

Fluctuations of the competition interface in presence of shocks

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Abstract

We consider last passage percolation (LPP) models with exponentially distributed random variables, which are linked to the totally asymmetric simple exclusion process (TASEP). The competition interface for LPP was introduced and studied by Ferrari and Pimentel in [23] for cases where the corresponding exclusion process had a rarefaction fan. Here we consider situations with a shock and determine the law of the fluctuations of the competition interface around its deterministic law of large number position. We also study the multipoint distribution of the LPP around the shock, extending our one-point result of [26].

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1 Introduction

The last passage percolation (LPP) model in two dimensions is a zero temperature limit of directed polymers and it is well known that it can be mapped to the totally asymmetric simple exclusion process (TASEP) when the random entries in the LPP model are taken to be (independent) exponentially (or geometrically) distributed. These models belong to the Kardar-Parisi-Zhang (KPZ) universality class of stochastic growth models [35] in $1+1$ dimensions (see surveys and lecture notes [8, 13, 25, 28, 39, 41]). In the context of growing interface the most studied observable is the height function of the surface. In terms of LPP, the height function at time t is given by the last passage time along the line with slope -1 at distance t from the origin. Generally, as time t goes to infinity, the correlation length scales like $t^{2/3}$, the height fluctuations like $t^{1/3}$. The limiting processes are also known [2, 6, 33, 38]. However, when there is a shock in TASEP, that means a non-differentiability in the limit shape of the height function, the situation is different. In that case the distribution function of the height function changes over distances of order $\mathcal{O}(t^{1/3})$ only. The reason of this phenomena is that (macroscopically) shocks are space-time lines where two characteristic lines come together and space-time regions which are correlated are connected through a characteristic line. This effectively implies that the LPP time around the shock is obtained by considering the maximum of two reduced LPP problems, which turn out to be asymptotically independent (this is due to the slow-decorrelation occurring along the characteristic lines [15, 24]). Due to the different first derivatives of the two law of large numbers the shock region has a width of order $t^{1/3}$ only. In our previous work [26] we determined the one-point distribution function of the rescaled height function in a $t^{1/3}$ neighborhood of the shock for some non-random initial conditions. Controlling local fluctuations in LPP with the results of Cator and Pimentel [10], we can extend this result in Theorem 3.1 to joint distributions around the shock.

The main contribution of this paper is however the study of another observable, namely of the competition interface, which is tightly related with the second class particle since its trajectory is the competition interface [23]. The competition interface was introduced by Ferrari and Pimentel [23]. Consider an LPP from a region $\mathcal{L} = \mathcal{L}^+ \cup \mathcal{L}^-$ with \mathcal{L}^\pm disjoint. Then the competition interface is the boundary between the region where the last passage time to \mathcal{L}^+ is larger than the one to \mathcal{L}^- . For example, one can take $\mathcal{L}^+ = (0, \mathbb{N})$ and $\mathcal{L}^- = (\mathbb{N}, 0)$, so that the competition interface starts from the origin. This case is studied in [23] and it is a special case of the model considered by Ferrari, Martin and Pimentel in [22]. In this case in terms of TASEP the system starts with step-initial conditions and develops a rarefaction fan,

while in terms of the height function the limit shape is differentiable. Then, the direction of the competition interface is randomly distributed and the second class particle has a speed which is uniformly distributed in $[-1, 1]$. In presence of a shock, the competition interface satisfies a (deterministic) law of large numbers [21,22], thus it is of interest to consider its fluctuations. It is well-known that for TASEP with product measure, density ρ on \mathbb{Z}_- and λ on \mathbb{N} ($\rho < \lambda$), the second class particle has Gaussian fluctuations in the $t^{1/2}$ scale around the macroscopic shock position [19,20,30]. Since the position of the competition interface is a random time change of the position of the second class particle [23] (see also Section 2.1.2), it is not straightforward to connect the two quantities. In Theorem 2.10 we give a direct derivation of the Gaussian fluctuation of the competition interface in this situation. The analogue result for the time-changed process (essentially for the second-class particle) is Theorem 4 of [22] (see also [11,12] for related studies for the Hammersley process / PNG growth model, with external sources). The Gaussian fluctuation reflects the randomness of the initial condition and does not depend on the fact that the growth model is in the KPZ universality class. Therefore it is of interest to consider situation without randomness in the initial conditions, where the effect of the KPZ fluctuations are not dominated by the ones of the initial condition. As an example, in Theorem 2.1 we determine the limiting distribution function of the position of the competition interface in the situation where the height function has a Airy_1 to Airy_1 shock transition, i.e., both to the left and to the right of the shock one has an Airy_1 process. Theorem 2.1 is a special case of Theorem 2.3 which applies to more general LPP problems.

The results obtained in this paper are based on the asymptotic independence of two LPP problems. For TASEP with stationary initial conditions [36], i.e., Bernoulli product measure with $\rho = \lambda$, this can not be applied anymore. This case has been however studied fairly well. Firstly, one can identify the competition interface with the maximizer of the (backwards) LPP problem [4]. Secondly, the second class particle starting from the origin has some explicitly known scaling function [3,27,37]. Recent progresses on the knowledge of the time-time correlation in the KPZ height function [17,18,29,34] shows that it would be relevant to study the competition interface and the second class particle as a process in time as well.

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2 Last Passage Percolation and Competition Interfaces

2.1 Models and main Result

We consider last passage percolation times on \mathbb{Z}^2 . Fix a starting set $S_A \subset \mathbb{Z}^2$ and an endset $S_E \subset \mathbb{Z}^2$. An up-right path from S_A to S_E is a sequence of points $\pi = (\pi(0), \pi(1), \dots, \pi(n)) \in \mathbb{Z}^{2n}$ such that $\pi(0) \in S_A, \pi(n) \in S_E$ and $\pi(i) - \pi(i-1) \in \{(0, 1), (1, 0)\}$. We denote by $\ell(\pi) = n$ the length of π . Take a family $\{\omega_{i,j}\}_{i,j \in \mathbb{Z}}$ of independent, nonnegative random variables, and define the last passage percolation (LPP) time from S_A to S_E to be

$$L_{S_A \rightarrow S_E} := \max_{\pi: S_A \rightarrow S_E} \sum_{0 \leq k \leq \ell(\pi)} \omega_{\pi(k)} \quad (2.1)$$

and define, say, $L_{S_A \rightarrow S_E}$ to be $-\infty$ when the maximum in (2.1) does not exist. We denote by π^{\max} any path for which the maximum in (2.1) is attained and call π^{\max} a maximizing path. In the LPP models we consider in this paper, π^{\max} is always a.s. unique. Last passage percolation is linked to the totally asymmetric simple exclusion process (TASEP) in the following way. In TASEP, each $i \in \mathbb{Z}$ is occupied by at most one particle, and each particle waits an exponential time (whose parameter may depend on the particle) before it jumps one step to the right iff its right neighbor is not occupied. Labeling particles from right to left

$$\dots < x_2(0) < x_1(0) < 0 \leq x_0(0) < x_{-1}(0) < \dots$$

we denote by $x_n(t)$ the position of particle n at time t and have that at all time $t \geq 0$, $x_{n+1}(t) < x_n(t)$, $n \in \mathbb{Z}$. TASEP can be translated into a LPP model as follows. Define

$$\mathcal{L} = \{(k + x_k(0), k) : k \in \mathbb{Z}\} \quad (2.2)$$

and weights

$$\omega_{i,j} := \begin{cases} 0 & \text{if } (i, j) \in \mathcal{L}, \\ \exp(v_j) & \text{else,} \end{cases} \quad (2.3)$$

where v_j is the parameter of the exponential clock attached to particle j . Then we have

$$\mathbb{P}\left(\bigcap_{k=1}^{\ell} \{x_{n_k}(t) \geq m_k - n_k\}\right) = \mathbb{P}\left(\bigcap_{k=1}^{\ell} \{L_{\mathcal{L} \rightarrow (m_k, n_k)} \leq t\}\right). \quad (2.4)$$

It will be important for us to consider separately the sets

$$\mathcal{L}^+ = \{(k + x_k(0), k) : k > 0\} \quad \text{and} \quad \mathcal{L}^- = \{(k + x_k(0), k) : k \leq 0\}. \quad (2.5)$$

We denote by $\pi_+^{\max}, \pi_-^{\max}$ the (a.s. unique) maximizing paths of $L_{\mathcal{L}^+ \rightarrow (m,n)}, L_{\mathcal{L}^- \rightarrow (m,n)}$. The aim of this section is to obtain limit results for the competition interface in the LPP model (2.2), (2.3). The competition interface is essentially the boundary of the region where the LPP to \mathcal{L}^+ is larger than the LPP to \mathcal{L}^- .

2.1.1 The competition interface

Here we study the LPP model defined by (2.2), (2.3) with a competition interface, which was introduced in [23]. To associate a competition interface to the LPP time, we consider TASEP with initial data satisfying $x_0(0) = 1$ and $x_1(0) < -1$. Note that for such initial data we have (with $\mathcal{L}^+, \mathcal{L}^-$ defined in (2.5)) almost surely $L_{\mathcal{L}^+ \rightarrow (i,j)} \neq L_{\mathcal{L}^- \rightarrow (i,j)}$ for $(i, j) \in \mathbb{N}^2$ ($\mathbb{N} = 1, 2, 3, \dots$). We then define two clusters via

$$\begin{aligned} \Gamma_+^\infty &:= \{(i, j) \in \mathbb{Z}^2 : L_{\mathcal{L}^+ \rightarrow (i,j)} > L_{\mathcal{L}^- \rightarrow (i,j)}\}, \\ \Gamma_-^\infty &:= \{(i, j) \in \mathbb{Z}^2 : L_{\mathcal{L}^- \rightarrow (i,j)} > L_{\mathcal{L}^+ \rightarrow (i,j)}\}. \end{aligned} \quad (2.6)$$

We can think of the points in Γ_+^∞ as painted red, and the points in Γ_-^∞ as painted blue. Each $(i, j) \in \mathbb{N}^2$ thus has a well-defined color. The two colors are separated through the competition interface ϕ : Set $\phi_0 := (0, 0)$ and define $\phi = (\phi_0, \phi_1, \phi_2, \dots)$ inductively via

$$\phi_{n+1} = \begin{cases} \phi_n + (1, 0) & \text{if } \phi_n + (1, 1) \in \Gamma_+^\infty, \\ \phi_n + (0, 1) & \text{if } \phi_n + (1, 1) \in \Gamma_-^\infty, \end{cases} \quad (2.7)$$

and we write $\phi_n = (I_n, J_n)$. We always have $\phi_1 = (1, 0)$ in our model since $L_{\mathcal{L}^- \rightarrow (1,1)} = \omega_{1,1} < \omega_{1,1} + \omega_{0,1} \leq L_{\mathcal{L}^+ \rightarrow (1,1)}$. Note that $I_n + J_n = n$ and $(k, n - k)$ is red for $0 \leq k < I_n$ and blue for $I_n < k \leq n$.

