad \hat o^2 = (t_2, t_2\tfrac{\xi_2}{\xi_1} + u_{j-1}). \tag{4.33}$$

Let $\hat u(i) = \Gamma^u_i(\pi_{\hat o^1,\hat o^2})$ for $i \in [t_1, t_2]$. Then, by the order of geodesics

$$u(i) \le \hat u(i) \text{ for } i \in [t_1, t_2], \tag{4.34}$$

so that

$$\{u(i) > u_j N^{2/3}\} \subseteq \{\hat{u}(i) > u_j N^{2/3}\} \text{ for } i \in [t_1, t_2]. \tag{4.35}$$

This gives

$$\mathbb{P}(B_{j,k} \cap A_{j-1}) \leq \mathbb{P}\big(\hat{u}(k2^{-j}N) > u_j N^{2/3}\big). \tag{4.36}$$

Since the law of $\hat{u}$ is the one of a point-to-point LPP over a time distance $t_2 - t_1 = 2^{-j+1}N$, we can apply Theorem 4.2 with $\tau = 1/2$, $N = t_2 - t_1$ M satisfying $(u_j - u_{j-1})N^{2/3} = M(\frac{1}{2}(t_j - t_{j-1}))^{2/3}$. This gives

$$\mathbb{P}\big(\hat{u}(k2^{-j}N) > u_j N^{2/3}\big) \leq e^{-c_1(u_1 2^{-(j-1)/2} 2^{2j/3})^3} \leq e^{-c_1 u_1^3 2^{j/2}}. \tag{4.37}$$

This bound applied to (4.31) leads to $\mathbb{P}(G) \leq Ce^{-cv^3}$ for some constants $C, c > 0$. □

Now we have all the ingredients to prove Theorem 2.8.

Proof of Theorem 2.8. Theorem 4.2 implies that with probability at least $1 - e^{-c_1 M^3/8}$, the geodesic from o to ξN does not deviate more than $\frac{1}{2}M(\tau N)^{2/3}$ away from the point $\tau\xi N$. Given this event, by order of geodesics, the geodesic from o to ξN is sandwiched between the geodesics from $\frac{1}{2}M(\tau N)^{2/3}\mathrm{e}_1$ to $\tau\xi N + \frac{1}{2}M(\tau N)^{2/3}\mathrm{e}_1$ and the one from $-\frac{1}{2}M(\tau N)^{2/3}\mathrm{e}_1$ to $\tau\xi N - \frac{1}{2}M(\tau N)^{2/3}\mathrm{e}_1$. By Theorem 4.4, the latter two geodesics fluctuates no more than $\frac{1}{2}M(\tau N)^{2/3}$, with probability at least $1 - Ce^{-cM^3/\tau^2}$, which implies the claim. □

4.2 Localization of the exit point

In this subsection, we estimate the location of the exit point for densities slightly larger or smaller than 1/2. This will allow us to sandwich the point-to-point geodesics by those of the stationary. Notice that to apply Lemma 3.2, it would be enough to set in the event $\mathcal{A}_1$ (resp. $\mathcal{A}_2$) below that the exit point is positive (resp. negative) and bounded by $15rN^{2/3}$ (resp. $-15rN^{2/3}$) as the exit point for the LPP $G_{o,x}$ is 0. However, with this slight modification (that the exit point is $rN^{2/3}$ from the origin), the proof is then applicable for more general initial conditions provided the exit points of the LPP with initial conditions h_0 on $\mathcal{L}$ is localized in $[-rN^{2/3}, rN^{2/3}]$ with high probability.

Fix $r > 0$ and let $s_r, t_r > 0$ which will be determined later. s_r and t_r represent the scale of space and time respectively, see Figure 4.2. Let $0 < a < 1$. Define the points

$$\begin{aligned} x^1 &= N\mathrm{e}_4 + as_r N^{2/3}\mathrm{e}_3, \\ x^2 &= N\mathrm{e}_4 - as_r N^{2/3}\mathrm{e}_3. \end{aligned} \tag{4.38}$$

Define the densities

$$\rho_+ = \tfrac{1}{2} + rN^{-1/3}, \quad \rho_- = \tfrac{1}{2} - rN^{-1/3}, \tag{4.39}$$

and, with $o = (0, 0)$, the events

$$\begin{aligned} \mathcal{A}_1 &= \big\{Z^{\rho_+}_{o,x^2} \geq rN^{2/3}, Z^{\rho_+}_{o,x^1} \leq 15rN^{2/3}\big\}, \\ \mathcal{A}_2 &= \big\{Z^{\rho_-}_{o,x^2} \geq -15rN^{2/3}, Z^{\rho_-}_{o,x^1} \leq -rN^{2/3}\big\}, \\ \mathcal{A} &= \mathcal{A}_1 \cup \mathcal{A}_2. \end{aligned} \tag{4.40}$$

The event $\mathcal{A}$ is highly probable for large r as shows the following lemma.

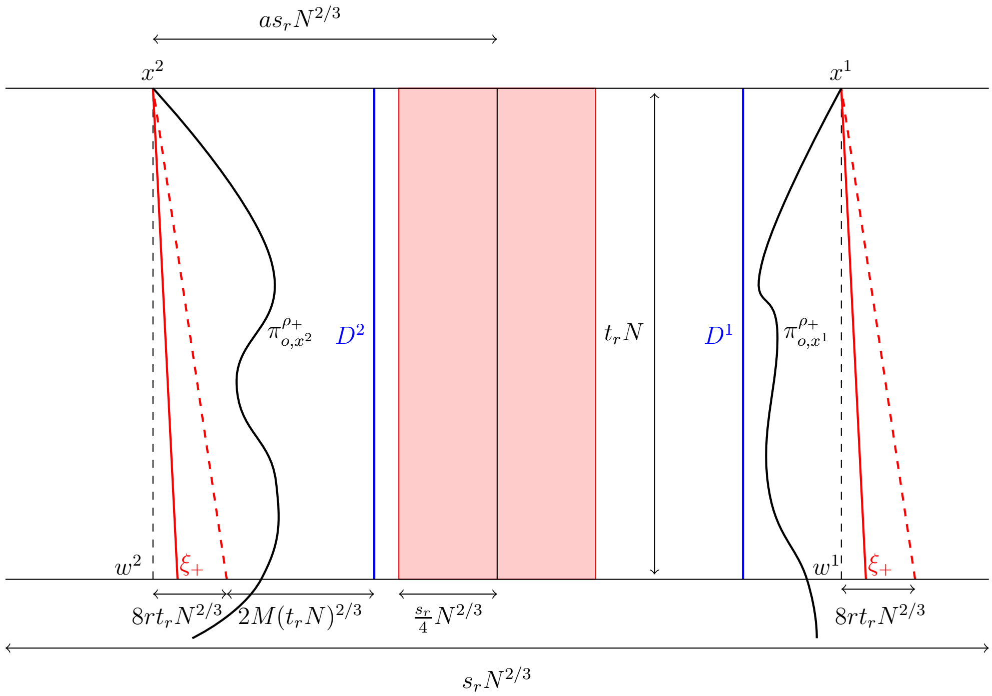

Figure 4.2: Illustration of the geometry around the end-point (N, N) magnified and rotated by $\pi/4$. Choosing t_r, s_r properly forces the geodesic $\pi^{\rho_+}_{o,x^2}$ to traverse to the left of D^2 and $\pi^{\rho_+}_{o,x^1}$ to the right of D^1.

Lemma 4.5. *Assume* $0 \leq s_r \leq 2r$ *and* $0 < a < 1$*. There exists* $c, N_0 > 0$ *such that for* $N > N_0$ *and* $0 < r < N^{1/3}/\log(N)$,

$$\mathbb{P}(\mathcal{A}) \geq 1 - e^{-cr^3}. \tag{4.41}$$

Proof. We will show the claim for $\mathcal{A}_1$, one can similarly prove the claim for $\mathcal{A}_2$. The result would then follow from union bound. To prove the claim for $\mathcal{A}_1$, by union bound it is enough to show that

$$\mathbb{P}\big(Z^{\rho_+}_{o,x^2} \geq rN^{2/3}\big) \geq 1 - e^{-cr^3}, \tag{4.42}$$

$$\mathbb{P}\big(Z^{\rho_+}_{o,x^1} \leq 15rN^{2/3}\big) \geq 1 - e^{-cr^3}. \tag{4.43}$$

Let $x^3 = x^2 - rN^{2/3}\mathrm{e}_1$. Then by stationarity of the model,

$$\mathbb{P}\big(Z^{\rho_+}_{o,x^2} < rN^{2/3}\big) = \mathbb{P}\big(Z^{\rho_+}_{o,x^3} < 0\big). \tag{4.44}$$

Now we want to use (4.5). For that denote $\tilde{N} = N - \frac{r}{2}N^{2/3}$ and write

$$x^3 = \xi(\rho_+)2\tilde{N} + \tilde{r}\tilde{N}^{2/3}\mathrm{e}_3. \tag{4.45}$$

Solving with respect to $\tilde{r}$ we obtain

$$\tilde{r} = \frac{7}{2}r - as_r + \mathcal{O}(r^2/N^{1/3}). \tag{4.46}$$

Applying (4.5) with $\nu \to \rho_+$, $N \to \tilde{N}$ and $r \to \tilde{r}$ gives

$$\mathbb{P}\big(Z^{\rho_+}_{o,x^3} < 0\big) \leq e^{-c_0\tilde{r}^3}. \tag{4.47}$$

Since $s_r \leq 2r$ and $r \leq N^{1/3}/\log(N)$, we have $\tilde{r} \geq r$ for all N large enough, which proves (4.42).

Let $x^4 = x^1 - 15rN^{2/3}\mathrm{e}_1$. Then by stationarity of the model,

$$\mathbb{P}\big(Z^{\rho_+}_{o,x^1} > 15rN^{2/3}\big) = \mathbb{P}\big(Z^{\rho_+}_{o,x^4} > 0\big). \tag{4.48}$$

We will apply this time (4.4). Denote $\tilde{N} = N - \frac{15}{2}rN^{2/3}$ and write

$$x^4 = \xi(\rho_+)2\tilde{N} - \hat{r}\tilde{N}^{2/3}\mathrm{e}_3. \tag{4.49}$$

Solving with respect to $\hat{r}$ we obtain

$$\hat{r} = \frac{7}{2}r - as_r + \mathcal{O}(r^2/N^{1/3}) \geq r \tag{4.50}$$

for all N large enough. Applying (4.4) with $\nu \to \rho_+$, $N \to \tilde{N}$ and $r \to \hat{r}$ proves (4.43). $\square$

4.3 Uniform sandwiching of geodesics terminating in $\mathcal{C}^{s_r/2,t_r}$.

Consider the following assumption.

Assumption 4.6. *Let $M_0 > 0$, $a = 3/8$, $s_r \leq \min\{r, 4\}$ and make the following assumptions on the parameters:*

$$r \leq N^{1/3}/\log(N), \quad M_0 \leq \tilde{M} := \tfrac{1}{16}s_r t_r^{-2/3} - 4rt_r^{1/3}. \tag{4.51}$$

We shall later discuss this assumption in Remark 4.10 below. Under Assumption 4.6, the geodesics $\pi^{1/2}_{o,x}$ and $\pi_{y,x}$, for $y \in \mathcal{R}^{r/2,1/4}$, are controlled by the ones with densities ρ_+ and ρ_- for all $x \in \mathcal{C}^{s_r/2,t_r}$. This is the content of the following result whose proof we defer to the end of this section.

Lemma 4.7. *Under Assumption 4.6, there exists $C, c > 0$ such that*

$$\mathbb{P}\Big(\pi^{\rho_-}_{o,x} \preceq \pi^{1/2}_{o,x}, \pi_{y,x} \preceq \pi^{\rho_+}_{o,x} \quad \forall x \in \mathcal{C}^{s_r/2,t_r}, y \in \mathcal{R}^{r/2,1/4}\Big) \geq 1 - Ce^{-c\tilde{M}^3} - 2e^{-cr^3} \tag{4.52}$$

for all N large enough.