We can prove limit results for $I_n - J_n$ for LPP models that fall into the framework of asymptotic independence (see Section 2.2), and now give the result for one such model, see Figure 1.

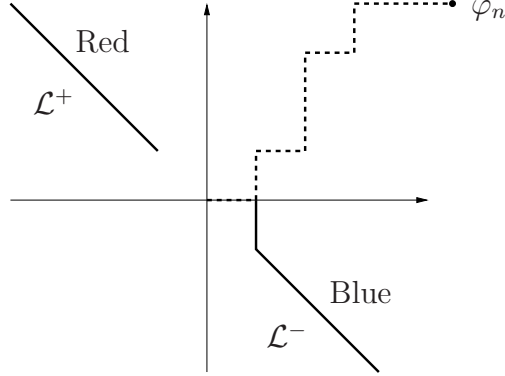


Figure 1: The LPP model of Theorem 2.1 and the associated competition interface (dashed line).

Theorem 2.1. *Let $(\phi_n)_{n \in \mathbb{N}} = ((I_n, J_n))_{n \in \mathbb{N}}$ be the competition interface (2.7) associated to the LPP model with $\mathcal{L} = \mathcal{L}^+ \cup \mathcal{L}^-$ where $\mathcal{L}^+, \mathcal{L}^-$ are given by (2.5) and*

$$x_k(0) = -2k, k \in \mathbb{Z} \setminus \{0\} \quad \text{and} \quad x_0(0) = 1. \quad (2.8)$$

The weights are

$$\omega_{i,j} := \begin{cases} 0 & \text{if } (i, j) \in \mathcal{L}, \\ \exp(1) & \text{if } (i, j) \in \mathcal{L}^c, j > 0, \\ \exp(\alpha) & \text{if } (i, j) \in \mathcal{L}^c, j \leq 0, \end{cases} \quad (2.9)$$

where $\alpha < 1$. Then

$$\begin{aligned} & \lim_{t \rightarrow \infty} \mathbb{P} \left(\frac{I_{[t]} - J_{[t]} - (\alpha - 1)t}{t^{1/3}} \leq s \right) \\ &= \int_0^\infty dx \int_{-\infty}^\infty dy F'_{\text{GOE}} \left(\frac{-y - \frac{2^{4/3}}{(2-\alpha)^{4/3}} s}{\sigma_1} \right) F'_{\text{GOE}} \left(\frac{x - y - \frac{2^{4/3}}{\alpha(2-\alpha)^{4/3}} s}{\sigma_2} \right) \frac{1}{\sigma_1 \sigma_2}, \end{aligned}$$

where F'_{GOE} is the density of the GOE Tracy-Widom distribution of random matrices [42] and $\sigma_1 = \frac{2^{2/3}}{(2-\alpha)^{1/3}}$ and $\sigma_2 = \frac{2^{2/3}(2-2\alpha+\alpha^2)^{1/3}}{\alpha^{2/3}(2-\alpha)}$.

The limit in Theorem 2.1 is the probability that the difference of two independent, GOE-distributed random variables χ_1, χ_2 is positive. In short, to prove Theorem 2.1, one first observes that the event $\{I_{[t]} - J_{[t]} \leq -t + 2M\}$, for $0 \leq M \leq t$ amounts to the event that $(M, t-M)$ is blue, i.e., $L_{\mathcal{L}^- \rightarrow (M, t-M)} - L_{\mathcal{L}^+ \rightarrow (M, t-M)} > 0$. The random variables

$L_{\mathcal{L}^+ \rightarrow (M, t-M)}, L_{\mathcal{L}^- \rightarrow (M, t-M)}$ given by (2.5), (2.8), (2.9) are asymptotically independent and converge (when rescaled) to χ_1, χ_2 , leading to Theorem 2.1. Hence, more generally, as soon as we can prove asymptotic independence, we can prove an analogue of Theorem 2.1. This generalization is given in Theorem 2.3.

Remark 2.2. This LPP corresponds to TASEP where, due to the slow rate $\alpha < 1$, a shock is generated. The competition interface moves with speed $\alpha - 1$ and has fluctuation of order $\mathcal{O}(t^{1/3})$. The situation considered in [23] corresponds to TASEP with so-called step-initial condition which generates a rarefaction fan (region of decreasing particle density). In this case the speed of the competition interface is randomly distributed.

2.1.2 Competition Interfaces and second class particles

One of the motivations to study the competition interface (2.7) is its direct connection to second class particles established in [23] and [22]. To define the second class particle one uses the basic coupling to start TASEP in $\eta, \eta' \in \{0, 1\}^{\mathbb{Z}}$ where η, η' differ only at the origin. Then they differ at one site for all time denoted by

$$X(t) = \sum_{x \in \mathbb{Z}} x \mathbb{1}_{\{\eta_t(x) \neq \eta'_t(x)\}}. \quad (2.10)$$

$X(t)$ is the position at time t of the second class particle which started at the origin. Consider now TASEP with Riemann initial data, i.e., for which the initial particle density ρ_0 has the form

$$\rho_0(\xi) = \begin{cases} \rho_- & \text{if } \xi < 0, \\ \rho_+ & \text{if } \xi \geq 0. \end{cases} \quad (2.11)$$

If now $\rho_- < \rho_+$, the solution of the Burgers equation with initial data (2.11) which gives the $t \rightarrow \infty$ density profile of TASEP is given by

$$\rho(\xi, 1) = \begin{cases} \rho_- & \text{if } \xi < 1 - \rho_- - \rho_+, \\ \rho_+ & \text{if } \xi \geq 1 - \rho_- - \rho_+. \end{cases} \quad (2.12)$$

The discontinuity in (2.12) is called shock and $X(t)/t$ converges a.s. to $1 - \rho_- - \rho_+$. Thus $X(t)$ provides an interpretation of the random shock position; for Riemann initial data with $\rho_- > \rho_+$ created by Bernoulli initial data, $X(t)/t$ is asymptotically uniformly distributed among the characteristics emanating from the origin, see e.g. Theorem 3 in [22] for both results.

In the construction given in Proposition 2.2 in [22], an initial configuration with a second class particle at the origin becomes an initial configuration with a hole at 0 and a particle at 1 (note that we only defined the competition interface for such initial data). Associating to such initial data the LPP model (2.4) and the competition interface (2.7), $I_n - J_n - 1$ becomes then the position of the second class particle after its $(n - 1)$ th jump ($n \geq 1$). More precisely, with $\tau_n = L_{\mathcal{L} \rightarrow \phi_n}$ we define for $t \geq 0$

$$(I(t), J(t)) = \phi_n \quad \text{if } t \in [\tau_n, \tau_{n+1}). \quad (2.13)$$

Then, in the coupling of [22] $I(t) - 1$ (resp. $J(t)$) equals the number of rightward (resp. leftward) jumps of the second class particle in $[0, t]$, note $\tau_1 = 0$ in our setting¹.

2.2 General Model and Theorem

In this section we consider the general LPP model defined by (2.3), (2.2) (2.5). We prove a generalization of Theorem 2.1 about the convergence of $I_n - J_n$ under three assumptions, see Theorem 2.3 below. As noted after Theorem 2.1, one looks at the probability that $L_{\mathcal{L}^- \rightarrow (M, t-M)} - L_{\mathcal{L}^+ \rightarrow (M, t-M)}$ is positive for suitable $M = M(t)$. Hence, $L_{\mathcal{L}^- \rightarrow (M, t-M)} - L_{\mathcal{L}^+ \rightarrow (M, t-M)}$ need to converge under the same rescaling (otherwise the probability of their difference being positive converges to 0 or 1). This is the content of Assumption 1. Furthermore, one needs to establish asymptotic independence of the two LPP times. Independence is equivalent to non-crossing of the two maximizing paths $\pi_+^{\max}, \pi_-^{\max}$ of $L_{\mathcal{L}^- \rightarrow (M, t-M)}, L_{\mathcal{L}^+ \rightarrow (M, t-M)}$. However, close to the endpoint $\pi_+^{\max}, \pi_-^{\max}$ will typically intersect. If, however, the fluctuations are not affected (on the leading order) by the randomness in a region of radius t^ν , $\nu < 1$ away from $(M, t - M)$ (Assumption 2), then the maximizers until that distance do are supported with high probability on disjoint set of points (Assumption 3). The framework of these Assumptions was used in [26] to prove the limit law of LPP times. Here we prove convergence of $I_n - J_n$ and obtain Theorem 2.1 as a corollary.

In the following, we do not always write the integer parts, since even any perturbation of order $o(t^{1/3})$ becomes irrelevant in the limit, see (2.31) below. For convenience, we assume that all appearing distribution functions are continuous, otherwise statements will only hold at continuity points.

¹In [22], the choice of \mathcal{L} and $\omega_{i,j}$ differs slightly from the one in (2.4), such that $(I(t) - J(t))_{t \geq 0}$ equals $(X(t))_{t \geq 0}$ in [22].

Assumption 1. Assume that there exists some μ such that

$$\lim_{t \rightarrow \infty} \mathbb{P} \left(\frac{L_{\mathcal{L}^+ \rightarrow (\eta t, t)} - \mu t}{t^{1/3}} \leq s \right) = G_1(s), \quad (2.14)$$

and

$$\lim_{t \rightarrow \infty} \mathbb{P} \left(\frac{L_{\mathcal{L}^- \rightarrow (\eta t, t)} - \mu t}{t^{1/3}} \leq s \right) = G_2(s), \quad (2.15)$$

where G_1 and G_2 are some (continuous) distribution functions.

Secondly, we assume that there is a point E^+ at distance of order t^ν , for some $1/3 < \nu < 1$, which lies on the characteristic from \mathcal{L}^+ to E and that there is slow-decorrelation as in Theorem 2.1 of [15].

Assumption 2. Assume that we have a point $E^+ = (\eta t - \kappa t^\nu, t - t^\nu)$ such that for some μ_0 , and $\nu \in (1/3, 1)$ it holds

$$\lim_{t \rightarrow \infty} \mathbb{P} \left(\frac{L_{E^+ \rightarrow (\eta t, t)} - \mu_0 t^\nu}{t^{\nu/3}} \leq s \right) = G_0(s), \quad (2.16)$$

and

$$\lim_{t \rightarrow \infty} \mathbb{P} \left(\frac{L_{\mathcal{L}^+ \rightarrow E^+} - \mu t + \mu_0 t^\nu}{t^{1/3}} \leq s \right) = G_1(s), \quad (2.17)$$

where G_0 and G_1 are (continuous) distribution functions.

Assumption 3. Let ν be as in Assumption 2. Consider the points $D_\gamma = ([\gamma \eta t], [\gamma t])$ with $\gamma \in [0, 1 - t^{\beta-1}]$ and $\beta \in (0, \nu)$. Assume that

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathbb{P} \left(\bigcup_{\substack{D_\gamma \\ \gamma \in [0, 1 - t^{\beta-1}]} \left\{ D_\gamma \in \pi_{L_{\mathcal{L}^+ \rightarrow E^+}}^{\max} \right\} \right) &= 0, \\ \lim_{t \rightarrow \infty} \mathbb{P} \left(\bigcup_{\substack{D_\gamma \\ \gamma \in [0, 1 - t^{\beta-1}]} \left\{ D_\gamma \in \pi_{L_{\mathcal{L}^- \rightarrow (\eta t, t)}}^{\max} \right\} \right) &= 0. \end{aligned} \quad (2.18)$$

Then, under the proceeding Assumptions, we have the following Theorem.

Theorem 2.3. Assume Assumptions 1,2,3 hold with

$$\eta = \eta_0 + u t^{-2/3}, \quad u \in \mathbb{R}. \quad (2.19)$$

Then, for any sequence $a_t = o(t^{1/3})$ we have

$$\lim_{t \rightarrow \infty} \mathbb{P} \left(I_{[t]} - J_{[t]} \leq -t \frac{1 - \eta_0}{1 + \eta_0} + \frac{2u}{(1 + \eta_0)^{4/3}} t^{1/3} + a_t \right) = \mathbb{P}_{G_2 \star G_{1,-}}((0, \infty)), \quad (2.20)$$

where $\mathbb{P}_{G_2 \star G_{1,-}}$ is the convolution of the probability measures induced by the distribution functions $G_2(s)$ and $1 - G_1(-s)$.

Proof of Theorem 2.1. In the proof of Corollary 2 of [26], it has been shown that the Assumptions 1,2,3 hold ($x_0(0) = 1$ rather than $x_0(0) = 0$ clearly do not affect the asymptotic behavior. This is easily seen by comparison and basic coupling, see remark at page 7 of [7]) with $\eta_0 = \frac{\alpha}{2-\alpha}$ and

$$G_1(\tau) = F_{\text{GOE}}\left(\frac{\tau + 2u}{\sigma_1}\right), \quad G_2(\tau) = F_{\text{GOE}}\left(\frac{\tau + 2u/\alpha}{\sigma_2}\right), \quad (2.21)$$

where σ_1, σ_2 were defined in Theorem 2.1. Then, taking $u = \frac{2^{1/3}}{(2-\alpha)^{4/3}}s$ gives

$$-t \frac{1 - \eta_0}{1 + \eta_0} + \frac{2u}{(1 + \eta_0)^{4/3}} t^{1/3} = (\alpha - 1)t + st^{1/3} + \mathcal{O}(t^{-1/3}) \quad (2.22)$$

from which the result follows using Theorem 2.3. \square

2.3 Proof of Theorem 2.3

In the following, we will often use the following elementary lemma from [5]. By “ \Rightarrow ” we designate convergence in distribution.

Lemma 2.4 (Lemma 4.1 in [5]). *Let D be a probability distribution and $(X_n)_{n \in \mathbb{N}}, (\tilde{X}_n)_{n \in \mathbb{N}}$ be sequences of random variables. If $X_n \geq \tilde{X}_n$ and $\tilde{X}_n, X_n \Rightarrow D$, then $X_n - \tilde{X}_n$ converges to zero in probability. If $X_n \Rightarrow D$ and $X_n - \tilde{X}_n$ converges to zero in probability, then $\tilde{X}_n \Rightarrow D$ as well.*

We denote by

$$L_{\mathcal{L}^+ \rightarrow (\eta t, t)}^{\text{resc}} = \frac{L_{\mathcal{L}^+ \rightarrow (\eta t, t)} - \mu t}{t^{1/3}} \quad (2.23)$$

the last passage time $L_{\mathcal{L}^+ \rightarrow (\eta t, t)}$ rescaled as required by Assumption 1. We define analogously $L_{\mathcal{L}^- \rightarrow (\eta t, t)}^{\text{resc}}, L_{E^+ \rightarrow (\eta t, t)}^{\text{resc}}$ and $L_{\mathcal{L}^+ \rightarrow E^+}^{\text{resc}}$ as the last passage times rescaled as required by Assumption 1 resp. 2.

To study the asymptotic behavior of $I_n - J_n$ we make the following observation.

Proposition 2.5. *Let ϕ be the competition interface of the LPP model (2.2), (2.3) with $x_0(0) = 1, x_1(0) < -1$. Let $n, M \in \mathbb{N}$ and $M \leq n - 1$. Then*

$$\begin{aligned} \mathbb{P}((M, n - M) \in \Gamma_-^\infty) &\leq \mathbb{P}(I_n - J_n \leq -n + 2M) \\ &\leq \mathbb{P}((M, n - M) + (1, -1) \in \Gamma_-^\infty). \end{aligned} \quad (2.24)$$

Proof. Note that since $I_n + J_n = n$ we have the equality of the events

$$\{I_n - J_n \leq -n + 2M\} = \{\phi_n \in \{(k, n - k), 0 \leq k \leq M\}\}, \quad (2.25)$$

with ϕ_n given in (2.13). The statement follows now from

$$\begin{aligned} \{(M, n - M) \in \Gamma_-^\infty\} &\subseteq \{\phi_n \in \{(k, n - k), 0 \leq k \leq M\}\} \\ &\subseteq \{(M, n - M) + (1, -1) \in \Gamma_-^\infty\}. \end{aligned} \quad (2.26)$$

□

With this observation, we can translate the behavior of $I_n - J_n$ into the difference of LPP times.

Proposition 2.6. *Let $\eta = \eta_0 + ut^{-2/3}$, with $u \in \mathbb{R}$ and suppose that*

$$L_{\mathcal{L}^- \rightarrow (\eta t, t)}^{\text{resc}} - L_{\mathcal{L}^+ \rightarrow (\eta t, t)}^{\text{resc}} \Rightarrow D \text{ as } t \rightarrow \infty. \quad (2.27)$$

Then, under Assumption 1, for any sequence $a_t = o(t^{1/3})$ we have

$$\lim_{t \rightarrow \infty} \mathbb{P} \left(I_{\lfloor t \rfloor} - J_{\lfloor t \rfloor} \leq -t \frac{1 - \eta_0}{1 + \eta_0} + \frac{2u}{(1 + \eta_0)^{4/3}} t^{1/3} + a_t \right) = \mathbb{P}_D((0, \infty)), \quad (2.28)$$

where \mathbb{P}_D is the probability measure with distribution D .

Proof. Let us define

$$\hat{\eta} = \eta_0 + u(t/(1 + \eta_0))^{-2/3}, \quad n(t) = \lfloor t \rfloor, \quad M(t) = \left\lfloor \frac{\hat{\eta} t}{1 + \hat{\eta}} \right\rfloor. \quad (2.29)$$

Then setting $\ell = \frac{t}{1 + \hat{\eta}}$ we have with $\eta(\ell) = \eta_0 + u\ell^{-2/3}$ and

$$(M(t), n(t) - M(t)) = (\eta(\ell)\ell, \ell) + c_\ell, \quad (2.30)$$

with $c_\ell = (c_\ell^1, c_\ell^2) = o(\ell^{1/3})$.