Define

$$\begin{aligned} c^1 &= \pi^{\rho_+}_{o,x^1} \cap L_{1-t_r}, \\ c^2 &= \pi^{\rho_+}_{o,x^2} \cap L_{1-t_r}. \end{aligned} \tag{4.53}$$

To ease the notation we also denote

$$\begin{aligned} w^2 &= (1-t_r)N\mathrm{e}_4 - as_rN^{2/3}\mathrm{e}_3 = x^2 - t_rN\mathrm{e}_4, \\ w^1 &= (1-t_r)N\mathrm{e}_4 + as_rN^{2/3}\mathrm{e}_3 = x^1 - t_rN\mathrm{e}_4. \end{aligned} \tag{4.54}$$

Lemma 4.8. *There exists $c, N_0, M_0 > 0$ such that for $t_rN > N_0$, $r \leq N^{1/3}/\log(N)$, $M \geq M_0$*

$$\mathbb{P}\Big(w^2 - M(t_rN)^{2/3}\mathrm{e}_3 \preceq c^2 \preceq w^2 + (8rt_rN^{2/3} + M(t_rN)^{2/3})\mathrm{e}_3\Big) \geq 1 - e^{-cM^3}, \tag{4.55}$$

$$\mathbb{P}\Big(w^1 - M(t_rN)^{2/3}\mathrm{e}_3 \preceq c^1 \preceq w^1 + (8rt_rN^{2/3} + M(t_rN)^{2/3})\mathrm{e}_3\Big) \geq 1 - e^{-cM^3}. \tag{4.56}$$

Proof. Let p^2 be the point of intersection of the characteristic ξ_+ starting from x^2 with the line L_{1-t_r}. We have

$$p^2 = w^2 + (4rt_rN^{2/3} + \mathcal{O}(r^3t_r))\mathrm{e}_3 = w^2 + 4rt_rN^{2/3}(1+o(1))\mathrm{e}_3 \tag{4.57}$$

for $r \leq N^{1/3}/\log(N)$ and N large enough, implying

$$w^2 \preceq p^2 \preceq w^2 + 8rt_rN^{2/3}\mathrm{e}_3 =: z^2. \tag{4.58}$$

By the order on geodesics $c^2 \preceq \pi^{\rho+}_{o,x^2+4rt_rN^{2/3}\mathrm{e}_3}$ and if $Z^{\rho+}_{z^2+M(t_rN)^{2/3}\mathrm{e}_3,x^2+4rt_rN^{2/3}\mathrm{e}_3} < 0$, then $\pi^{\rho+}_{o,x^2+4rt_rN^{2/3}\mathrm{e}_3} \preceq z^2 + M(t_rN)^{2/3}\mathrm{e}_3$. Thus

$$\mathbb{P}(c^2 \preceq z^2 + M(t_rN)^{2/3}\mathrm{e}_3) \geq \mathbb{P}(Z^{\rho+}_{z^2+M(t_rN)^{2/3}\mathrm{e}_3,x^2+4rt_rN^{2/3}\mathrm{e}_3} < 0). \tag{4.59}$$

Using (4.4), the latter is bounded from above by $1 - e^{-c_0M^3}$ provided $M \geq M_0$ and $t_rN \geq N_0$.

A similar bound can be obtained for

$$\mathbb{P}(w^2 - M(t_rN)^{2/3}\mathrm{e}_3 \preceq c^2) \tag{4.60}$$

using (4.5). Thus we have shown that (4.55) holds. The proof of (4.56) is almost identical and thus we do not repeat the details. □

Set

$$\begin{aligned} q^1 &= (1-t_r)N\mathrm{e}_4 + N^{2/3}(as_r - 2Mt_r^{2/3})\mathrm{e}_3, \\ q^2 &= (1-t_r)N\mathrm{e}_4 + N^{2/3}(8rt_r - as_r + 2Mt_r^{2/3})\mathrm{e}_3 \end{aligned} \tag{4.61}$$

and define the lines (see Figure 4.2)

$$\begin{aligned} D^1 &= \{q^1 + \alpha t_rN\mathrm{e}_4 : 0 < \alpha < 1\}, \\ D^2 &= \{q^2 + \alpha t_rN\mathrm{e}_4 : 0 < \alpha < 1\}. \end{aligned} \tag{4.62}$$

Lemma 4.9. *There exist $N_1, c, C > 0$ such that for every $N \geq N_1$ and $M \leq N^{1/3}/\log(N)$, $r \leq N^{1/3}/\log(N)$,*

$$\mathbb{P}\Big(D^1 \preceq \pi^{\rho+}_{o,x^1}\Big) \geq 1 - Ce^{-cM^3}, \tag{4.63}$$

$$\mathbb{P}\Big(\pi^{\rho+}_{o,x^2} \preceq D^2\Big) \geq 1 - Ce^{-cM^3}. \tag{4.64}$$

Proof. We will show (4.64) as (4.63) can be proven similarly. Let

$$u^2 = q^2 - M(t_rN)^{2/3}\mathrm{e}_3. \tag{4.65}$$

By Theorem 4.4 we have

$$\mathbb{P}\big(\pi_{u^2,x^2} \preceq D^2\big) \geq 1 - Ce^{-cM^3}. \tag{4.66}$$

Recall the definition (4.53) of c^1. By Lemma 4.8

$$\mathbb{P}\big(c^2 \preceq u^2\big) \geq 1 - e^{-cM^3}, \tag{4.67}$$

which implies that

$$\mathbb{P}\big(\pi^{\rho+}_{o,x^2} \preceq \pi_{u^2,x^2}\big) \geq 1 - Ce^{-cM^3}. \tag{4.68}$$

(4.66) and (4.68) imply (4.64). □

Remark 4.10. Now we can discuss the origin of the conditions in Assumption 4.6. The bound on r comes from Lemma 4.9. The condition on M is a consequence of the conditions $q^2 \preceq (1-t_r)N\mathrm{e}_4 - \frac{1}{4}s_r N^{2/3}\mathrm{e}_3$ and also $(1-t_r)N\mathrm{e}_4 + \frac{1}{4}s_r N^{2/3}\mathrm{e}_3 \preceq q^1$. As we want M to grow to infinity we need to take $t_r \ll s_r/r$.

For $0 < \tau < 1$ and $\sigma \in \mathbb{R}_+$, define the anti-diagonal segment

$$\mathcal{I}^{\sigma,\tau} = \{(1-\tau)N\mathrm{e}_4 + i\mathrm{e}_3, i \in [-\tfrac{\sigma}{2}N^{2/3}, \tfrac{\sigma}{2}N^{2/3}]\}, \tag{4.69}$$

located right below the cylinder $\mathcal{C}^{\sigma,\tau}$. Define the events

$$\begin{aligned} \mathcal{O} &= \{Z^{\rho_-}_{o,x} \in [-15rN^{2/3}, -rN^{2/3}], Z^{\rho_+}_{o,x} \in [rN^{2/3}, 15rN^{2/3}] \quad \forall x \in \mathcal{C}^{s_r/2,t_r}\}, \\ \mathcal{B} &= \Big\{\{\pi^{\rho_-}_{o,x} \cap \mathcal{I}^{s_r,t_r} \neq \emptyset\} \cap \{\pi^{\rho_+}_{o,x} \cap \mathcal{I}^{s_r,t_r} \neq \emptyset\} \quad \forall x \in \mathcal{C}^{s_r/2,t_r}\Big\}. \end{aligned} \tag{4.70}$$

Corollary 4.11. *Under Assumption 4.6 there exists $C, c, N_0 > 0$ such that for $N > N_0$*

$$\mathbb{P}(\mathcal{O}) \geq 1 - 2e^{-cr^3} - Ce^{-c\tilde{M}^3}, \tag{4.71}$$

$$\mathbb{P}(\mathcal{B}) \geq 1 - e^{-c\tilde{M}^3}. \tag{4.72}$$

Proof. It might be helpful to take a look at Figure 4.2 while reading the proof. We prove in details the statements for ρ_+, since the proof for ρ_- is almost identical.

By our choice of parameters,

$$D^2 \preceq \mathcal{C}^{s_r/2,t_r} \preceq D^1. \tag{4.73}$$

By Lemma 4.9 and order of geodesics, with probability at least $1 - Ce^{-cM^3}$,

$$\pi^{\rho_+}_{o,x^2} \preceq \pi^{\rho_+}_{o,x} \preceq \pi^{\rho_+}_{o,x^1} \tag{4.74}$$

for all $x \in \mathcal{C}^{s_r/2,t_r}$. By Lemma 4.5 the exit point of the geodesics to x^1 and x^2 for the stationary model with density ρ_+ lies between $rN^{2/3}$ and $15rN^{2/3}$ with probability at least $1 - e^{-cr^3}$, which leads to (4.71).

To prove (4.72), we now choose $M = \tilde{M}$ from Assumption 4.6, which implies

$$w^1 + (8rt_rN^{2/3} + \tilde{M}(t_rN)^{2/3})\mathrm{e}_3 \preceq w^1 + \tfrac{1}{2}s_r\mathrm{e}_3 \tag{4.75}$$

and

$$w^2 - \tfrac{1}{2}s_r\mathrm{e}_3 \preceq w^2 - \tilde{M}(t_rN)^{2/3}\mathrm{e}_3. \tag{4.76}$$

Thus by Lemma 4.8 we know that the crossing of $\pi^{\rho_+}_{o,x^1}$ and $\pi^{\rho_+}_{o,x^2}$ with $\mathcal{I}^{s_r,t_r}$ occurs with probability at least $1 - e^{-c\tilde{M}^3}$. This, together with (4.74) implies (4.72). □

Proof of Lemma 4.7. Consider the straight line going from $(0, rN^{2/3})$ to x^1 parameterized by $(u, l_1(u))_{u\geq 0}$ with

$$l_1(u) = rN^{2/3} + u\frac{N - (\frac{3}{8}s_r + r)N^{2/3}}{N + \frac{3}{8}s_rN^{2/3}}, \tag{4.77}$$

and the straight line $(u, l_2(u))_{u\geq 0}$, which overlaps in its first part with the boundary of $\mathcal{R}^{r/2,1/4}$, defined through

$$l_2(u) = \frac{r}{2}N^{2/3} + u. \tag{4.78}$$

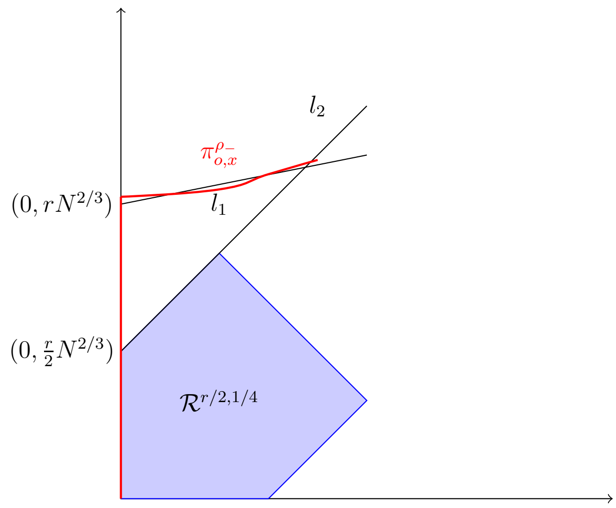

Figure 4.3: Illustration of the setup in the proof of Lemma 4.7 .

By our assumption $s_r \leq r$,

$$\inf_{0\leq u\leq \frac{N}{4}} (l_1(u) - l_2(u)) \geq \frac{r}{2}N^{2/3} - \frac{\frac{7}{4}rN^{2/3}}{N + \frac{3}{8}rN^{2/3}} \frac{N}{4} \geq \frac{1}{16}rN^{2/3}. \tag{4.79}$$

It follows from (4.79) and Theorem 4.4 that for some $C, c > 0$ (see Figure 4.3)

$$\mathbb{P}(\Gamma^l_u(\pi_{(0,rN^{2/3}),x^1}) < l_2(u) \text{ for some } 0 \leq u \leq \tfrac{N}{4}) \leq Ce^{-cr^3}. \tag{4.80}$$

One can obtain an analogue of Lemma 4.9 for ρ_-, by considering the reflection of the setup in Figure 4.2 along the vertical line crossing the origin. Thus, by order of geodesics and (4.73), with probability at least $1-C_1e^{-c_1M^3}$, $\pi^{\rho_-}_{o,x} \preceq \pi^{\rho_-}_{o,x^1}$ for all $x \in \mathcal{C}^{s_r/2,t_r}$. Furthermore, on the event $\mathcal{A}_2$, $\pi^{\rho_-}_{o,x^1}$ starts from a point above $(0, rN^{2/3})$. Combining these two facts with (4.80) we get

$$\mathbb{P}\Big(\pi^{\rho_-}_{o,x} \preceq \pi_{(0,rN^{2/3}),x^1} \preceq \mathcal{R}^{r/2,1/4} \quad \forall x \in \mathcal{C}^{s_r/2,t_r}\Big) \geq 1 - C_1e^{-c_1M^3} - Ce^{-cr^3}. \tag{4.81}$$

A similar result can be obtained for $\pi^{\rho_+}_{o,x}$, which combined with (4.81) gives

$$\mathbb{P}\Big(\pi^{\rho_-}_{o,x} \preceq \mathcal{R}^{r/2,1/4} \preceq \pi^{\rho_+}_{o,x} \quad \forall x \in \mathcal{C}^{s_r/2,t_r}\Big) \geq 1 - 2C_1e^{-c_1M^3} - 2Ce^{-cr^3}. \tag{4.82}$$

By the order of geodesics, on the event of (4.82), every geodesic starting in $\mathcal{R}^{r/2,1/4}$ and ending at x, is sandwiched between $\pi^{\rho_-}_{o,x}$ and $\pi^{\rho_+}_{o,x}$.

Next, for each point $x \in \mathcal{C}^{s_r/2,t_r}$, its associated density $\rho(x)$ satisfies $|\rho(x) - \frac{1}{2}| \leq N^{-1/3}$ for all $s_r \leq 4(1-t_r)$ and N large enough. By (4.3) of Lemma 4.1, with probability at least $1 - e^{-c_0r^3}$ the exit point of $\pi^{1/2}_{o,x}$ is also between $-rN^{2/3}$ and $rN^{2/3}$. Thus by appropriate choice of constants C, c, the sandwitching of $\pi^{1/2}_{o,x}$ in (4.52) holds. □

5 Lower bound for the probability of no coalescence

5.1 Point-to-point case: proof of Theorem 2.2

Using the results of Section 4 we first relate the bound for the coalescing point of $\pi^{1/2}_{o,x}$ and $\pi_{o,x}$ to that of the coalescing point of $\pi^{\rho+}_{o,x}$ and $\pi^{\rho-}_{o,x}$.

Lemma 5.1. *Under Assumption 4.6, there exists $C, c > 0$ such that*

$$\begin{aligned}\mathbb{P}\Big(C_p(\pi^{1/2}_{o,x}, \pi_{y,x}) \leq L_{1-t_r} \quad \forall x \in \mathcal{C}^{s_r/2,t_r}, y \in \mathcal{R}^{r/2,1/4}\Big) &\geq 1 - Ce^{-c\tilde{M}^3} - 2e^{-cr^3} \\ - \mathbb{P}\Big(\exists x \in \mathcal{C}^{s_r,t_r} : C_p(\pi^{\rho+}_{o,x}, \pi^{\rho-}_{o,x}) \geq \mathcal{I}^{s_r/2,t_r}, \mathcal{B}\Big)&\end{aligned} \tag{5.1}$$

for all N large enough, where $\tilde{M} = \frac{1}{16}s_r t_r^{-2/3} - 4rt_r^{1/3}$.

Proof. We bound the probability of the complement event. Define the event

$$\mathcal{G} = \{\pi^{\rho-}_{y,x} \preceq \pi^{1/2}_{o,x}, \pi_{y,x} \preceq \pi^{\rho+}_{y,x} \quad \forall x \in \mathcal{C}^{s_r/2,t_r}, y \in \mathcal{R}^{r/2,1/4}\}. \tag{5.2}$$

Then

$$\begin{aligned}\mathbb{P}\Big(\exists x \in \mathcal{C}^{s_r/2,t_r}, y \in \mathcal{R}^{r/2,1/4} : C_p(\pi^{1/2}_{o,x}, \pi_{y,x}) > L_{1-t_r}\Big) &\leq \mathbb{P}(\mathcal{B}^c) + \mathbb{P}(\mathcal{G}^c) \\ + \mathbb{P}\Big(\{\exists x \in \mathcal{C}^{s_r/2,t_r}, y \in \mathcal{R}^{r/2,1/4} : C_p(\pi^{1/2}_{o,x}, \pi_{y,x}) > L_{1-t_r}\} \cap \mathcal{B} \cap \mathcal{G}\Big)&.\end{aligned} \tag{5.3}$$

Note that if $\mathcal{B}$ and $\mathcal{G}$ hold, then both geodesics $\pi^{1/2}_{o,x}$ and $\pi_{y,x}$ are sandwiched between $\pi^{\rho-}_{o,x}$ and $\pi^{\rho+}_{o,x}$ and their crossings with the line L_{1-t_r} occurs in the segment $\mathcal{I}^{s_r,t_r}$. Furthermore, if $C_p(\pi^{1/2}_{o,x}, \pi_{y,x}) > L_{1-t_r}$ and $\mathcal{G}$ holds, then also $C_p(\pi^{\rho+}_{o,x}, \pi^{\rho-}_{o,x}) > L_{1-t_r}$. This, together with Corollary 4.11 and Lemma 4.7 proves the claim. ☐

Thus, to prove Theorem 2.2 it remains to get an upper bound for the last probability in (5.1).

For $x, y, z \in \mathbb{Z}^2$ such that $x \leq y \leq z$, let $\gamma_{x,z}$ be an up-right path going from x to z. Define the exit point of $\gamma_{x,z}$ with respect to the point y

$$\mathcal{Z}_y(\gamma_{x,z}) = \sup\{u \in \gamma_{x,z} : u_1 = y_1 \text{ or } u_2 = y_2\}. \tag{5.4}$$

Define the sets

$$\begin{aligned}\mathcal{H} &= \{(1-t_r)N\mathrm{e}_4 + \tfrac{s_r}{2}N^{2/3}\mathrm{e}_3 - i\mathrm{e}_1, \quad 1 \leq i \leq \tfrac{s_r}{2}N^{2/3}\}, \\ \mathcal{V} &= \{(1-t_r)N\mathrm{e}_4 - \tfrac{s_r}{2}N^{2/3}\mathrm{e}_3 - i\mathrm{e}_2, \quad 1 \leq i \leq \tfrac{s_r}{2}N^{2/3}\},\end{aligned} \tag{5.5}$$

and the point

$$v^c = [(1-t_r)N - \tfrac{s_r}{2}N^{2/3}]\mathrm{e}_4. \tag{5.6}$$

Define the event

$$E_1 = \{\mathcal{Z}_{v^c}(\pi^{\rho+}_{o,x}) \in \mathcal{H} \cup \mathcal{V}, \mathcal{Z}_{v^c}(\pi^{\rho-}_{o,x}) \in \mathcal{H} \cup \mathcal{V} \quad \forall x \in \mathcal{C}^{s_r/2,t_r}\}. \tag{5.7}$$

Note that (see Figure 5.1)

$$E_1 = \mathcal{B}, \tag{5.8}$$

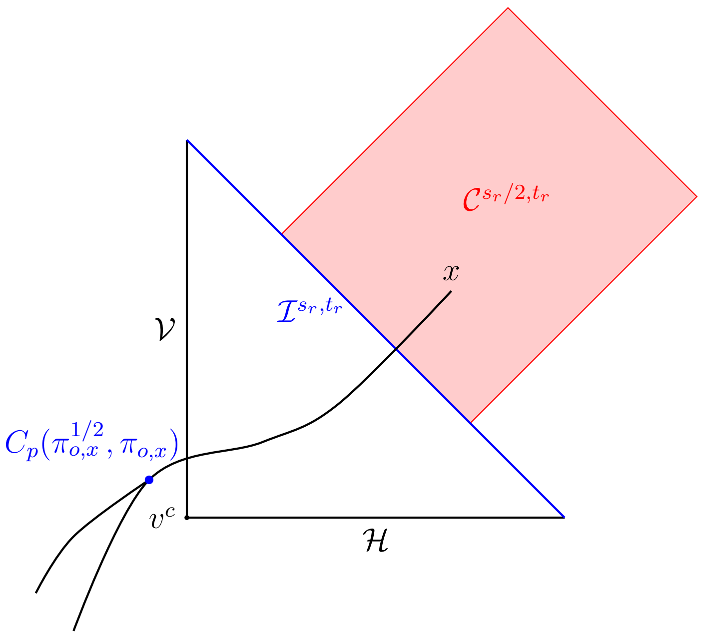

Figure 5.1: On the event $E_1 \cap E_2$, the geodesics $\pi_{o,x}^{\rho+}$ and $\pi_{o,x}^{\rho-}$ coalesce before crossing $\mathcal{I}^{s_r,t_r}$.

since to cross the set $\mathcal{I}^{s_r,t_r}$ the geodesic must cross either $\mathcal{H}$ or $\mathcal{V}$ and, viceversa, if the geodesic crosses $\mathcal{H} \cup \mathcal{V}$, then it crosses also $\mathcal{I}^{s_r,t_r}$.

On the edges of $\mathbb{Z}^2_{\geq 0}$ define the random field B through

$$B_{x,x+\mathrm{e}_k} = G(\pi_{o,x+\mathrm{e}_k}) - G(\pi_{o,x}), \quad k = 1, 2 \tag{5.9}$$

for all $x > o$. Similarly we define

$$\begin{aligned} B^{\rho+}_{x,x+\mathrm{e}_k} &= G(\pi^{\rho+}_{o,x+\mathrm{e}_k}) - G(\pi^{\rho+}_{o,x}), \quad k = 1, 2, \\ B^{\rho-}_{x,x+\mathrm{e}_k} &= G(\pi^{\rho-}_{o,x+\mathrm{e}_k}) - G(\pi^{\rho-}_{o,x}), \quad k = 1, 2. \end{aligned} \tag{5.10}$$

One can couple the random fields $B, B^{\rho-}$ and $B^{\rho+}$ (see [26, Theorem 2.1]) such that

$$\begin{aligned} B^{\rho-}_{x-\mathrm{e}_1,x} \leq B_{x-\mathrm{e}_1,x} \leq B^{\rho+}_{x-\mathrm{e}_1,x}, \\ B^{\rho+}_{x-\mathrm{e}_2,x} \leq B_{x-\mathrm{e}_2,x} \leq B^{\rho-}_{x-\mathrm{e}_2,x}. \end{aligned} \tag{5.11}$$

Let $o \leq v \leq x$. Then, since each geodesic has to pass either by one site on the right or above v, we have

$$\begin{aligned} G_{o,x} = G_{o,v} + \max \Bigg\{ &\sup_{0 \leq l \leq x_1 - v_1} \sum_{i=0}^{l} B_{v+i\mathrm{e}_1, v+(i+1)\mathrm{e}_1} + G_{v+(l+1)\mathrm{e}_1+\mathrm{e}_2, x}, \\ &\sup_{0 \leq l \leq x_2 - v_2} \sum_{i=0}^{l} B_{v+i\mathrm{e}_2, v+(i+1)\mathrm{e}_2} + G_{v+(l+1)\mathrm{e}_2+\mathrm{e}_1, x} \Bigg\}. \end{aligned} \tag{5.12}$$

Thus setting $v = v^c$, on the event $E_1 \cap \mathcal{G}$, for every $x \in \mathcal{C}^{s_r/2,t_r}$

$$\begin{aligned} G_{o,x} = G_{o,v^c} + \max \Bigg\{ &\sup_{u \in \mathcal{H}} \sum_{i=0}^{u_1 - v^c_1} B_{v^c+i\mathrm{e}_1, v^c+(i+1)\mathrm{e}_1} + G_{u+\mathrm{e}_2, x}, \\ &\sup_{u \in \mathcal{V}} \sum_{i=0}^{u_2 - v^c_2} B_{v^c+i\mathrm{e}_2, v^c+(i+1)\mathrm{e}_2} + G_{u+\mathrm{e}_1, x} \Bigg\}. \end{aligned} \tag{5.13}$$

$G^{\rho+}_{o,x}$ and $G^{\rho-}_{o,x}$ can be decomposed in the same way.