What we have to show is that if $L_{\mathcal{L}^- \rightarrow (\eta(\ell)\ell, \ell)}^{\text{resc}} - L_{\mathcal{L}^+ \rightarrow (\eta(\ell)\ell, \ell)}^{\text{resc}} \Rightarrow D$, then also for any $b_\ell = (b_\ell^1, b_\ell^2) = o(\ell^{1/3})$

$$L_{\mathcal{L}^- \rightarrow (\eta(\ell)\ell, \ell) + b_\ell}^{\text{resc}} - L_{\mathcal{L}^+ \rightarrow (\eta(\ell)\ell, \ell) + b_\ell}^{\text{resc}} \Rightarrow D. \quad (2.31)$$

Indeed, given (2.31), it follows from Proposition 2.5 that

$$\lim_{t \rightarrow \infty} \mathbb{P} (I_{\lfloor t \rfloor} - J_{\lfloor t \rfloor} \leq -n(t) + 2M(t)) = \mathbb{P}_D((0, \infty)). \quad (2.32)$$

Furthermore, if $a_t = o(t^{1/3})$ there is an integer $\tilde{M}(t)$ such that

$$-n(t) + 2M(t) + \lfloor a_t \rfloor = -n(t) + 2\tilde{M}(t), \quad M(t) - \tilde{M}(t) = o(t^{1/3}). \quad (2.33)$$

Applying Proposition 2.5 with $n(t)$, $\tilde{M}(t)$ and then using (2.31) we obtain

$$\lim_{t \rightarrow \infty} \mathbb{P} \left(I_{\lfloor t \rfloor} - J_{\lfloor t \rfloor} \leq -n(t) + 2M(t) + \lfloor a_t \rfloor \right) = \mathbb{P}_D((0, \infty)), \quad (2.34)$$

which is the statement to be proven. So let us now prove (2.31). Writing

$$\begin{aligned} X_\ell &= L_{\mathcal{L}^- \rightarrow (\eta(\ell)\ell, \ell)}^{\text{resc}}, \quad Y_\ell = L_{\mathcal{L}^+ \rightarrow (\eta(\ell)\ell, \ell)}^{\text{resc}}, \\ Z_\ell &= \frac{L_{\mathcal{L}^- \rightarrow (\eta(\ell)\ell, \ell) + b_\ell} - L_{\mathcal{L}^- \rightarrow (\eta(\ell)\ell, \ell)}}{\ell^{1/3}} + \frac{L_{\mathcal{L}^+ \rightarrow (\eta(\ell)\ell, \ell)} - L_{\mathcal{L}^+ \rightarrow (\eta(\ell)\ell, \ell) + b_\ell}}{\ell^{1/3}}, \end{aligned} \quad (2.35)$$

we have

$$L_{\mathcal{L}^- \rightarrow (\eta(\ell)\ell, \ell) + b_\ell}^{\text{resc}} - L_{\mathcal{L}^+ \rightarrow (\eta(\ell)\ell, \ell) + b_\ell}^{\text{resc}} = X_\ell - Y_\ell + Z_\ell, \quad (2.36)$$

so it suffices to show $Z_\ell \Rightarrow 0$. Let $L = \max \left\{ \left| \frac{b_\ell^+}{\eta} \right|, |b_\ell^2| \right\}$. Then

$$\left| L_{\mathcal{L}^- \rightarrow (\eta(\ell)\ell, \ell) + b_\ell}^{\text{resc}} - L_{\mathcal{L}^- \rightarrow (\eta(\ell)\ell, \ell)}^{\text{resc}} \right| \leq \left| \widehat{L}_{\mathcal{L}^- \rightarrow (\eta(\ell)(\ell-L), \ell-L)}^{\text{resc}} - \widehat{L}_{\mathcal{L}^- \rightarrow (\eta(\ell)(\ell+L), \ell+L)}^{\text{resc}} \right| \quad (2.37)$$

where we defined

$$\widehat{L}_{\mathcal{L}^- \rightarrow (\eta(\ell)(\ell \pm L), \ell \pm L)}^{\text{resc}} = \frac{L_{\mathcal{L}^- \rightarrow (\eta(\ell)(\ell \pm L), \ell \pm L)} - \mu \ell}{\ell^{1/3}}. \quad (2.38)$$

Now

$$\widehat{L}_{\mathcal{L}^- \rightarrow (\eta(\ell)(\ell-L), \ell-L)}^{\text{resc}} \leq \widehat{L}_{\mathcal{L}^- \rightarrow (\eta(\ell)(\ell+L), \ell+L)}^{\text{resc}}, \quad (2.39)$$

and $L_{\mathcal{L}^- \rightarrow (\eta(\ell)(\ell-L), \ell-L)}^{\text{resc}}, L_{\mathcal{L}^- \rightarrow (\eta(\ell)(\ell+L), \ell+L)}^{\text{resc}}$ both converge to the same distribution by Assumption 1. Indeed, setting $\ell_\pm = \ell \pm L$ we get

$$\begin{aligned} \widehat{L}_{\mathcal{L}^- \rightarrow (\eta(\ell)(\ell \pm L), \ell \pm L)}^{\text{resc}} &= \frac{L_{\mathcal{L}^- \rightarrow (\eta(\ell_\pm)\ell_\pm, \ell_\pm)} - \mu \ell_\pm}{\ell_\pm^{1/3}} + \mathcal{O}(L\ell^{-1/3}) \\ &+ \mathcal{O}(L\ell_\pm^{-4/3})(L_{\mathcal{L}^- \rightarrow (\eta(\ell)\ell_\pm, \ell_\pm)} - \mu \ell_\pm) \\ &+ \frac{L_{\mathcal{L}^- \rightarrow (\eta(\ell)\ell_\pm, \ell_\pm)} - L_{\mathcal{L}^- \rightarrow (\eta(\ell_\pm)\ell_\pm, \ell_\pm)}}{\ell_\pm^{1/3}} \end{aligned} \quad (2.40)$$

and all terms except the first one on the right-hand side of (2.40) converge to zero in probability. Hence by Lemma 2.4 $\widehat{L}_{\mathcal{L}^- \rightarrow (\eta(\ell)(\ell-L), \ell-L)}^{\text{resc}} - \widehat{L}_{\mathcal{L}^- \rightarrow (\eta(\ell)(\ell+L), \ell+L)}^{\text{resc}}$ converges to zero in probability and so does the left-hand side of (2.37). An analogous argument shows that $\frac{L_{\mathcal{L}^+ \rightarrow (\eta(\ell)\ell, \ell)} - L_{\mathcal{L}^+ \rightarrow (\eta(\ell)\ell, \ell) + b_\ell}}{\ell^{1/3}}$ converges to zero in probability. This implies that $Z_\ell \Rightarrow 0$. \square

In view of the previous proposition, we need to determine the limit law of $L_{\mathcal{L}^- \rightarrow (\eta t, t)}^{\text{resc}} - L_{\mathcal{L}^+ \rightarrow (\eta t, t)}^{\text{resc}}$. For this, we proceed as in [26], reducing the problem to the limit law of two independent random variables in the following three Propositions.

Proposition 2.7. *Suppose Assumptions 1,2 hold, and let D be a probability distribution. Then if*

$$L_{\mathcal{L}^- \rightarrow (\eta t, t)}^{\text{resc}} - \frac{L_{\mathcal{L}^+ \rightarrow E^+} + L_{E^+ \rightarrow (\eta t, t)} - \mu t}{t^{1/3}} \Rightarrow D \text{ as } t \rightarrow \infty, \quad (2.41)$$

also

$$L_{\mathcal{L}^- \rightarrow (\eta t, t)}^{\text{resc}} - L_{\mathcal{L}^+ \rightarrow (\eta t, t)}^{\text{resc}} \Rightarrow D \text{ as } t \rightarrow \infty. \quad (2.42)$$

Proof. We set

$$\begin{aligned} X_n &= L_{\mathcal{L}^- \rightarrow (\eta t, t)}^{\text{resc}} - \frac{L_{\mathcal{L}^+ \rightarrow E^+} + L_{E^+ \rightarrow (\eta t, t)} - \mu t}{t^{1/3}}, \\ \tilde{X}_n &= L_{\mathcal{L}^- \rightarrow (\eta t, t)}^{\text{resc}} - L_{\mathcal{L}^+ \rightarrow (\eta t, t)}^{\text{resc}}. \end{aligned} \quad (2.43)$$

By Lemma 2.4 it suffices to show $X_n - \tilde{X}_n$ converges to 0 in probability. Now

$$X_n - \tilde{X}_n = L_{\mathcal{L}^+ \rightarrow (\eta t, t)}^{\text{resc}} - \frac{L_{\mathcal{L}^+ \rightarrow E^+} + L_{E^+ \rightarrow (\eta t, t)} - \mu t}{t^{1/3}}, \quad (2.44)$$

and since

$$L_{\mathcal{L}^+ \rightarrow (\eta t, t)}^{\text{resc}} \geq \frac{L_{\mathcal{L}^+ \rightarrow E^+} + L_{E^+ \rightarrow (\eta t, t)} - \mu t}{t^{1/3}}, \quad (2.45)$$

to show that $X_n - \tilde{X}_n$ converges to zero in probability, it suffices again by Lemma 2.4 to show that $L_{\mathcal{L}^+ \rightarrow (\eta t, t)}^{\text{resc}}$ and $\frac{L_{\mathcal{L}^+ \rightarrow E^+} + L_{E^+ \rightarrow (\eta t, t)} - \mu t}{t^{1/3}}$ converge to the same distribution as $t \rightarrow \infty$. By Assumption 1, in the large- t limit, $\mathbb{P}(L_{\mathcal{L}^+ \rightarrow (\eta t, t)}^{\text{resc}} \leq s) \rightarrow G_1(s)$ and by Assumption 2 $\mathbb{P}(L_{\mathcal{L}^+ \rightarrow E^+}^{\text{resc}} \leq s) \rightarrow G_1(s)$. Furthermore, by Assumption 2 for any $\varepsilon > 0$

$$\mathbb{P}\left(\left|\frac{L_{E^+ \rightarrow (\eta t, t)} - \mu_0 t^\nu}{t^{\nu/3}}\right| \geq \varepsilon t^{(1-\nu)/3}\right) \rightarrow 0 \text{ as } t \rightarrow \infty. \quad (2.46)$$

Thus $\frac{L_{E^+ \rightarrow (\eta t, t)} - \mu_0 t^\nu}{t^{1/3}}$ converges to zero in probability as $t \rightarrow \infty$. Consequently

$$\begin{aligned} \mathbb{P}\left(\frac{L_{\mathcal{L}^+ \rightarrow E^+} + L_{E^+ \rightarrow (\eta t, t)} - \mu t}{t^{1/3}} \leq s\right) &= \mathbb{P}\left(L_{\mathcal{L}^+ \rightarrow E^+}^{\text{resc}} + \frac{L_{E^+ \rightarrow (\eta t, t)} - \mu_0 t^\nu}{t^{1/3}} \leq s\right) \\ &\rightarrow G_1(s) \text{ as } t \rightarrow \infty. \end{aligned} \quad (2.47)$$

□

For the next reduction, define for some set B and point C $\tilde{L}_{B \rightarrow C}$ to be the last passage time from B to C except that we take the maximum only over paths not containing any point $\bigcup_{\gamma \in [0, 1-t^{\beta-1}]} D_\gamma$ with D_γ as in Assumption 3. We rescale $\tilde{L}_{B \rightarrow C}$ just as $L_{B \rightarrow C}$.