This shows that on the event $E_1 \cap \mathcal{G}$ the restriction of the geodesics $\pi_{o,x}, \pi^{\rho+}_{o,x}, \pi^{\rho-}_{o,x}$ to $\mathbb{Z}^2_{>v^c}$ is a function of the weights (5.9)–(5.10) and the bulk weights east-north to v^c. More precisely, define $\mathcal{E}^B_{v^c}$ to be the set of edges in the south-west boundary of $\mathbb{Z}^2_{>v^c}$ restricted to $\mathcal{H} \cup \mathcal{V}$ i.e.

$$\begin{aligned}
\mathcal{E}^H_{v^c} &= \{(x - \mathrm{e}_1, x) : x \in \mathcal{H} \setminus v^c\}, \\
\mathcal{E}^V_{v^c} &= \{(x - \mathrm{e}_2, x) : x \in \mathcal{V} \setminus v^c\}, \\
\mathcal{E}^B_{v^c} &= \mathcal{E}^H_{v^c} \cup \mathcal{E}^V_{v^c}.
\end{aligned} \tag{5.14}$$

The representation (5.13) show that on the event $E_1 \cap \mathcal{G}$, for every $x \in \mathcal{C}^{s_r/2,t_r}$, the restrictions of the geodesics $\pi_{o,x}, \pi^{\rho+}_{o,x}, \pi^{\rho-}_{o,x}$ to $\mathbb{Z}_{>v^c}$ are functions of the bulk weights

$$\{\omega_x\}_{x \in \mathbb{Z}^2_{>v^c}} \tag{5.15}$$

and the boundary weights

$$\{B_e\}_{e \in \mathcal{E}^B_{v^c}}, \{B^{\rho+}_e\}_{e \in \mathcal{E}^B_{v^c}} \text{ and } \{B^{\rho-}_e\}_{e \in \mathcal{E}^B_{v^c}} \tag{5.16}$$

respectively. Define the event

$$E_2 = \{\exists e \in \mathcal{E}^B_{v^c} : B^{\rho+}_e \neq B^{\rho-}_e\}. \tag{5.17}$$

Lemma 5.2. *We have*

$$\{\exists x \in \mathcal{C}^{s_r/2,t_r} : C_p(\pi^{\rho+}_{o,x}, \pi^{\rho-}_{o,x}) \geq \mathcal{I}^{s_r,t_r}, \mathcal{B}\} \subseteq E_1 \cap E_2. \tag{5.18}$$

Proof. The representation (5.13) for the stationary models imply that $G^{\rho-}_{o,x}$ and $G^{\rho+}_{o,x}$ are functions of the weights in (5.15) and the stationary weights in (5.16). This implies that

$$E_1 \cap E_2^c \subseteq \{C_p(\pi^{\rho+}_{o,x}, \pi^{\rho-}_{o,x}) \leq \mathcal{H} \cup \mathcal{V} \leq \mathcal{I}^{s_r,t_r} \quad \forall x \in \mathcal{C}^{s_r/2,t_r}, E_1\}. \tag{5.19}$$

Indeed, on the event E_1 the geodesics $\pi^{\rho-}_{o,x}$ and $\pi^{\rho+}_{o,x}$ must cross $\mathcal{H} \cup \mathcal{V}$. Their exit points $\mathcal{Z}_{v^c}(\pi^{\rho-}_{o,x}), \mathcal{Z}_{v^c}(\pi^{\rho+}_{o,x})$ from the set $\mathcal{H} \cup \mathcal{V}$ is determined by the weights $\{B^{\rho+}_e\}_{e \in \mathcal{E}^B_{v^c}}$ and $\{B^{\rho-}_e\}_{e \in \mathcal{E}^B_{v^c}}$ and the bulk weights that are north-east to the point v^c. As the only difference in the input to determine the exist points is the boundary weights, if these are equal (which happens on E_2^c) then the exit points are equal and therefore the geodesics must have coalesced.

Thus on the event E_1, the coalescing can be northwest of $\mathcal{I}^{s_r,t_r}$ only if the event E_2 occurs. Using (5.8), namely $E_1 = \mathcal{B}$, and (5.19) the result follows. ☐

Thus it remains to find an upper bound for $\mathbb{P}(E_2)$. For $m > 0$, let us define

$$\mathcal{A}^m = \left\{B^{\rho+}_{v^c + i\mathrm{e}_1, v^c + (i+1)\mathrm{e}_1} = B^{\rho-}_{v^c + i\mathrm{e}_1, v^c + (i+1)\mathrm{e}_1}, 0 \leq i \leq m\right\}. \tag{5.20}$$

Recall that $\rho_+ = 1/2 + rN^{-1/3}$. In [7] the following bound is proven.

Lemma 5.3 (Lemma 5.9 of [7])**.** *Let $m \geq 1$. For $0 < \theta < \rho_+$, it holds*

$$\begin{aligned}
\mathbb{P}(\mathcal{A}^m) \geq 1 &- \frac{2rN^{-1/3}}{\frac{1}{2} + rN^{-1/3}} \\
&+ \frac{\frac{1}{2} - rN^{-1/3}}{\frac{1}{2} + rN^{-1/3}} \left[1 + \frac{2r\theta N^{-1/3} + \theta^2}{\frac{1}{4} - (r^2N^{-2/3} + 2rN^{-1/3}\theta + \theta^2)}\right]^m \frac{1}{1 + 2\theta r^{-1}N^{1/3}}.
\end{aligned} \tag{5.21}$$

Corollary 5.4. *There exists $C > 0$, such that for every $r > 0$ and $0 < \eta < 1/4000$,*

$$\mathbb{P}(\mathcal{A}^{\eta r^{-2}N^{2/3}}) \geq 1 - C\eta^{1/2}. \tag{5.22}$$

for all N large enough.

Proof. We set $\theta = \eta^{-1/2}rN^{-1/3}$ and plug this into (5.21). Taking $N \to \infty$ we obtain

$$\lim_{N\to\infty} \mathbb{P}(\mathcal{A}^{\eta r^{-2}N^{2/3}}) \geq 1 - \frac{e^{4+8\sqrt{\eta}}\sqrt{\eta}}{1+2\sqrt{\eta}}. \tag{5.23}$$

Taking for instance $C = 62$, then for all $0 < \eta < 1/4000$, we have $1 \geq C\sqrt{\eta} \geq \frac{e^{4+8\sqrt{\eta}}\sqrt{\eta}}{1+2\sqrt{\eta}}$, which implies the result. □

We are now able to bound the probability of E_2.

Corollary 5.5. *There exists $C > 0$, such that for every $r > 0$ and $0 < s_r < 1/(2000r^2)$*

$$\mathbb{P}(E_2) \leq Cs_r^{1/2}r \tag{5.24}$$

for all N large enough.

Proof. Define

$$\begin{aligned} E_2^H &= \{\exists e \in \mathcal{E}_{v^c}^H : B_e^{\rho+} \neq B_e^{\rho-}\}, \\ E_2^V &= \{\exists e \in \mathcal{E}_{v^c}^V : B_e^{\rho+} \neq B_e^{\rho-}\}, \end{aligned} \tag{5.25}$$

and note that

$$E_2 = E_2^H \cup E_2^V. \tag{5.26}$$

By the symmetry of the problem, it is enough to show

$$\mathbb{P}(E_2^H) \leq Cs_r^{1/2}r. \tag{5.27}$$

Apply Corollary 5.4 with $m = \frac{s_r}{2}N^{2/3}$ i.e. with $\eta = \frac{s_r r^2}{2}$. Then $\mathbb{P}\big((E_2^H)^c\big) = \mathbb{P}(\mathcal{A}^{\eta r^{-2}N^{2/3}})$ and (5.22) gives the claimed result. □

Corollary 5.6. *Consider the parameters satisfying Assumption 4.6 and $s_r r^2 < 1/4000$. Then, there exist constants $c, C > 0$ such that*

$$\mathbb{P}\Big(C_p(\pi_{o,x}^{1/2}, \pi_{y,x}) \leq L_{1-t_r} \quad \forall x \in \mathcal{C}^{s_r/2,t_r}, y \in \mathcal{R}^{r/2,1/4}\Big) \geq 1 - Ce^{-c\tilde{M}^3} - e^{-cr^3} - Cs_r^{1/2}r \tag{5.28}$$

for all N large enough, where $\tilde{M} = \frac{1}{16}s_r t_r^{-2/3} - 4rt_r^{1/3}$

Proof. By Lemma 5.2

$$\mathbb{P}\Big(\exists x \in \mathcal{C}^{s_r/2,t_r} : C_p(\pi_{o,x}^{\rho_+}, \pi_{o,x}^{\rho_-}) \geq \mathcal{I}^{s_r,t_r}, \mathcal{B}\Big) \leq \mathbb{P}(E_2). \tag{5.29}$$

The result follows from Corollary 5.5 and Lemma 5.1. □

Proof of Theorem 2.2. To end the proof of Theorem 2.2, we just need to express the parameters r, s_r, t_r, M in terms of δ so that Assumption 4.6 is in force. For a small $\delta > 0$ consider the scaling

$$\begin{aligned} s_r &= 2\delta, \\ t_r &= \delta^{3/2}/(\log(1/\delta))^3, \\ r &= \tfrac{1}{4}\log(1/\delta). \end{aligned} \tag{5.30}$$

Then $M \tilde{=} \frac{1}{8}(\log(1/\delta))^2 - \sqrt{\delta} \gg r$ as $\delta \to 0$. Let us verify the assumptions. For all $\delta \leq 0.05$, the last inequality in (4.51) holds true and also $s_r < r$. Finally, for $\delta \leq \exp(-4M_0)$, we have $M = \tilde{M} \geq M_0$ as well. For small δ, the largest error term in (5.28) is $Cs_r^{1/2}r$, which however goes to 0 as δ goes to 0. ☐

Remark 5.7. Of course, (5.30) is not the only possible choice of parameters. For instance, one can take $t_r = \delta^{3/2}/\log(1/\delta)$, but then we have to take smaller values of r (and $\tilde{M}$), e.g., $r = \frac{1}{16}\sqrt{\log(1/\delta)}$, for which the last inequality in (4.51) holds for $\delta \leq 0.01$, but the decay of the estimate e^{-cr^3} and e^{-cM^3} are much slower.

5.2 General initial conditions: proof of Theorem 2.5

In this section we prove Theorem 2.5. The strategy of the proof is identical to the one of Theorem 2.2 and thus we will focus only on the differences.

Proof of Theorem 2.5. Here we think of the stationary LPP as leaving from the line $\mathcal{L} = \{x \in \mathbb{Z}^2 | x_1 + x_2 = 0\}$ rather than the point $o = (0,0)$. The first ingredient of the proof is a version of Lemma 4.5 in the new setting. Indeed, the statement of Lemma 4.5 holds with $Z^{\rho\pm}_{o,x^k}$ replaced by $Z^{\rho\pm}_{\mathcal{L},x^k}$ as well[2]. In the proof we use the line-to-point versions of (4.44) and (4.48). Due to the property $\{Z^{\rho+}_{o,x^3} < 0\} = \{Z^{\rho+}_{\mathcal{L},x^3} < 0\}$, we can still apply the bounds from Lemma 4.1.

In the proof of Theorem 2.2 we used comparison between the geodesics from the point-to-point and the ones of two stationary models, with density ρ_+ and ρ_-. From Lemma 4.5 we had a control on the probability that the geodesics starts ordered. We have chosen the events (4.40) such that when it occurs, then the geodesics with ρ_+ and ρ_- are at least at a distance $rN^{1/3}$ from the origin. For the point-to-point case, this was not necessarily needed (it was enough that they would be on the correct side of the origin).

This generalization was though to simplify the work for general initial condition. Indeed, in this case, the exit point of $Z^{h_0}_{\mathcal{L},x}$ is not the origin anymore. But by Assumption 2.4 it is between $-rN^{2/3}$ and $rN^{2/3}$ with $r = \log(\delta^{-1})$ with probability at least $1 - Q(\delta)$. On the events where this holds, the geodesics starts ordered as for the point-to-point case and the proof is identical.

This is the reason why in Theorem 2.5 we have the extra term $Q(\delta)$ in the bound. ☐

[2]Since the geodesics have slope very close to 1, one might expect that $rN^{2/3}$ and $15rN^{2/3}$ should be replaced by their half. However the statement doesn't have to be changed since we did not choose these boundary to be sharp.

6 Upper bound for the probability of no coalescence

In this section we will prove Theorem 2.6, but for this we need some preparations.

6.1 Coupling of stationary models with distinct densities

Let $\mathbf{a} = (a_j)_{j\in\mathbb{Z}}$ and $\mathbf{s} = (s_j)_{j\in\mathbb{Z}}$ be two independent sequences of i.i.d. exponential random variables of intensity β and α respectively, where $0 < \beta < \alpha < 1$. We think of a_j as the inter-arrival time between customer j and customer $j-1$, and of s_j as the service time of customer j. The waiting time of the j'th customer is given by

$$w_j = \sup_{i\le j}\Big(\sum_{k=i}^{j} s_{k-1} - a_k\Big)^+. \tag{6.1}$$

The distribution of w_0 (and by stationarity the distribution of any w_j for $j \in \mathbb{Z}$) is given by

$$\mathbb{P}(w_0 \in dw) := f(dw) = \big(1 - \frac{\beta}{\alpha}\big)\delta_0(dw) + \frac{(\alpha-\beta)\beta}{\alpha} e^{-(\alpha-\beta)w} dw. \tag{6.2}$$

The queueing map $D : \mathbb{R}_+^{\mathbb{Z}} \times \mathbb{R}_+^{\mathbb{Z}} \to \mathbb{R}_+^{\mathbb{Z}}$ takes the sequence of interarrival times and the service times and maps them to the inter-departure times

$$\begin{aligned} \mathbf{d} &= D(\mathbf{a}, \mathbf{s}),\\ d_j &= (w_{j-1} + s_{j-1} - a_j)^- + s_j. \end{aligned} \tag{6.3}$$

We denote by $\nu^{\beta,\alpha}$ the distribution of $(D(\mathbf{a}, \mathbf{s}), \mathbf{s})$ on $\mathbb{R}_+^{\mathbb{Z}} \times \mathbb{R}_+^{\mathbb{Z}}$, that is

$$\nu^{\beta,\alpha} \sim (D(\mathbf{a}, \mathbf{s}), \mathbf{s}). \tag{6.4}$$

By Burke's Theorem [19] $D(\mathbf{a}, \mathbf{s})$ is a sequence of i.i.d. exponential random variables of intensity β, consequently, the measure $\nu^{\beta,\alpha}$ is referred to as a stationary measure of the queue. One can write

$$d_j = e_j + s_j, \tag{6.5}$$

where e_j is called the j'th *idle time* and is given by

$$e_j = (w_{j-1} + s_{j-1} - a_j)^-. \tag{6.6}$$

e_j is the time between the departure of customer $j-1$ and the arrival of customer j in which the sever is idle. Define

$$x_j = s_{j-1} - a_j, \tag{6.7}$$

and the summation operator

$$S_l^k = \sum_{i=k}^{l} x_i. \tag{6.8}$$

Summing e_j we obtain the cumulative idle time (see Chapter 9.2, Eq. 2.7 of [52]).

Lemma 6.1 (Lemma A1 of [7])**.** *For any $k \le l$*

$$\sum_{i=k}^{l} e_i = \Big(\inf_{k\le i\le l} w_{k-1} + S_i^k\Big)^-. \tag{6.9}$$

It has long been known that the LPP on the lattice can be seen as queues in tandem. In particular, the stationary distribution for LPP can be seen as a stationary distribution of queues in tandem. In [26] Fan an Seppäläinen found the multi-species stationary distribution for LPP. For $0 < \beta < \alpha < 1$, let $I^\beta = \{I_i^\beta\}_{i\in\mathbb{Z}}$ and $I^\alpha = \{I_i^\alpha\}_{i\in\mathbb{Z}}$ be two i.i.d. random sequences such that

$$I_1^\beta \sim \mathrm{Exp}(1-\beta) \text{ and } I_1^\alpha \sim \mathrm{Exp}(1-\alpha). \tag{6.10}$$

Let $x \in \mathbb{Z}$ such that $o = (0,0) \leq N\mathrm{e}_4 + x\mathrm{e}_1$. Let G^α, G^β be stationary LPP as in Section 3.2 with the weights in (6.10). Define the sequences $I^{\beta,x}$ and $I^{\alpha,x}$ by

$$\begin{aligned} I_i^{\beta,x} &= G^\beta_{(1-t_r)N\mathrm{e}_4+i\mathrm{e}_1} - G^\beta_{(1-t_r)N\mathrm{e}_4+(i-1)\mathrm{e}_1} \quad \text{for } i > x,\\ I_i^{\alpha,x} &= G^\alpha_{(1-t_r)N\mathrm{e}_4+i\mathrm{e}_1} - G^\alpha_{(1-t_r)N\mathrm{e}_4+(i-1)\mathrm{e}_1} \quad \text{for } i > x. \end{aligned} \tag{6.11}$$

The multi-species results in [26], in particular Theorem 2.3 of [26], show that if we take $(I^\alpha, I^\beta) \sim \nu^{1-\alpha,1-\beta}$ then

$$(I^{\alpha,x}, I^{\beta,x}) \sim \nu^{1-\alpha,1-\beta}|_{\mathbb{R}^{x+\mathbb{Z}_+}}. \tag{6.12}$$

where $\nu^{1-\alpha,1-\beta}|_{\mathbb{R}^{x+\mathbb{Z}_+}}$ is the restriction of $\nu^{\beta,\alpha}$ to $\mathbb{R}^{x+\mathbb{Z}_+}$.

6.2 Control on the stationary geodesics at the $(1-t_r)N$.

As main ingredient in the proof, in Proposition 6.2 below, we show that with positive probability the geodesics ending at $N\mathrm{e}_4$ for the stationary models with density ρ_+ and ρ_- do not coalesce before time $(1-t_r)N$.

Let $r_0 > 0$ to be determined later, $z_0 = -r_0 t_r^{2/3} N^{2/3}$, $z_1 = r_0 t_r^{2/3} N^{2/3}$, and define

$$\mathcal{H}^\rho = \sup\{i \in \mathbb{Z} | (1-t_r)N\mathrm{e}_4 + (i,0) \in \pi^\rho_{o,N\mathrm{e}_4}\} \tag{6.13}$$

be the exit point of the geodesic $\pi^\rho_{o,N\mathrm{e}_4}$ with respect to the horizontal line $(\mathbb{Z}, (1-t_r)N)$, which geometrically is at position $\widetilde{\mathcal{H}}^\rho = (1-t_r)N\mathrm{e}_4 + (\mathcal{H}^\rho, 0)$. In this section we prove the following result.

Proposition 6.2. *Under the choice of parameters in* (5.30) *and for* $r_0 = \delta^{-1}(\log\delta^{-1})^{-1}$, *there exist* $C, \delta_0 > 0$ *such that for* $\delta < \delta_0$ *and* N *large enough*

$$\mathbb{P}(\mathcal{H}^{\rho_-} \in I_-, \mathcal{H}^{\rho_+} > 0) \geq C\delta^{1/2}. \tag{6.14}$$

Define

$$I = I_- \cup I_+ \text{ where } I_- = \{z_0, \ldots, 0\}, I_+ = \{1, \ldots, z_1\}. \tag{6.15}$$

Let $o_2 = Ne_4$, define

$$\hat{I}_i = \widehat{G}_{o_2,(1-t_r)N\mathrm{e}_4-(i+1)\mathrm{e}_1} - \widehat{G}_{o_2,(1-t_r)N\mathrm{e}_4-i\mathrm{e}_1} \quad \text{for } i \in \mathbb{Z} \tag{6.16}$$

and let $(a_j)_{j\in\mathbb{Z}}$ and $(s_j)_{j\in\mathbb{Z}}$ be two independent sequences of i.i.d. random variables, independent of $\hat{I}$, such that

$$a_0 \sim \mathrm{Exp}(1-\rho_+), \quad s_0 \sim \mathrm{Exp}(1-\rho_-). \tag{6.17}$$

For $i \in \mathbb{Z}$ define the shifted random varaibles

$$X_i^1 = \hat{I}_i - 2, \quad X_i^2 = s_i - 2, \quad X_i^3 = a_i - 2. \tag{6.18}$$

Finally define the random walks

$$\begin{aligned} S_i^{a,x} &= \sum_{k=x}^{i} X_k^a, \quad \text{for } a \in \{1,2,3\}, \\ S_i^{a,b,x} &= S_i^{a,x} - S_i^{b,x} \quad \text{for } a,b \in \{1,2,3\}. \end{aligned} \tag{6.19}$$

In particular we have

$$S_i^{2,3,x} = \sum_{k=x}^{i} (s_k - a_k). \tag{6.20}$$

We also define unbiased versions of $S^{a,x}$ for $a \in \{2,3\}$

$$\bar{S}_i^{2,x} = \sum_{k=x}^{i} (s_k - (1-\rho_-)^{-1}), \quad \bar{S}_i^{3,x} = \sum_{k=x}^{i} (a_k - (1-\rho_+)^{-1}). \tag{6.21}$$

A simple computation gives, for $i > x$,

$$S_i^{2,x} \le \bar{S}_i^{2,x} \le S_i^{2,x} + (i-x)\frac{rN^{-1/3}}{\frac{1}{4} + \frac{1}{2}rN^{-1/3}}, \tag{6.22}$$

$$\bar{S}_i^{3,x} \le S_i^{3,x} \le \bar{S}_i^{3,x} + (i-x)\frac{rN^{-1/3}}{\frac{1}{4} - \frac{1}{2}rN^{-1/3}}. \tag{6.23}$$

Next we define events we will refer to in the proof of Proposition 6.2. We will use S^{2,z_0} to represent the increments of the ρ^--geodesic starting from the origin and terminating at the interval I. For $M > 0$ define

$$\mathcal{E}_2 = \Big\{ \sup_{i \in I} |S_i^{2,z_0}| \le M(z_1 - z_0)^{1/2} \Big\}. \tag{6.24}$$

$\mathcal{E}_2$ is the event on which the increments of the $\pi_{o,z}^{\rho_-}$ for $z \in I$ are not too big. We will use $S^{2,1,z_0}$ to represents the change in passage time of the sum of two paths meeting at I as we change the meeting point on I: γ_1 starting from the origin using ρ^- boundary weights and terminating at I, and γ_2 starting from the end point of γ_1 and terminating at Ne_4. Define the set

$$\mathcal{E}_1 = \Big\{ \mathcal{H}^{\rho_-} \in I_-, \sup_{i \in I} |S_i^{2,1,z_0}| \le 2M(z_1 - z_0)^{1/2} \Big\}. \tag{6.25}$$

$\mathcal{E}_1$ is the event on which $S^{2,1,z_0}$ attains its maximum on I_- and it is not too big. For $x > 0$, define the set

$$\mathcal{E}_{3,x} = \Big\{ \inf_{i \in I_-} (x + S_i^{2,3,z_0}) > -2M(z_1 - z_0)^{1/2}, \inf_{i \in I_+} (x + S_i^{2,3,z_0}) < -7M(z_1 - z_0)^{1/2} \Big\}. \tag{6.26}$$

$\mathcal{E}_{3,x}$ is the event that the difference of the increments of the geodesics $\pi_{o,z}^{\rho_+}$ and $\pi_{o,z}^{\rho_-}$ is not too big on I_- and is large on I_+. Note that on the event $\mathcal{E}_2$

$$S_i^{2,1,z_0} > -M(z_1 - z_0)^{1/2} - S_i^{1,z_0} \text{ and } S_i^{2,1,z_0} < M(z_1 - z_0)^{1/2} - S_i^{1,z_0}, \tag{6.27}$$

which implies

$$\mathcal{E}_1 \cap \mathcal{E}_2 \supseteq \{\mathcal{H}^{\rho-} \in I_-, \sup_{i \in I} |S_i^{1,z_0}| \le M(z_1 - z_0)^{1/2}\} \cap \mathcal{E}_2. \tag{6.28}$$

Similarly, on the event $\mathcal{E}_2$

$$S_i^{2,3,z_0} > -M(z_1 - z_0)^{1/2} - S_i^{3,z_0} \text{ and } S_i^{2,3,z_0} < M(z_1 - z_0)^{1/2} - S_i^{3,z_0}, \tag{6.29}$$

so that, for any $x \in [0, (z_1 - z_0)^{1/2}]$ and $M \ge 1$,

$$\begin{aligned}
\mathcal{E}_{3,x} \cap \mathcal{E}_2 &\supseteq \{\inf_{i \in I_-}(x - S_i^{3,z_0}) > -M(z_1 - z_0)^{1/2}, \inf_{i \in I_+}(x - S_i^{3,z_0}) < -8M(z_1 - z_0)^{1/2}\} \cap \mathcal{E}_2 \\
&\supseteq \{\sup_{i \in I_-} S_i^{3,z_0} < M(z_1 - z_0)^{1/2}, \sup_{i \in I_+} S_i^{3,z_0} > 9M(z_1 - z_0)^{1/2}\} \cap \mathcal{E}_2 \\
&\supseteq \{\sup_{i \in I_-} S_i^{3,z_0} < M(z_1 - z_0)^{1/2}, \sup_{i \in I_+} S_i^{3,z_0} - S_0^{3,z_0} > 9M(z_1 - z_0)^{1/2} - S_0^{3,z_0}\} \cap \mathcal{E}_2 \\
&\supseteq \{\sup_{i \in I_-} |S_i^{3,z_0}| < M(z_1 - z_0)^{1/2}, \sup_{i \in I_+} S_i^{3,0} > 10M(z_1 - z_0)^{1/2}\} \cap \mathcal{E}_2 \\
&\supseteq \{\sup_{i \in I_-} |S_i^{3,z_0}| < M(z_1 - z_0)^{1/2}, S_{z_1}^{3,0} > 10M(z_1 - z_0)^{1/2}\} \cap \mathcal{E}_2.
\end{aligned} \tag{6.30}$$

In the first line we used (6.29). For the second inclusion, we used the fact that the first event is larger than the same at $x = 0$ and the second event is larger than the same at $x = (z_1 - z_0)^{1/2}$.