Proposition 2.8. *Suppose that Assumptions 1,2,3 hold. Then if*

$$Y_{t,1} := \tilde{L}_{\mathcal{L}^- \rightarrow (\eta t, t)}^{\text{resc}} - \frac{\tilde{L}_{\mathcal{L}^+ \rightarrow E^+} + L_{E^+ \rightarrow (\eta t, t)} - \mu t}{t^{1/3}} \Rightarrow D \text{ as } t \rightarrow \infty, \quad (2.48)$$

also

$$Y_{t,2} := L_{\mathcal{L}^- \rightarrow (\eta t, t)}^{\text{resc}} - \frac{L_{\mathcal{L}^+ \rightarrow E^+} + L_{E^+ \rightarrow (\eta t, t)} - \mu t}{t^{1/3}} \Rightarrow D \text{ as } t \rightarrow \infty. \quad (2.49)$$

Proof. For $\varepsilon > 0$ we have

$$\begin{aligned} \mathbb{P}(|Y_{t,1} - Y_{t,2}| \geq \varepsilon) &\leq \mathbb{P}\left(\left|\frac{L_{\mathcal{L}^+ \rightarrow E^+} - \tilde{L}_{\mathcal{L}^+ \rightarrow E^+}}{t^{1/3}}\right| \geq \varepsilon/2\right) \\ &\quad + \mathbb{P}\left(\left|\frac{L_{\mathcal{L}^- \rightarrow (\eta t, t)} - \tilde{L}_{\mathcal{L}^- \rightarrow (\eta t, t)}}{t^{1/3}}\right| \geq \varepsilon/2\right). \end{aligned} \quad (2.50)$$

On the other hand,

$$\{L_{\mathcal{L}^+ \rightarrow E^+} - \tilde{L}_{\mathcal{L}^+ \rightarrow E^+} = 0\} \subset \bigcup_{\substack{D_\gamma \\ \gamma \in [0, 1-t^{\beta-1}]}} \{D_\gamma \in \pi_{L_{\mathcal{L}^+ \rightarrow E^+}}^{\max}\} \quad (2.51)$$

because if the maximizers do not reach any points D_γ , then the two last passage time are identical. The same holds for E^+ replaced with $(\eta t, t)$. However, by Assumption 3, the probability of the event in the r.h.s. of (2.51) goes to 0 as $t \rightarrow \infty$. Hence by Lemma 2.4 if $Y_{t,1} \Rightarrow D$, also $Y_{t,2} \Rightarrow D$. \square

Finally, we have the following.

Proposition 2.9. *If we have that*

$$\tilde{L}_{\mathcal{L}^- \rightarrow (\eta t, t)}^{\text{resc}} - \tilde{L}_{\mathcal{L}^+ \rightarrow E^+}^{\text{resc}} \Rightarrow D \text{ as } t \rightarrow \infty, \quad (2.52)$$

then also

$$\tilde{L}_{\mathcal{L}^- \rightarrow (\eta t, t)}^{\text{resc}} - \frac{\tilde{L}_{\mathcal{L}^+ \rightarrow E^+} + L_{E^+ \rightarrow (\eta t, t)} - \mu t}{t^{1/3}} \Rightarrow D \text{ as } t \rightarrow \infty. \quad (2.53)$$

Proof. Simply note that $X_t = \frac{L_{E^+ \rightarrow (\eta t, t)} - \mu t}{t^{1/3}}$ converges to zero in probability by (2.46), hence if (2.52) holds we have by Lemma 2.4

$$\tilde{L}_{\mathcal{L}^- \rightarrow (\eta t, t)}^{\text{resc}} - (\tilde{L}_{\mathcal{L}^+ \rightarrow E^+}^{\text{resc}} + X_t) = \tilde{L}_{\mathcal{L}^- \rightarrow (\eta t, t)}^{\text{resc}} - \frac{\tilde{L}_{\mathcal{L}^+ \rightarrow E^+} + L_{E^+ \rightarrow (\eta t, t)} - \mu t}{t^{1/3}} \Rightarrow D. \quad (2.54)$$

\square

Proof of Theorem 2.3. By construction $\tilde{L}_{\mathcal{L}^- \rightarrow (\eta t, t)}^{\text{resc}}, \tilde{L}_{\mathcal{L}^+ \rightarrow E^+}^{\text{resc}}$ are independent, since they depend on disjoint sets of $\omega_{i,j}$'s, which are independent random variables. Furthermore, as shown in the proof of Proposition 2.8, $\tilde{L}_{\mathcal{L}^- \rightarrow (\eta t, t)}^{\text{resc}} - L_{\mathcal{L}^- \rightarrow (\eta t, t)}^{\text{resc}}$ and $\tilde{L}_{\mathcal{L}^+ \rightarrow E^+}^{\text{resc}} - L_{\mathcal{L}^+ \rightarrow E^+}^{\text{resc}}$ converge to zero in probability and hence by Assumptions 1,2 and Lemma 2.4

$$\mathbb{P}(\tilde{L}_{\mathcal{L}^- \rightarrow (\eta t, t)}^{\text{resc}} \leq s) \rightarrow G_2(s) \quad \text{and} \quad \mathbb{P}(\tilde{L}_{\mathcal{L}^+ \rightarrow E^+}^{\text{resc}} \leq s) \rightarrow G_1(s) \quad (2.55)$$

as $t \rightarrow \infty$. Now (2.55) and independence imply that the vector $(\tilde{L}_{\mathcal{L}^- \rightarrow (\eta t, t)}^{\text{resc}}, \tilde{L}_{\mathcal{L}^+ \rightarrow E^+}^{\text{resc}})$ converges in law to the product measure $\mathbb{P}_{G_2} \otimes \mathbb{P}_{G_1}$. Consequently, by the continuous mapping theorem, $(\tilde{L}_{\mathcal{L}^- \rightarrow (\eta t, t)}^{\text{resc}} - \tilde{L}_{\mathcal{L}^+ \rightarrow E^+}^{\text{resc}})$ converges to $\mathbb{P}_{G_2} \star \mathbb{P}_{G_1, -}$. Combining this with Propositions 2.6, 2.7, 2.8 and 2.9 leads to the proof of Theorem 2.3. \square

2.4 Application to Bernoulli initial data

In this section we explain that Theorem 2.3 applies to random shock Bernoulli initial data as well. In this case, the asymptotic independence of LPP times underlying Theorem 2.3 had already been established in [14] and does not require the detailed control over fluctuations of maximizing paths as in Assumption 3. We rehash the proof of asymptotic independence given in [14] and explain how to obtain the result from it. For the processes $(I(t), J(t))$ from (2.13), a central limit theorem is also available, see Theorem 4 in [22].

Consider TASEP $(\eta_t)_{t \geq 0}$ where the $(\eta_0(i), i \in \mathbb{Z})$ are independent Bernoulli random variables with parameter ρ_+ (resp. ρ_-) for $i \geq 0$ (resp. $i < 0$). Theorem 2.3 applies when $\rho_+ > \rho_-$, i.e., in the shock case. For $\rho_+ < \rho_-$ we have a rarefaction fan and the competition interface has a random asymptotic direction, see Theorem 2(c) of [22], while for $\rho_+ = \rho_-$ the competition interface has a deterministic direction but there is no underlying independence of LPP times, see [2]. Hence, Theorem 2.3 does not apply in these cases.

Let now \mathcal{L} and the $\{\omega_{i,j}\}$ be given by (2.2), (2.3) (all particles have jump rate 1). While the set \mathcal{L} is random in our case, an application of Burke's Theorem [9] yields that the correspondence (2.4) holds for a point-to-point problem with certain boundary weights (see [37]). Namely, neglecting an asymptotically irrelevant set of weights which are equal to zero (see Propo-

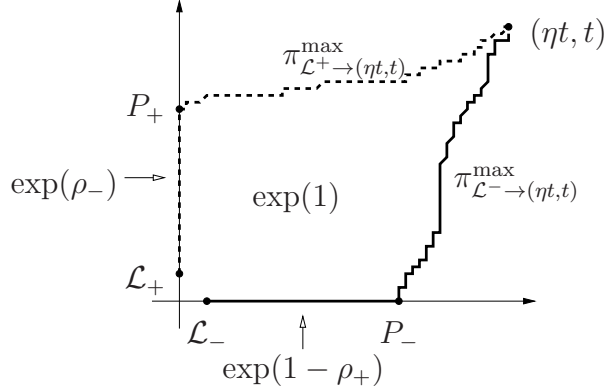


Figure 2: The LPP model of Theorem 2.10 for $\rho_+ = 3/4 = 1 - \rho_-$. The maximizing paths $\pi_{\mathcal{L}^+ \rightarrow (\eta t, t)}^{\max}$ and $\pi_{\mathcal{L}^- \rightarrow (\eta t, t)}^{\max}$ follows the boundary until points close to P_+ resp. P_- after which they enter into the bulk.

sition 2.2 of [27]), one can choose in (2.4) $\mathcal{L} = \{(0, 0)\}$ and weights²

$$\omega_{i,j} := \begin{cases} 0 & \text{if } i = j = 0, \\ \exp(1) & \text{if } i, j > 0, \\ \exp(\rho_-) & \text{if } i = 0, j \geq 1, \\ \exp(1 - \rho_+) & \text{if } j = 0, i \geq 1, \end{cases} \quad (2.56)$$

see Figure 2. Choosing $\mathcal{L}^+ = \{(0, 1)\}$ and $\mathcal{L}^- = \{(1, 0)\}$ we can again define the competition interface via $\phi_0 = (0, 0)$ and (2.7). We then obtain the following result.