Proof of Proposition 6.2. Recall (6.11) and abbreviate $I_i^{\rho+} = I_i^{\rho+,z_0}$ and $I_i^{\rho-} = I_i^{\rho-,z_0}$ for $z_0 \le i \le z_1$. Note that

$$\Big\{\mathcal{H}^{\rho-} \in I_-, \sup_{i \in I_-} \sum_{k=z_0}^{i} (I_k^{\rho+} - \hat{I}_k) < \sup_{i \in I_+} \sum_{k=z_0}^{i} (I_k^{\rho+} - \hat{I}_k)\Big\} \subseteq \Big\{\mathcal{H}^{\rho-} \in I_-, \mathcal{H}^{\rho+} > 0\Big\}. \tag{6.31}$$

Indeed, if $\mathcal{H}^{\rho-} \in I_-$ then also $\mathcal{H}^{\rho+} \ge z_0$, and the second condition implies that $\mathcal{H}^{\rho+} \notin I_-$, because $\sum_{k=z_0}^{i}(I_k^{\rho+} - \hat{I}_k)$ is the difference between the LPP for paths passing by $(1 - t_r)N\mathrm{e}_4 + (i, 0)$ and the LPP for paths passing by $(1 - t_r)N\mathrm{e}_4 + (z_0, 0)$. Using a decomposition as in (6.5), we can write

$$\begin{aligned}
&\Big\{\mathcal{H}^{\rho-} \in I_-, \sup_{i \in I_-} \sum_{k=z_0}^{i} (I_k^{\rho+} - \hat{I}_k) < \sup_{i \in I_+} \sum_{k=z_0}^{i} (I_k^{\rho+} - \hat{I}_k)\Big\} \\
=&\Big\{\mathcal{H}^{\rho-} \in I_-, \sup_{i \in I_-} \sum_{k=z_0}^{i} e_k + \sum_{k=z_0}^{i} (I_k^{\rho-} - \hat{I}_k) < \sup_{i \in I_+} \sum_{k=z_0}^{i} e_k + \sum_{k=z_0}^{i} (I_k^{\rho-} - \hat{I}_k)\Big\}
\end{aligned} \tag{6.32}$$

where $e_j = \hat{I}_j^{\rho+} - \hat{I}_j^{\rho-}$. By (6.9) the following event has the same probability as (6.32)

$$\begin{aligned}
\mathcal{E}_4 = \Big\{&\mathcal{H}^{\rho-} \in I_-, \\
&\sup_{i \in I_-} \Big(\inf_{z_0 \le l \le i} w_{z_0 - 1} + S_l^{2,3,z_0}\Big)^- + S_i^{2,1,z_0} < \sup_{i \in I_+} \Big(\inf_{z_0 \le l \le i} w_{z_0 - 1} + S_l^{2,3,z_0}\Big)^- + S_i^{2,1,z_0}\Big\}.
\end{aligned} \tag{6.33}$$

It follows that

$$\mathbb{P}(\mathcal{H}^{\rho-} \in I_-, \mathcal{H}^{\rho+} > 0) \ge \mathbb{P}(\mathcal{E}_4). \tag{6.34}$$

For $x > 0$, define

$$\mathcal{E}_{4,x} = \Big\{ \sup_{i \in I_-} S_i^{2,1,z_0} > \sup_{i \in I_+} S_i^{2,1,z_0}, \\ \sup_{i \in I_-} \Big(\inf_{z_0 \le l \le i} x + S_l^{2,3,z_0} \Big)^- + S_i^{2,1,z_0} < \sup_{i \in I_+} \Big(\inf_{z_0 \le l \le i} x + S_l^{2,3,z_0} \Big)^- + S_i^{2,1,z_0} \Big\}. \quad (6.35)$$

Note that

$$\mathcal{E}_2 \cap \mathcal{E}_1 \cap \mathcal{E}_{3,x} \subseteq \mathcal{E}_{4,x}. \quad (6.36)$$

In our case, the value of x in $\mathcal{E}_{4,x}$ is random and distributed according to (6.2), namely

$$f(dw) = \big(1 - \frac{\beta}{\alpha}\big)\delta_0(dw) + \frac{(\alpha-\beta)\beta}{\alpha} e^{-(\alpha-\beta)w} dw. \quad (6.37)$$

Therefore we have

$$\mathbb{P}(\mathcal{E}_4) = \int_0^\infty \mathbb{P}(\mathcal{E}_4 | w_{z_0 - 1} = w) f(dw) = \int_0^\infty \mathbb{P}(\mathcal{E}_{4,w}) f(dw) \\ \ge \int_0^\infty \mathbb{P}(\mathcal{E}_2, \mathcal{E}_1, \mathcal{E}_{3,w}) f(dw) \ge \int_0^{(z_1 - z_0)^{1/2}} \mathbb{P}(\mathcal{E}_2, \mathcal{E}_1, \mathcal{E}_{3,w}) f(dw), \quad (6.38)$$

where in the second equality we used the fact that the processes $\{S_j^{i,z_0}\}_{j \ge z_0, i \in \{1,2,3\}}$ are independent of w_{z_0-1}.

Taking M large enough, Lemma 6.5 below gives

$$\mathbb{P}(\mathcal{E}_2) \ge 1/2 \quad (6.39)$$

for all N large enough. (6.39) and Lemma 6.6 below imply that there exists $c_2 > 0$ such that for any $w \in [0, (z_1 - z_0)^{1/2}]$, $\delta < \delta_0$, and large enough N

$$\mathbb{P}(\mathcal{E}_2, \mathcal{E}_1, \mathcal{E}_{3,w}) \ge \tfrac{1}{2} c_2. \quad (6.40)$$

Plugging (6.40) in (6.38)

$$\mathbb{P}(\mathcal{E}_4) \ge \tfrac{1}{2} c_2 \Big(1 - \frac{\frac{1}{2} - rN^{-1/3}}{\frac{1}{2} + rN^{-1/3}} e^{-2rN^{-1/3}(z_1 - z_0)^{1/2}} \Big). \quad (6.41)$$

Note that

$$2rN^{-1/3}(z_1 - z_0)^{1/2} = 2r(2r_0)^{1/2} t_r^{1/3} = 2^{3/2} r_0^{1/2} \delta^{1/2} \to 0 \quad (6.42)$$

with $r_0 = \delta^{-1}(\log \delta^{-1})^{-1}$. Therefore, there exists $C > 0$ such that for $\delta < \delta_0$ and large enough N

$$\mathbb{P}(\mathcal{E}_4) \ge C\delta^{1/2}. \quad (6.43)$$

Using (6.43) in (6.34) we obtain the result. □

From here to the end of the section we prove results that were used in the proof of Proposition 6.2. The following lemma shows that with probability close to 1/2, $\pi^{\rho-}_{o,Ne_4}$ will cross the interval I from its left half.

Lemma 6.3. *Under* (5.30)*, there exists* $\delta_0, c > 0$ *such that for* $\delta < \delta_0$ *and for large enough* N

$$\mathbb{P}(\mathcal{H}^{\rho-} \in I_-) \ge \frac{1}{2} - e^{-cr_0^3}. \quad (6.44)$$

Proof. Let $v = (1-t_r)Ne_4 - r_0 t_r^{2/3} N^{2/3} e_1$. Note that

$$\{Z^{\rho_-}_{v,Ne_4} \in [0,-z_0]\} = \{\mathcal{H}^{\rho_-} \in I_-\}. \tag{6.45}$$

Moreover

$$\mathbb{P}(Z^{\rho_-}_{v,Ne_4} \in [0,-z_0]) = \mathbb{P}(Z^{\rho_-}_{v,Ne_4} \leq -z_0) - \mathbb{P}(Z^{\rho_-}_{v,Ne_4} < 0). \tag{6.46}$$

The exit point Z^{ρ}_{v,Ne_4} is stochastically monotone in ρ, that is,

$$\mathbb{P}(Z^{\rho}_{v,Ne_4} \leq x) \geq \mathbb{P}(Z^{\lambda}_{v,Ne_4} \leq x) \quad \text{for } \rho \leq \lambda. \tag{6.47}$$

It follows that

$$\begin{aligned}\mathbb{P}(Z^{\rho_-}_{v,Ne_4} \leq -z_0) &\geq \mathbb{P}(Z^{1/2}_{v,Ne_4} \leq -z_0) = \mathbb{P}(Z^{1/2}_{v-z_0,Ne_4} \leq 0)\\ &= \mathbb{P}(Z^{1/2}_{(1-t_r)Ne_4,Ne_4} \leq 0) = 1/2,\end{aligned} \tag{6.48}$$

where the last equality follows from symmetry. Consider the characteristic ρ_- emanating from Ne_4 and its intersection point c^0 with the set $\{(1-t_r)Ne_4 + ie_1\}_{i\in\mathbb{Z}}$. A simple approximation of the characteristic $\frac{(\rho_-)^2}{(1-\rho_-)^2}$ shows that

$$c^0_1 = (1-t_r)N - 8rt_r N^{2/3} + \mathcal{O}(N^{1/3}). \tag{6.49}$$

It follows that

$$\begin{aligned}c^0_1 - v_1 &\geq r_0 t_r^{2/3} N^{2/3} - 8rt_r N^{2/3}(1+o(1))\\ &= \Big(r_0\delta(\log(\delta^{-1}))^{-2} - 2\delta^{3/2}(\log(\delta^{-1}))^{-2}(1+o(1))\Big)N^{2/3}.\end{aligned} \tag{6.50}$$

For δ small enough

$$c^0_1 - v_1 \geq \frac{1}{2} r_0 \delta (\log(\delta^{-1}))^{-2} N^{2/3} \tag{6.51}$$

and

$$\frac{c^0_1 - v_1}{(t_r N)^{2/3}} \geq \frac{r_0}{2}. \tag{6.52}$$

It follows from Lemma 4.1 that

$$\mathbb{P}(Z^{\rho_-}_{v,Ne_4} < 0) \leq e^{-cr_0^3}. \tag{6.53}$$

Plugging (6.48) and (6.53) in (6.46) and using (6.45) implies the result. □

Our next result controls the maximum of S^{1,z_0} on I.

Lemma 6.4. *There exists $c > 0$ such that for any fixed $y > 9r_0^{1/2}$*

$$\mathbb{P}\Big(\sup_{z_0 \leq i \leq z_1} |S^{1,z_0}_i| \leq y(z_1 - z_0)^{1/2}\Big) \geq 1 - 2Ce^{-c(y-9r_0^{1/2})^2} - 2e^{-cr_0^3}. \tag{6.54}$$

for all N large enough.

Proof. Let

$$\lambda_\pm = \frac{1}{2} \pm r_0(t_r N)^{-1/3}. \tag{6.55}$$

We first show that

$$\mathbb{P}(\widehat{Z}^{\lambda_+}_{N\mathrm{e}_4,x} > 0, \widehat{Z}^{\lambda_-}_{N\mathrm{e}_4,x} < 0 \quad \forall x \in I) \geq 1 - 2e^{-cr_0^3}. \tag{6.56}$$

We will show that

$$\mathbb{P}(\widehat{Z}^{\lambda_+}_{N\mathrm{e}_4,x} > 0 \quad \forall x \in I) \geq 1 - e^{-cr_0^3}, \tag{6.57}$$

a similar result can be shown for $\hat{Z}^{\lambda_-}_{N\mathrm{e}_4,x}$ and the result follows by union bound. Note that

$$\mathbb{P}(\widehat{Z}^{\lambda_+}_{N\mathrm{e}_4,x} > 0 \quad \forall x \in I) = \mathbb{P}(\widehat{Z}^{\lambda_+}_{N\mathrm{e}_4,(1-t_r)N\mathrm{e}_4+z_1\mathrm{e}_1} > 0). \tag{6.58}$$

Consider the characteristic λ_+ emanating from Ne_4 and its intersection point d with the set $\{(1-t_r)Ne_4 + ie_1\}_{i\in\mathbb{Z}}$. A simple approximation of the characteristic $\frac{(\lambda_+)^2}{(1-\lambda_+)^2}$ shows that

$$d_1 = (1-t_r)N + 8r_0(t_rN)^{2/3} + \mathcal{O}(N^{1/3}) \geq (1-t_r)N + 2r_0(t_rN)^{2/3} \tag{6.59}$$

for all N large enough. It follows that

$$d_1 - [(1-t_r)N + z_1] \geq r_0(t_rN)^{2/3}. \tag{6.60}$$

As in previous proofs, applying Lemma 4.5 we conclude (6.57) and therefore (6.56).