Theorem 2.10. *Consider the LPP model defined by (2.56) with $\mathcal{L}^+ = \{(0, 1)\}$ and $\mathcal{L}^- = \{(1, 0)\}$. Let $(I_n, J_n)_{n \geq 1}$ be the competition interface in this model. Take $\eta = \frac{(1-\rho_+)(1-\rho_-)}{\rho_- \rho_+}$ and let $v_+ = \frac{1}{\rho_-^2} - \frac{\eta}{(1-\rho_-)^2}$, $v_- = \frac{\eta}{(1-\rho_+)^2} - \frac{1}{\rho_+^2}$ and finally $m_+ = 1/(1 - \rho_-)$, $m_- = 1/(1 - \rho_+)$. We then have*

$$\lim_{t \rightarrow \infty} \mathbb{P} \left(I_{[t]} - J_{[t]} \leq -t \frac{1 - \eta}{1 + \eta} + \frac{2ut^{1/2}}{(1 + \eta)^{3/2}} \right) = \int_{u(m_+ - m_-)}^{\infty} dx \frac{e^{-\frac{x^2}{2(v_- + v_+)}}}{\sqrt{2\pi(v_- + v_+)}}. \quad (2.57)$$

²In the cited papers [14, 27, 37], the parameter put in the exponential was its mean rather than the jump rate, which are inverse of each other.

To obtain Theorem 2.10, note first that Proposition 2.5 is a general statement and applies here too. Denote now

$$L_{\mathcal{L}^\pm \rightarrow (\eta t + ut^{1/2}, t)}^{\text{resc}} = \frac{L_{\mathcal{L}^\pm \rightarrow (\eta t + ut^{1/2}, t)} - \frac{t}{\rho_+ \rho_-}}{t^{1/2}}. \quad (2.58)$$

Furthermore, denote by $N(m, v)$ a Gaussian random variable with mean m and variance v and write $N_\pm = N(m_\pm u, v_\pm)$. We now have

$$\lim_{t \rightarrow \infty} \mathbb{P}(L_{\mathcal{L}^\pm \rightarrow (\eta t + ut^{1/2}, t)}^{\text{resc}} \leq s) = \mathbb{P}(N_\pm \leq s) \quad (2.59)$$

where (2.59) follows for \mathcal{L}^+ from Proposition 2.8 part (b) in [14] (a random matrix theory variant of this result already appeared in Theorem 1.1 of [1]) by a simple change of variable, for $L_{\mathcal{L}^- \rightarrow (\eta t + ut^{1/2}, t)}$ one has to look at the transposed LPP model with weights $\tilde{\omega}_{i,j} = \omega_{j,i}$ and transposed endpoint to apply Proposition 2.8 of [14]. We can see (2.59) as confirming the analogue of Assumption 1 in this new setting. In particular, Proposition 2.6 still holds, i.e., we have

$$(2.57) = \lim_{t \rightarrow \infty} \mathbb{P}(L_{\mathcal{L}^- \rightarrow (\eta t + ut^{1/2}, t)}^{\text{resc}} - L_{\mathcal{L}^+ \rightarrow (\eta t + ut^{1/2}, t)}^{\text{resc}} > 0). \quad (2.60)$$

Thus one has to establish the asymptotic independence of $L_{\mathcal{L}^+ \rightarrow (\eta t + ut^{1/2}, t)}$ and $L_{\mathcal{L}^- \rightarrow (\eta t + ut^{1/2}, t)}$, which is done in [14] using coupling arguments. Namely, one defines the points

$$\begin{aligned} P_+ &= \left(0, t - \frac{\eta t}{(1/\rho_- - 1)^2} + \frac{ut^{1/2}\rho_-}{1 - \rho_-} \right), \\ P_- &= \left(\eta t - \frac{t}{(1/(1 - \rho_+) - 1)^2} + ut^{1/2}, 0 \right), \end{aligned} \quad (2.61)$$

and the random variables

$$\begin{aligned} X_{\mathcal{L}^+ \rightarrow (\eta t + ut^{1/2}, t)} &= L_{\mathcal{L}^+ \rightarrow P_+} + L_{P_+ \rightarrow (\eta t + ut^{1/2}, t)}, \\ X_{\mathcal{L}^- \rightarrow (\eta t + ut^{1/2}, t)} &= L_{\mathcal{L}^- \rightarrow P_-} + L_{P_- \rightarrow (\eta t + ut^{1/2}, t)}. \end{aligned} \quad (2.62)$$

The choice of P_+, P_- is such that $X_{\mathcal{L}^- \rightarrow (\eta t + ut^{1/2}, t)}$ and $X_{\mathcal{L}^+ \rightarrow (\eta t + ut^{1/2}, t)}$ have the same leading order as $L_{\mathcal{L}^\pm \rightarrow (\eta t + ut^{1/2}, t)}$. Denote by $X_{\mathcal{L}^\pm \rightarrow (\eta t + ut^{1/2}, t)}^{\text{resc}}$ the random variables $X_{\mathcal{L}^\pm \rightarrow (\eta t + ut^{1/2}, t)}$ rescaled as in (2.58). Note now that $L_{\mathcal{L}^+ \rightarrow P_+}$ and $L_{\mathcal{L}^- \rightarrow P_-}$ follow a simple central limit theorem. By this and law of large numbers $L_{\mathcal{L}^+ \rightarrow P_+} - (t - \frac{\eta t}{(1/\rho_- - 1)^2})/\rho_-$ and $L_{\mathcal{L}^- \rightarrow P_-} - (\eta t - \frac{t}{(1/(1 - \rho_+) - 1)^2})/(1 - \rho_+)$ converge, when divided by $t^{1/2}$, to

N_+, N_- . On the other hand $L_{P_+ \rightarrow (\eta t + ut^{1/2}, t)}$ and $L_{P_- \rightarrow (\eta t + ut^{1/2}, t)}$ have only $t^{1/3}$ fluctuations. After centering by their leading order and dividing by $t^{1/2}$, $L_{P_+ \rightarrow (\eta t + ut^{1/2}, t)}$ and $L_{P_- \rightarrow (\eta t + ut^{1/2}, t)}$ converge in probability to 0. This implies that

$$\lim_{t \rightarrow \infty} \mathbb{P}(X_{\mathcal{L}^\pm \rightarrow (\eta t + ut^{1/2}, t)}^{\text{resc}} \leq s) = \mathbb{P}(N_\pm \leq s). \quad (2.63)$$

But $L_{\mathcal{L}^+ \rightarrow P_+}$ and $L_{\mathcal{L}^- \rightarrow P_-}$ are by definition independent. Therefore we obtain for independent N_+, N_-

$$\lim_{t \rightarrow \infty} \mathbb{P}\left(X_{\mathcal{L}^- \rightarrow (\eta t + ut^{1/2}, t)}^{\text{resc}} - X_{\mathcal{L}^+ \rightarrow (\eta t + ut^{1/2}, t)}^{\text{resc}} > 0\right) = \mathbb{P}(N_- - N_+ > 0). \quad (2.64)$$

Finally, since $X_{\mathcal{L}^\pm \rightarrow (\eta t + ut^{1/2}, t)} \leq L_{\mathcal{L}^\pm \rightarrow (\eta t + ut^{1/2}, t)}$, Equation (2.63) together with Lemma 2.4 imply that $L_{\mathcal{L}^+ \rightarrow (\eta t + ut^{1/2}, t)}^{\text{resc}} - X_{\mathcal{L}^+ \rightarrow (\eta t + ut^{1/2}, t)}^{\text{resc}}$ and $L_{\mathcal{L}^- \rightarrow (\eta t + ut^{1/2}, t)}^{\text{resc}} - X_{\mathcal{L}^- \rightarrow (\eta t + ut^{1/2}, t)}^{\text{resc}}$ converge to 0 in probability, hence (2.64) yields Theorem 2.10.

3 Multipoint Distributions

In this section we consider an LPP model for which, in the TASEP picture there is a shock, and we ask the question of the joint distribution of the LPP times around the shock line, namely

$$\lim_{t \rightarrow \infty} \mathbb{P}\left(\bigcap_{k=1}^m \{L_{\mathcal{L} \rightarrow (\eta_0 t + u_k t^{1/3}, t)} \leq \mu t + s_k t^{1/3}\}\right). \quad (3.1)$$

To define the model, take $\beta > 0$ and

$$\mathcal{L}^+ = (\lfloor -\beta t \rfloor, 0) \quad \text{and} \quad \mathcal{L}^- = (0, \lfloor -\beta t \rfloor), \quad (3.2)$$

as well as $\eta_0 = 1$ and $\omega_{i,j} \sim \exp(1)$, see Figure 3. The physical picture in the background is the following: the LPP from \mathcal{L}^\pm to different points at a distance $\mathcal{O}(t)$ have fluctuations of order $\mathcal{O}(t^{1/3})$ and the correlation length scales as $\mathcal{O}(t^{2/3})$ due to KPZ scaling theory. Consider thus two different end-point at distance $ut^{2/3}$. Then, the law of large number from \mathcal{L}^+ and \mathcal{L}^- change over that distance at first approximation linearly in $ut^{2/3}$ with two different prefactors (except along the diagonal). This means that in order to see the effect of both boundaries \mathcal{L}^\pm , we need to consider $u \sim t^{-1/3}$, otherwise one of the two LPP problem dominates completely the other. However, since on that small scale the rescaled LPP process should have Gaussian increments (as it is proven for the corresponding limit processes, the Airy processes [16, 31, 40]) changes in the fluctuation of the two LPP problem will be $\mathcal{O}(t^{1/6})$, thus irrelevant with respect to the $\mathcal{O}(t^{1/3})$ fluctuations. This leads to the following result.

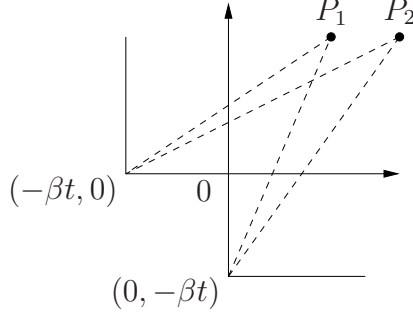


Figure 3: The LPP model (3.2). The endpoints are at distance $\mathcal{O}(t^{1/3})$: $P_k = (t + u_k t^{1/3}, t)$.