Define

$$\begin{aligned} \hat{I}^{\lambda_-}_i &= G^{\lambda_-}_{Ne_4,(1-t_r)Ne_4+ie_1} - G^{\lambda_-}_{Ne_4,(1-t_r)Ne_4+(i+1)e_1}, \\ \hat{I}^{\lambda_+}_i &= G^{\lambda_+}_{Ne_4,(1-t_r)Ne_4+ie_1} - G^{\lambda_+}_{Ne_4,(1-t_r)Ne_4+(i+1)e_1}. \end{aligned} \tag{6.61}$$

Using the Comparison Lemma and Lemma 4.1, see Section 3.4, we obtain

$$\mathbb{P}(\hat{I}^{\lambda_-}_i \leq \hat{I}_i \leq \hat{I}^{\lambda_+}_i \quad i \in I) \geq 1 - 2e^{-cr_0^3}. \tag{6.62}$$

Therefore, we also have

$$\mathbb{P}\Big(\sum_{k=z_0}^{i}(\hat{I}^{\lambda_-}_i - 2) \leq S^{1,z_0}_i \leq \sum_{k=z_0}^{i}(\hat{I}^{\lambda_+}_i - 2)\Big) \geq 1 - 2e^{-cr_0^3}. \tag{6.63}$$

Denote $\widehat{S}_i := \sum_{k=z_0}^{i}(\hat{I}^{\lambda_+}_k - (1-\lambda_+)^{-1})$, which is a martingale starting at time $i = z_0$. Then

$$\begin{aligned} &\mathbb{P}\Big(\sup_{z_0\leq i\leq z_1}\sum_{k=z_0}^{i}(\hat{I}^{\lambda_+}_k - 2) \leq yr_0^{1/2}(t_rN)^{1/3}\Big) \\ &\geq \mathbb{P}\Big(\sup_{z_0\leq i\leq z_1}\widehat{S}_i + (z_1-z_0)((1-\lambda_+)^{-1}-2) \leq yr_0^{1/2}(t_rN)^{1/3}\Big) \\ &\geq \mathbb{P}\Big(\sup_{z_0\leq i\leq z_1} r_0^{-1/2}(t_rN)^{-1/3}\widehat{S}_i \leq y - 9r_0^{1/2}\Big), \end{aligned} \tag{6.64}$$

where in the third line we used

$$(z_1-z_0)((1-\lambda_+)^{-1}-2) = 8r_0(t_rN)^{1/3} + \mathcal{O}(1) \leq 9r_0(t_rN)^{1/3} \tag{6.65}$$

for all N large enough. The scaling is chosen such that, setting $i = z_0 + 2\tau r_0(t_rN)^{2/3}$, the scaled random walk $r_0^{-1/2}(t_rN)^{-1/3}\widehat{S}_i$ converges as $N \to \infty$ weakly to a Brownian motion on the interval $\tau \in [0, 1]$ for some (finite) diffusion constant. Thus there exists constants $C, c > 0$ such that for any given $y > 9r_0^{1/2}$

$$\mathbb{P}\Big(\sup_{z_0 \leq i \leq z_1} \sum_{k=z_0}^{i} (\hat{I}_k^{\lambda_+} - 2) \leq y r_0^{1/2}(t_rN)^{1/3}\Big) \geq 1 - Ce^{-c(y-9r_0^{1/2})^2} \tag{6.66}$$

for all N large enough. Similarly we show that for any given $y > 9r_0^{1/2}$

$$\mathbb{P}\Big(\inf_{z_0 \leq i \leq z_1} \sum_{k=z_0}^{i} (\hat{I}_k^{\lambda_-} - 2) \geq -y r_0^{1/2}(t_rN)^{1/3}\Big) \geq 1 - Ce^{-c(y-9r_0^{1/2})^2} \tag{6.67}$$

for all N large enough. From (6.63), (6.66) and (6.67) it follows that

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{z_0 \leq i \leq z_1} |S_i^{1,z_0}| \leq y(z_1 - z_0)^{1/2}\Big) = \mathbb{P}\Big(\sup_{z_0 \leq i \leq z_1} |S_i^{1,z_0}| \leq y r_0^{1/2}(t_rN)^{1/3}\Big) \\
&\geq \mathbb{P}\Big(\inf_{z_0 \leq i \leq z_1} \sum_{k=z_0}^{i} (\hat{I}_k^{\lambda_-} - 2) \geq -y r_0^{1/2}(t_rN)^{1/3}, \sup_{z_0 \leq i \leq z_1} \sum_{k=z_0}^{i} (\hat{I}_k^{\lambda_+} - 2) \leq y r_0^{1/2}(t_rN)^{1/3}\Big) \\
&\geq 1 - 2e^{-c(y-9r_0^{1/2})^2} - 2e^{-cr_0^3}.
\end{aligned} \tag{6.68}$$

□

Next we control the fluctuations of the random walk S^{2,z_0}.

Lemma 6.5. *Let* $\mathcal{C} = \{\sup_{z_0 \leq i \leq z_1} |S_i^{2,z_0}| \leq y(z_1 - z_0)^{1/2}\}$. *Under the choice of parameters in* (5.30)*, there exists* $c, \delta_0 > 0$ *such that for* $\delta < \delta_0$*, and any fixed* $y > r_0^{1/2}$

$$\mathbb{P}(\mathcal{C}) > 1 - Ce^{-c(y-r_0^{1/2})^2} \tag{6.69}$$

for all N *large enough.*

Proof. By (6.22) we have

$$\mathbb{P}(\mathcal{C}) > \mathbb{P}\Big(\sup_{z_0 \leq i \leq z_1} |(z_1 - z_0)^{-1/2}\bar{S}_i^{2,z_0}| \leq y - (z_1 - z_0)^{1/2}\frac{rN^{-1/3}}{\frac{1}{4} + \frac{1}{2}rN^{-1/3}}\Big). \tag{6.70}$$

Note that

$$\begin{aligned}
(z_1 - z_0)^{1/2}\frac{rN^{-1/3}}{\frac{1}{4} + \frac{1}{2}rN^{-1/3}} &= (2r_0)^{1/2}(t_rN)^{1/3}\frac{rN^{-1/3}}{\frac{1}{4} + \frac{1}{2}rN^{-1/3}} = (2r_0)^{1/2}\frac{t_r^{1/3}r}{\frac{1}{4} + \frac{1}{2}rN^{-1/3}} \\
&= (2r_0)^{1/2}\frac{\frac{1}{4}\delta^{1/2}}{\frac{1}{4} + \frac{1}{2}rN^{-1/3}}.
\end{aligned} \tag{6.71}$$

Thus for all N large enough and δ small enough

$$\mathbb{P}(\mathcal{C}) > \mathbb{P}\Big(\sup_{z_0 \leq i \leq z_1} |(z_1 - z_0)^{-1/2}\bar{S}_i^{2,z_0}| \leq y - r_0^{1/2}\Big). \tag{6.72}$$

Also notice that $(z_1 - z_0)^{-1/2}\bar{S}_i^{2,z_0}$ converges weakly to a Brownian motion as $N \to \infty$. Using Doob maximum inequality one deduces that for N large enough (6.69) indeed holds. □

Next we are going to prove that, conditioned on $\mathcal{E}_2$, $\mathcal{E}_1$ and $\mathcal{E}_{3,x}$ occur with positive probability. This is the core argument behind the proof of Proposition 6.2. Given that the increments of $\pi^{\rho_-}_{o,z}$ are not too big across I, which is $\mathcal{E}_2$, then $\mathcal{E}_1$ would imply that the geodesic $\pi^{\rho_-}_{o,Ne_4}$ crosses through I_- and that if it changed its traversal point of I it would not lose much. $\mathcal{E}_3$ would then imply that it is much preferable for $\pi^{\rho_+}_{o,Ne_4}$ to cross outside I_-.

Lemma 6.6. *There exists $c_2, r_0 > 0$ and $M \geq 1$ such that for $x \in [0, (z_1 - z_0)^{1/2}]$*

$$\mathbb{P}(\mathcal{E}_1, \mathcal{E}_{3,x} | \mathcal{E}_2) > c_2. \tag{6.73}$$

for all N large enough.

Proof. Define

$$\begin{aligned} \mathcal{F}_1 &= \{\mathcal{H}^{\rho_-} \in I_-, \sup_{i \in I} |S_i^{1,z_0}| \leq M(z_1 - z_0)^{1/2}\}, \\ \mathcal{F}_3 &= \{\sup_{i \in I_-} |S_i^{3,z_0}| < M(z_1 - z_0)^{1/2}, S_{z_1}^{3,0} > 10M(z_1 - z_0)^{1/2}\}. \end{aligned} \tag{6.74}$$

By the independence of S^{1,z_0} and S^{3,z_0}

$$\mathbb{P}(\mathcal{E}_1, \mathcal{E}_{3,x} | \mathcal{E}_2) \geq \mathbb{P}(\mathcal{F}_1, \mathcal{F}_3 | \mathcal{E}_2) = \mathbb{P}(\mathcal{F}_1)\mathbb{P}(\mathcal{F}_3). \tag{6.75}$$

Next we want to derive lower bounds for $\mathbb{P}(\mathcal{F}_1)$ and $\mathbb{P}(\mathcal{F}_3)$.

Note that

$$\mathbb{P}(\mathcal{F}_1) \geq \mathbb{P}(\mathcal{H}^{\rho_-} \in I_-) - \mathbb{P}(\sup_{i \in I} |S_i^{1,z_0}| \geq M(z_1 - z_0)^{1/2}). \tag{6.76}$$

By Lemma 6.3 and Lemma 6.4 there exists $r_0 > 0$ and $M > 9r_0^{1/2}$ for which

$$\mathbb{P}(\mathcal{H}^{\rho_-} \in I_-) \geq 1/4 \quad \text{and} \quad \mathbb{P}(\sup_{i \in I} |S_i^{1,z_0}| \geq M(z_1 - z_0)^{1/2}) \leq 1/8, \tag{6.77}$$

so that

$$\mathbb{P}(\mathcal{F}_1) \geq 1/8. \tag{6.78}$$

Let us now try to find a lower bound for $\mathbb{P}(\mathcal{F}_3)$.

$$\begin{aligned} \mathbb{P}(\mathcal{F}_3) &= \mathbb{P}\Big(\sup_{i \in I_-} |S_i^{3,z_0}| < M(z_1 - z_0)^{1/2}, S_{z_1}^{3,0} > 10M(z_1 - z_0)^{1/2}\Big) && (6.79) \\ &= \mathbb{P}\Big(\sup_{i \in I_-} |S_i^{3,z_0}| < M(z_1 - z_0)^{1/2}\Big)\mathbb{P}\Big(S_{z_1}^{3,0} > 10M(z_1 - z_0)^{1/2}\Big) && (6.80) \end{aligned}$$

since the processes $\{S_i^{3,z_0}\}_{i \in I_-}$ and $\{S_i^{3,0}\}_{i \in I_+}$ are independent. Note that the centered random walks $\{(z_1 - z_0)^{-1/2}\bar{S}_i^{3,z_0}\}_{i \in I_-}$ and $\{(z_1 - z_0)^{-1/2}\bar{S}_i^{3,0}\}_{i \in I_+}$ converge weakly to a Brownian motion. Furthermore, the difference coming from the non-zero drift of S_i^{3,z_0} is, by (6.23), bounded by

$$\sup_{i \in I} |(z_1 - z_0)^{-1/2} S_i^{3,z_0} - (z_1 - z_0)^{-1/2} \bar{S}_i^{3,z_0}| \leq (z_1 - z_0)^{1/2} \frac{rN^{-1/3}}{\frac{1}{4} - \frac{1}{2}rN^{-1/3}} \tag{6.81}$$

which is $o\Big((z_1-z_0)^{1/2}\Big)$ as $\delta\to 0$(similarly to (6.71)). Thus by choosing M large enough, there exists $c_2>0$ such that for N large enough

$$\begin{aligned}&\mathbb{P}\Big(\sup_{i\in I_-}|S_i^{3,z_0}|<M(z_1-z_0)^{1/2}\Big)\geq 1/2,\\&\mathbb{P}\Big(S_{z_1}^{3,0}>10M(z_1-z_0)^{1/2}\Big)\geq 16c_2.\end{aligned}\tag{6.82}$$

It follows that

$$\mathbb{P}(\mathcal{F}_3)\geq 8c_2 \tag{6.83}$$

Plugging (6.78) and (6.83) in (6.75) we obtain the result. □

6.3 Proof of Theorem 2.6

Finally we prove the second main result of this paper.