Theorem 3.1. *Let $u_1 < u_2 < \dots < u_m$ be real numbers. Consider the LPP model (3.2), $\mathcal{L} = \mathcal{L}^- \cup \mathcal{L}^+$ and points $P_k = (t + u_k t^{1/3}, t)$. Then*

$$\begin{aligned} & \lim_{t \rightarrow \infty} \mathbb{P} \left(\bigcap_{k=1}^m \{L_{\mathcal{L} \rightarrow P_k} \leq (1 + \sqrt{1 + \beta})^2 t + s_k t^{1/3}\} \right) \\ &= F_{\text{GUE}} \left(\frac{\min_k \{s_k - \mu_+ u_k\}}{\sigma} \right) F_{\text{GUE}} \left(\frac{\min_k \{s_k - \mu_- u_k\}}{\sigma} \right), \end{aligned} \quad (3.3)$$

with $\mu_+ = 1 + 1/\sqrt{1 + \beta}$, $\mu_- = 1 + \sqrt{1 + \beta}$, and $\sigma = \frac{(1 + \beta + \sqrt{1 + \beta})^{4/3}}{(1 + \beta)^{1/6}}$.

As explained in Section 2.2, our Assumptions imply that $L_{\mathcal{L}^+ \rightarrow (\eta_0 t + u_k t^{1/3}, t)}$ and $L_{\mathcal{L}^- \rightarrow (\eta_0 t + u_k t^{1/3}, t)}$ are asymptotically independent. Consequently the limit (3.1) for $m = 1$ is given by

$$\begin{aligned} & \lim_{t \rightarrow \infty} \mathbb{P} \left(\frac{L_{\mathcal{L} \rightarrow (\eta_0 t + u_k t^{1/3}, t)} - \mu t}{t^{1/3}} \leq s_k \right) \\ &= \lim_{t \rightarrow \infty} \mathbb{P} \left(\max \left\{ \frac{L_{\mathcal{L}^+ \rightarrow (\eta_0 t + u_k t^{1/3}, t)} - \mu t}{t^{1/3}}, \frac{L_{\mathcal{L}^- \rightarrow (\eta_0 t + u_k t^{1/3}, t)} - \mu t}{t^{1/3}} \right\} \leq s_k \right) \\ &= G_1(s_k) G_2(s_k) \end{aligned} \quad (3.4)$$

as shown in Theorem 2.1 in [26]. For the particular model (3.2) the Assumptions 1,2,3 were checked in [26], see Section 4.3.

To determine the limit (3.1) for general m , the first step is to establish an extended no-crossing result, which guarantees asymptotic independence of the vectors $(L_{\mathcal{L}^+ \rightarrow (\eta_0 t + u_k t^{1/3}, t)})_{k=1, \dots, m}$ and $(L_{\mathcal{L}^- \rightarrow (\eta_0 t + u_k t^{1/3}, t)})_{k=1, \dots, m}$. To obtain then the limit law of each of these vectors, one then needs to control the rescaled local fluctuations as well,

$$\lim_{t \rightarrow \infty} \frac{L_{\mathcal{L}^\pm \rightarrow (\eta_0 t + u_k t^{1/3}, t)} - L_{\mathcal{L}^\pm \rightarrow (\eta_0 t, t)}}{t^{1/3}}. \quad (3.5)$$

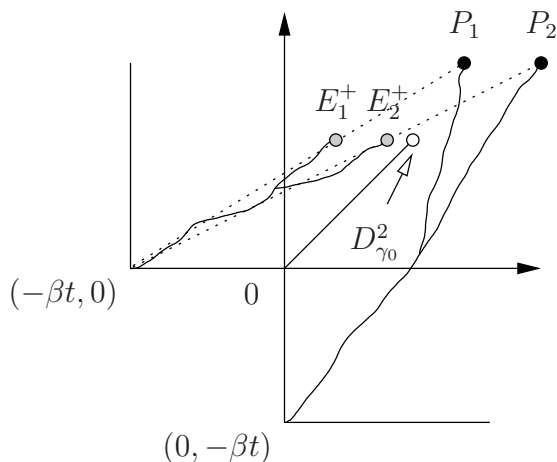


Figure 4: The LPP model (3.2) The maximizing paths $\pi_{\mathcal{L}^+ \rightarrow E_1^+}^{\max}, \pi_{\mathcal{L}^+ \rightarrow E_2^+}^{\max}, \pi_{\mathcal{L}^- \rightarrow P_1}^{\max}, \pi_{\mathcal{L}^- \rightarrow P_2}^{\max}$ cross the line $(0, 0)D_{\gamma_0}^2$ with vanishing probability, where $D_{\gamma_0}^2 = \gamma_0 P_2$ with $\gamma_0 = 1 - t^{\nu-1}(1 + \beta/2)$.

We give the extended no-crossing result in Proposition 3.3, while (3.5) has been obtained by Cator and Pimentel in [10] (they give a much more refined result actually), see Proposition 3.10.

For $P_k = (\eta(u_k)t, t)$ with $\eta(u_k) = \frac{1 + u_k t^{-2/3}}{1 + \beta/2}$, (2.16), (2.17) of Assumption 2 hold for points E_k^+ on the line $(-\beta t, 0)P_k$, i.e., we may take

$$E_k^+ = (t - t^\nu(1 + \beta) + u_k t^{1/3} - u_k t^{\nu-2/3}, t - t^\nu), \quad (3.6)$$

see Figure 4 for an illustration.

This, as well as the validity of Assumption 1, follow easily from the following theorem of Johansson for point-to-point last passage times, which also identifies the limit distribution G_1 in this case.

Proposition 3.2 (Point-to-Point LPP: Convergence to F_{GUE} , Theorem 1.6 in [32]). *Let $0 < \eta < \infty$. Then,*

$$\lim_{\ell \rightarrow \infty} \mathbb{P}(L_{(0,0) \rightarrow (\lfloor \eta \ell \rfloor, \lfloor \ell \rfloor)} \leq \mu_{\text{pp}} \ell + s \sigma_\eta \ell^{1/3}) = F_{\text{GUE}}(s) \quad (3.7)$$

where $\mu_{\text{pp}} = (1 + \sqrt{\eta})^2$, and $\sigma_\eta = \eta^{-1/6}(1 + \sqrt{\eta})^{4/3}$ with F_{GUE} the GUE Tracy Widom distribution function from random matrix theory.

The following is the extended no-crossing result.

Proposition 3.3. *Let $\nu \in (1/3, 1)$, $u_1 < \dots < u_m$ real numbers and take $\gamma \in [0, 1 - t^{\nu-1}(1 + \beta/2)]$. Let $\mathcal{L}^+, \mathcal{L}^-$ be as in Theorem 3.1 and consider the points $D_\gamma^m = \gamma P_m = (\lfloor \gamma(t + u_m t^{1/3}) \rfloor, \lfloor \gamma t \rfloor)$ and E_k^+ as in (3.6). Then*

$$\lim_{t \rightarrow \infty} \mathbb{P} \left(\bigcup_{\gamma \in [0, 1 - t^{\nu-1}(1 + \beta/2)]} \bigcup_{k=1}^m \{D_\gamma^m \in \pi_{\mathcal{L}^+ \rightarrow E_k^+}^{\max}\} \right) = 0 \quad (3.8)$$

and

$$\lim_{t \rightarrow \infty} \mathbb{P} \left(\bigcup_{\gamma \in [0, 1 - t^{\nu-1}(1 + \beta/2)]} \bigcup_{k=1}^m \{D_\gamma^m \in \pi_{\mathcal{L}^- \rightarrow P_k}^{\max}\} \right) = 0. \quad (3.9)$$

The control over the local fluctuations for point-to-point problems is given in the following Proposition.

Proposition 3.4 (Corollary of Theorem 5 in [10]). *Let $s > 0$ and $u \in \mathbb{R}$. We have in the sense of convergence in probability,*

$$\lim_{t \rightarrow \infty} \frac{L_{(0,0) \rightarrow (st+ut^{1/3}, t)} - \mu_s ut^{1/3} - L_{(0,0) \rightarrow (st, t)}}{t^{1/3}} = 0, \quad (3.10)$$

where $\mu_s = 1 + s^{-1/2}$.

Proof. This follows directly from the more refined Theorem 5 in [10], which in particular tells that the process

$$\Delta_t(u) = \frac{L_{(0,0) \rightarrow (st+ut^{1/3}, t)} - \mu_s ut^{1/3} - L_{(0,0) \rightarrow (st, t)}}{\mu_s t^{1/6}} \quad (3.11)$$

converges to Brownian Motion in the sense of weak convergence of probability measures in the space of càdlàg functions, implying (3.10). \square

Remark 3.5. *Assumptions 1,2, the multipoint version of Assumption 3 (here Proposition 3.3), and the control of the local fluctuations of $L_{\mathcal{L}^\pm \rightarrow (\eta t + ut^{1/3}, t)}$ (here Proposition 3.4) are sufficient to prove Theorem 3.1 for a general LPP model too.*

3.1 Proof of Theorem 3.1

To prove Theorem 3.1, we need the multidimensional generalization of Lemma 2.4, and another elementary lemma from [14]. As before, we denote by “ \Rightarrow ” convergence in distribution and by “ $\stackrel{d}{=}$ ” equality in distribution. Let X_n, \tilde{X}_n be random variables with values in \mathbb{R}^m . We say that $X_n \geq \tilde{X}_n$ if $X_n(k) \geq \tilde{X}_n(k)$ for each coordinate $X_n(k), \tilde{X}_n(k)$. We write $X_n \Rightarrow F$ where

F is a distribution function on \mathbb{R}^m , if for every continuity point (s_1, \dots, s_m) of F we have that $\mathbb{P}(\cap_{k=1}^m \{X_n(k) \leq s_k\})$ converges to $F(s_1, \dots, s_m)$. We say that $X_n - \tilde{X}_n$ converges to 0 in probability as $n \rightarrow \infty$, if for every $\varepsilon > 0$ we have $\mathbb{P}(\|X_n - \tilde{X}_n\|_\infty > \varepsilon)$ converges to 0 as $n \rightarrow \infty$. Finally, for X, Y in \mathbb{R}^m we define $\max(X, Y) := Z$ coordinatewise, i.e., $Z(k) = \max\{X(k), Y(k)\}$. With this notation, we have the following lemmas.