Proof of Theorem 2.6. Let

$$\rho'_+=\frac{1}{2}+\frac{1}{120}rN^{-1/3},\quad \rho'_-=\frac{1}{2}-\frac{1}{120}rN^{-1/3}, \tag{6.84}$$

and define

$$\mathcal{A}'=\Big\{-\frac{r}{8}N^{2/3}\leq Z^{\rho'_-}_{o,N\mathrm{e}_4}\leq Z^{\rho'_+}_{o,N\mathrm{e}_4}\leq \frac{r}{8}N^{2/3}\Big\}. \tag{6.85}$$

By Lemma 4.5

$$\mathbb{P}(\mathcal{A}')\geq 1-e^{-cr^3} \tag{6.86}$$

Define

$$y^1=\frac{1}{4}rN^{2/3}\mathrm{e}_1,\quad y^2=\frac{1}{4}rN^{2/3}\mathrm{e}_2. \tag{6.87}$$

Let $o^2=N\mathrm{e}_4$. Similar to (6.13) we define

$$H^j=\sup\{i:(i,(1-t_r)N)\in\pi_{y^j,o^2}\}\quad\text{for } i\in\{1,2\}. \tag{6.88}$$

Note that on the event $\mathcal{A}'$, the geodesics $\pi^{\rho'_-}_{o,o^2},\pi^{\rho'_+}_{o,o^2}$ are sandwiched between the geodesics $\pi_{y^1,o^2},\pi_{y^2,o^2}$, which implies that if the geodesics $\pi^{\rho'_-}_{o,o^2},\pi^{\rho'_+}_{o,o^2}$ did not coalesce then neither did $\pi_{y^1,o^2},\pi_{y^2,o^2}$ i.e.

$$\mathcal{A}'\cap\{\mathcal{H}^{\rho'_-}\in I_-,\mathcal{H}^{\rho'_+}>0\}\subseteq\{C_p(\pi_{y^1,o^2},\pi_{y^2,o^2})>L_{1-t_r}\}. \tag{6.89}$$

Indeed, on the event $\mathcal{A}'$

$$-y_2^2<Z^{\rho_-}_{o,o^2}\leq Z^{\rho_+}_{o,o^2}\leq y_1^1 \tag{6.90}$$

so that

$$\pi_{y_2,o^2}\preceq\pi^{\rho'_-}_{o,o^2}\preceq\pi^{\rho'_+}_{o,o^2}\preceq\pi_{y_1,o^2} \tag{6.91}$$

which implies that under $\mathcal{A}'\cap\{\mathcal{H}^{\rho'_-}\in I_-,\mathcal{H}^{\rho'_+}>0\}$

$$L_{1-t_r}\leq C_p(\pi^{\rho'_+}_{o,o^2},\pi^{\rho'_-}_{o,o^2})\leq C_p(\pi_{y^1,o^2},\pi_{y^2,o^2}) \tag{6.92}$$

Note that as $y^1, y^2 \in \mathcal{R}^{r/2,1/4}$ and $o^2 \in \mathcal{C}^{s_r/2,t_r}$

$$\{C_p(\pi_{y^1,o^2}, \pi_{y^2,o^2}) > L_{1-t_r}\} \subseteq \{\exists x \in \mathcal{C}^{s_r/2,t_r}, y \in \mathcal{R}^{r/2,1/4} : C_p(\pi^{1/2}_{o,x}, \pi_{y,x}) > L_{1-t_r}\}. \tag{6.93}$$

Indeed, if the geodesics π_{y^1,o^2} and π_{y^2,o^2} do not meet before the time horizon L_{1-t_r}, at least one of the them did not coalesce with the geodesic $\pi^{1/2}_{o,o^2}$ before L_{1-t_r}. It follows from (6.89), (6.93) and (6.86) that

$$\mathbb{P}(\exists x \in \mathcal{C}^{s_r/2,t_r}, y \in \mathcal{R}^{r/2,1/4} : C_p(\pi^{1/2}_{o,x}, \pi_{y,x}) > L_{1-t_r}) \geq C\delta^{1/2}. \tag{6.94}$$

□

6.4 Proof of Theorem 2.7

Proof of Theorem 2.7. Note that

$$\begin{aligned}&\{C_p(\pi^{1/2}_{o,x}, \pi_{y,x}) \leq L_{1-\tau} \quad \forall x \in \mathcal{C}^{\delta,\tau}, y \in \mathcal{R}^{\frac{1}{8}\log\delta^{-1},1/4}\}\\ &\subseteq \{C_p(\pi_{w,x}, \pi_{y,x}) \leq L_{1-\tau} \quad \forall x \in \mathcal{C}^{\delta,\tau}, w, y \in \mathcal{R}^{\frac{1}{8}\log\delta^{-1},1/4}\}.\end{aligned} \tag{6.95}$$

Indeed, on the event that any geodesic starting from $\mathcal{R}^{\frac{1}{8}\log\delta^{-1},1/4}$ and terminating in $\mathcal{C}^{\delta,\tau}$ coalesces with the stationary geodesic before the time horizon $L_{1-\tau}$ any two geodesics starting from $\mathcal{R}^{\frac{1}{8}\log\delta^{-1},1/4}$ and terminating in $\mathcal{C}^{\delta,\tau}$ must coalesce as well. Theorem 2.2 and (6.95) imply the lower bound in Theorem 2.7.

Next note that

$$\begin{aligned}&\{\exists x \in \mathcal{C}^{\delta,\tau}, y \in \mathcal{R}^{\frac{1}{8}\log\delta^{-1},1/4} : C_p(\pi^{1/2}_{o,x}, \pi_{y,x}) > L_{1-\tau}, |Z^{1/2}_{o,x}| \leq \frac{1}{16}\log(\delta^{-1})N^{2/3}\}\\ &\subseteq \{C_p(\pi_{w,x}, \pi_{y,x}) \leq L_{1-\tau} \quad \forall x \in \mathcal{C}^{\delta,\tau}, w, y \in \mathcal{R}^{\frac{1}{8}\log\delta^{-1},1/4}\}^c.\end{aligned} \tag{6.96}$$

To illustrate the validity of (6.96), assume w.l.o.g. that $x = Ne_4, y = o$ such that $C_p(\pi^{1/2}_{o,e_4N}, \pi_{o,e_4N}) > L_{1-\tau}$ and that $\frac{1}{16}\log(\delta^{-1})N^{2/3} \geq Z^{1/2}_{o,Ne_4} = a > 0$. It follows that

$$C_p(\pi_{ae_1,Ne_4}, \pi_{o,Ne_4}) > L_{1-\tau} \tag{6.97}$$

holds. The event in (6.97) is contained in the event in the last line of (6.96) which implies (6.96).

The inclusion (6.96) implies that

$$\begin{aligned}&\mathbb{P}\Big(C_p(\pi^{1/2}_{o,x}, \pi_{y,x}) \leq L_{1-\tau} \quad \forall x \in \mathcal{C}^{\delta,\tau}, y \in \mathcal{R}^{\frac{1}{8}\log\delta^{-1},1/4}\Big)\\ &+ \mathbb{P}\Big(|Z^{1/2}_{o,x}| > \frac{1}{16}\log(\delta^{-1})N^{2/3} \quad \text{for some } x \in \mathcal{C}^{\delta,\tau}\Big)\\ &\geq \mathbb{P}\Big(C_p(\pi_{w,x}, \pi_{y,x}) \leq L_{1-\tau} \quad \forall x \in \mathcal{C}^{\delta,\tau}, w, y \in \mathcal{R}^{\frac{1}{8}\log\delta^{-1},1/4}\Big).\end{aligned} \tag{6.98}$$

Next we claim that for some $c > 0$

$$\mathbb{P}\Big(|Z^{1/2}_{o,x}| > \frac{1}{16}\log(\delta^{-1})N^{2/3} \quad \text{for some } x \in \mathcal{C}^{\delta,\tau}\Big) \leq e^{-c\log(\delta^{-1})^3}. \tag{6.99}$$

Indeed, it follows by (4.71) with $r = \frac{1}{15}\frac{1}{16}\log(\delta^{-1})$ that

$$\begin{aligned}&\mathbb{P}\Big(Z^{1/2}_{o,x} > \frac{1}{16}\log(\delta^{-1})N^{2/3} \quad \text{for some } x \in \mathcal{C}^{\delta,\tau}\Big)\\ &\leq \mathbb{P}\Big(Z^{\rho_+}_{o,x} > \frac{1}{16}\log(\delta^{-1})N^{2/3} \quad \text{for some } x \in \mathcal{C}^{\delta,\tau}\Big) \leq e^{-c\log(\delta^{-1})^3}.\end{aligned} \tag{6.100}$$

A similar bound can be obtained for the lower tail to obtain (6.99). Using the upper bound in Theorem 2.6 and (6.99) in (6.98) we obtain the upper bound in Theorem 2.7. □

A An estimate

The proof to the following can be found in Example (b) in Section XII.5 of Feller [27] (see also Example 2.6 of [20]).

Lemma A.1. *Let* $0 < \beta < \alpha < 1$. *Let*

$$S_n = \sum_{i=1}^{n} X_i \tag{A.1}$$

where $\{X_i\}_{i\geq 1}$ *are i.i.d. with law*

$$X_i \sim \mathrm{Exp}(\alpha) - \mathrm{Exp}(\beta). \tag{A.2}$$

Then

$$\mathbb{P}\Big(\sup_{i\geq 1} S_i > \lambda\Big) \leq \frac{\beta}{\alpha} e^{-(\alpha-\beta)\lambda} \quad \textit{for all } \lambda > 0. \tag{A.3}$$

B Localization for random initial conditions

In this appendix we give a sketch of the proof of Proposition 2.10, which is essentially a special case of one situation of Lemma 5.2 of [33].

Denote by $I(M) = N\mathrm{e}_4 + M(2N)^{2/3}\mathrm{e}_3$ and $J(v) = v(2N)^{2/3}\mathrm{e}_3$, then we have $I(M) = x^1$ for $M = 2^{-2/3}\frac{3}{4}\delta$. Define the scaled random variables

$$L_N(v) = \frac{L_{J(v)\to x^1} - 4N}{2^{4/3}N^{1/3}}, \quad \text{and} \quad \mathcal{W}_N(v) = \frac{h_0(J(v))}{2^{4/3}N^{1/3}}. \tag{B.1}$$

Let us set $\alpha = 2^{-2/3}\log(\delta^{-1})$, so that $L_N(\alpha) \simeq -(\alpha - M)^2 \simeq -\alpha^2$ for $\delta \ll 1$. We have

$$\begin{aligned}&\mathbb{P}\left(Z^{h_0}_{\mathcal{L},x^1} > \log(\delta^{-1})N^{2/3}\right)\\ \leq&\mathbb{P}\left(\max_{v\leq\alpha}(L_N(v) + \mathcal{W}_N(v)) \leq -\tfrac{1}{4}\alpha^2\right) + \mathbb{P}\left(\max_{v>\alpha}(L_N(v) + \mathcal{W}_N(v)) > -\tfrac{1}{4}\alpha^2\right).\end{aligned} \tag{B.2}$$

Since $M < \alpha$ for $\delta \ll 1$, the first term of (B.2) is bounded by

$$\mathbb{P}(L_N(M) + \mathcal{W}_N(M) \leq -\tfrac{1}{4}\alpha^2) \leq \mathbb{P}(L_N(M) \leq -\tfrac{1}{8}\alpha^2) + \mathbb{P}(\mathcal{W}_N(M) \leq -\tfrac{1}{8}\alpha^2). \tag{B.3}$$

$L_N(M)$ has asymptotically GUE Tracy-Widom distributions with $\mathbb{P}(L_N(M) \leq -s) \leq C_1 e^{-c_1 s^{3/2}}$ for $s \gg 1$ (see e.g. Proposition 3 and (56) of [3]). Thus the first term of (B.3) is bounded by $C_1 e^{-\tilde{c}_1\alpha^3}$. $\mathcal{W}_N(v)$ is a rescaled sum of iid. random variables converging to a Brownian motion. Using the exponential Chebyshev's inequality one easily obtains that the second term of (B.3) is bounded by $C_2 e^{-c_2\alpha^4/M} \ll C_2 e^{-c_2\alpha^3}$.

The second term of (B.2), is bounded by

$$\mathbb{P}\left(\max_{v>\alpha}(L_N(v) + \tfrac{1}{2}(v-M)^2) \geq -\tfrac{1}{8}\alpha^2\right) + \mathbb{P}\left(\max_{v>\alpha}(\mathcal{W}_N(v) - \tfrac{1}{2}(v-M)^2) > -\tfrac{1}{8}\alpha^2\right). \tag{B.4}$$

The bound can be obtained through the decay of the kernel for point-to-half line (related with TASEP with half-flat initial condition), see Theorem 2.6 and Lemma 2.7 of [22], which leads to the bound (4.19) of [33]. In our setting, this gives that the first term in (B.4) is bounded by $C_3e^{-c_3\alpha^2}$ (the actual decay would be with α^3 instead of α^2, but one would need to do a more careful bound on the kernel). It is also possible to avoid the kernel estimates, as it was made in the proof of Lemma 4.3 of [32], but it is more complicated. Finally, the second term of (B.4) is bounded using standard Doob's maximal inequality, which leads to $C_4e^{-c_4\alpha^3}$ as well.

Putting all the estimates together, since $\alpha \simeq \log(\delta^{-1})$, our claim is proven.

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