Lemma 3.6 (Lemma 3.5 in [14], multidimensional version of Lemma 2.4). *With the above notation for random variables in \mathbb{R}^m , Lemma 2.4 holds true.*

Lemma 3.7 (Lemma 3.6 in [14]). *Let D_1, D_2, D_3 be probability distributions. Assume that $X_n \geq \tilde{X}_n$ and $X_n, \tilde{X}_n \Rightarrow D_1$, and equally that $Y_n \geq \tilde{Y}_n$ and $Y_n, \tilde{Y}_n \Rightarrow D_2$. Let $Z_n = \max\{X_n, Y_n\}$ and $\tilde{Z}_n = \max\{\tilde{X}_n, \tilde{Y}_n\}$. Then if $\tilde{Z}_n \Rightarrow D_3$, also $Z_n \Rightarrow D_3$.*

To prove Theorem 3.1, we reduce the problem to two independent random variables with the following two propositions.

Proposition 3.8. *Let $\mu = (1 + \sqrt{1 + \beta})^2$, take E_k^+ from (3.6) and define \mathbb{R}^m -valued random variables*

$$\begin{aligned} X_n^1 &= (t^{-1/3}(L_{\mathcal{L}^+ \rightarrow P_k} - \mu t))_{k=1, \dots, m}, \\ \tilde{X}_n^1 &= (t^{-1/3}(L_{\mathcal{L}^+ \rightarrow E_k^+} + L_{E_k^+ \rightarrow P_k} - \mu t))_{k=1, \dots, m}, \\ Y_n^1 &= \tilde{Y}_n^1 = (t^{-1/3}(L_{\mathcal{L}^- \rightarrow P_k} - \mu t))_{k=1, \dots, m}. \end{aligned} \quad (3.12)$$

Then, for D a probability distribution, if $\max(\tilde{X}_n^1, \tilde{Y}_n^1) \Rightarrow D$, also $\max(X_n^1, Y_n^1) \Rightarrow D$. Furthermore, with σ as in Theorem 3.1

$$\begin{aligned} X_n^1, \tilde{X}_n^1 &\Rightarrow F_{\text{GUE}} \left(\frac{\min_k \{s_k - u_k(1 + 1/\sqrt{1 + \beta})\}}{\sigma} \right), \\ Y_n^1, \tilde{Y}_n^1 &\Rightarrow F_{\text{GUE}} \left(\frac{\min_k \{s_k - u_k(1 + \sqrt{1 + \beta})\}}{\sigma} \right). \end{aligned} \quad (3.13)$$

Proof. We prove (3.13) first and take the point $P = (t, t)$. For every $k = 1, \dots, m$, we write

$$\frac{L_{\mathcal{L}^+ \rightarrow P_k} - \mu t}{t^{1/3}} = \frac{L_{\mathcal{L}^+ \rightarrow P} - \mu t}{t^{1/3}} + \frac{L_{\mathcal{L}^+ \rightarrow P_k} - L_{\mathcal{L}^+ \rightarrow P}}{t^{1/3}}, \quad (3.14)$$

and note $L_{\mathcal{L}^+ \rightarrow P} \stackrel{d}{=} L_{(0,0) \rightarrow ((1+\beta)t, t)}$. Applying Proposition 3.4 with $s = 1 + \beta$ and Proposition 3.2 to (3.14) we obtain $X_n^1 \Rightarrow F_{\text{GUE}} \left(\frac{\min_k \{s_k - u_k(1 + 1/\sqrt{1 + \beta})\}}{\sigma} \right)$.

Similarly, for $E^+ = (t - t^\nu(1 + \beta), t - t^\nu)$, we decompose

$$\begin{aligned} & \frac{L_{\mathcal{L}^+ \rightarrow E_k^+} + L_{E_k^+ \rightarrow P_k} - \mu t}{t^{1/3}} \\ &= \frac{L_{\mathcal{L}^+ \rightarrow E^+} - \mu(t - t^\nu)}{t^{1/3}} + \frac{L_{\mathcal{L}^+ \rightarrow E_k^+} - L_{\mathcal{L}^+ \rightarrow E^+}}{t^{1/3}} + \frac{L_{E_k^+ \rightarrow P_k} - \mu t^\nu}{t^{1/3}}. \end{aligned} \quad (3.15)$$

Since the points E_k^+ satisfy Assumption 2 of our LPP model, the last summand in (3.15) converges to 0, while the second summand converges to $u_k(1 + 1/\sqrt{1 + \beta})$ by Proposition 3.4. Furthermore, we have that $\mathbb{P}\left(\frac{L_{\mathcal{L}^+ \rightarrow E^+} - \mu(t - t^\nu)}{t^{1/3}} \leq s\right)$ converges to $F_{\text{GUE}}(s)$ by Proposition 3.2. So in total we get $\tilde{X}_n^1 \Rightarrow F_{\text{GUE}}\left(\frac{\min_k\{s_k - u_k(1 + 1/\sqrt{1 + \beta})\}}{\sigma}\right)$. Finally, replacing \mathcal{L}^+ by \mathcal{L}^- in (3.14) and noting that $L_{\mathcal{L}^- \rightarrow P} \stackrel{d}{=} L_{(0,0) \rightarrow ((1 + \beta)^{-1}\ell, \ell)}$ with $\ell = (1 + \beta)t$, by the same argument we get $Y_n^1 \Rightarrow F_{\text{GUE}}\left(\frac{\min_k\{s_k - u_k(1 + \sqrt{1 + \beta})\}}{\sigma}\right)$. The first assertion of the Proposition follows now from Lemma 3.7, since $X_n^1 \geq \tilde{X}_n^1$ and $Y_n^1 = \tilde{Y}_n^1$. \square

In the following, we denote for sets A, B by $\tilde{L}_{A \rightarrow B}$ the LPP time from A to B with the difference that the maximum in (2.1) is only taken over up-right paths that do not contain any point $D_\gamma^m = (\lfloor \gamma(t + u_m t^{1/3}) \rfloor, \lfloor \gamma t \rfloor)$, $\gamma \leq 1 - t^{\nu-1}(1 + \beta/2)$.

Proposition 3.9. *Let D be a probability distribution and define $X_n^2 = \tilde{X}_n^1$, $Y_n^2 = Y_n^1$ and*

$$\begin{aligned} \tilde{X}_n^2 &= \left(t^{-1/3}(\tilde{L}_{\mathcal{L}^+ \rightarrow E_k^+} + L_{E_k^+ \rightarrow P_k} - \mu t)\right)_{k=1, \dots, m}, \\ \tilde{Y}_n^2 &= \left(t^{-1/3}(\tilde{L}_{\mathcal{L}^- \rightarrow P_k} - \mu t)\right)_{k=1, \dots, m}. \end{aligned} \quad (3.16)$$

Then if $\max(\tilde{X}_n^2, \tilde{Y}_n^2) \Rightarrow D$, also $\max(X_n^2, Y_n^2) \Rightarrow D$. Furthermore,

$$\begin{aligned} \tilde{X}_n^2 &\Rightarrow F_{\text{GUE}}\left(\frac{\min_k\{s_k - u_k(1 + 1/\sqrt{1 + \beta})\}}{\sigma}\right), \\ \tilde{Y}_n^2 &\Rightarrow F_{\text{GUE}}\left(\frac{\min_k\{s_k - u_k(1 + \sqrt{1 + \beta})\}}{\sigma}\right). \end{aligned} \quad (3.17)$$

Proof. We prove (3.17) first. First we note that $X_n^2 \geq \tilde{X}_n^2$ and $Y_n^2 \geq \tilde{Y}_n^2$. Now, for $\varepsilon > 0$,

$$\mathbb{P}(\|X_n^2 - \tilde{X}_n^2\|_\infty > \varepsilon) \leq \mathbb{P}\left(\bigcup_{\gamma \in [0, 1 - t^{\nu-1}(1 + \beta/2)]} \bigcup_{k=1}^m \{D_\gamma^m \in \pi_{\mathcal{L}^+ \rightarrow E_k^+}^{\max}\}\right), \quad (3.18)$$

which goes to 0 as $t \rightarrow \infty$ by Proposition 3.3. Thus, by (3.13) and Lemma 3.6, it follows that $\tilde{X}_n^2 \Rightarrow F_{\text{GUE}}\left(\frac{\min_k\{s_k - u_k(1 + 1/\sqrt{1+\beta})\}}{\sigma}\right)$. An analogous proof shows that $\tilde{Y}_n^2 \Rightarrow F_{\text{GUE}}\left(\frac{\min_k\{s_k - u_k(1 + \sqrt{1+\beta})\}}{\sigma}\right)$. The first assertion of the proposition follows now from (3.13),(3.17) and Lemma 3.7. \square

With the reductions from Propositions 3.8, 3.9 at hand, we can now proceed to the proof of Theorem 3.1.

Proof of Theorem 3.1. For the sake of brevity, we define

$$\begin{aligned} F(s_1, \dots, s_m) = & F_{\text{GUE}}\left(\frac{\min_k\{s_k - u_k(1 + 1/\sqrt{1+\beta})\}}{\sigma}\right) \\ & \times F_{\text{GUE}}\left(\frac{\min_k\{s_k - u_k(1 + \sqrt{1+\beta})\}}{\sigma}\right) \end{aligned} \quad (3.19)$$

and the rescaled LPP times

$$\tilde{L}_{\mathcal{L}^+ \rightarrow E_k^+}^{\text{resc}} = \frac{\tilde{L}_{\mathcal{L}^+ \rightarrow E_k^+} - \mu(t - t^\nu)}{t^{1/3}}, \quad \tilde{L}_{\mathcal{L}^- \rightarrow P_k}^{\text{resc}} = \frac{\tilde{L}_{\mathcal{L}^- \rightarrow P_k} - \mu t}{t^{1/3}}. \quad (3.20)$$

First of all, note that

$$(t^{-1/3}(L_{\mathcal{L} \rightarrow P_k} - \mu t))_{k=1, \dots, m} = \max\{X_n^1, Y_n^1\}, \quad (3.21)$$

so that by Propositions 3.8, 3.9 we have to show

$$\max\{\tilde{X}_n^2, \tilde{Y}_n^2\} \Rightarrow F(s_1, \dots, s_m) \quad \text{as } t \rightarrow \infty. \quad (3.22)$$

Define the random variable X_t^k via

$$\frac{L_{E_k^+ \rightarrow P_k} - \mu t^\nu}{t^{1/3}} = \frac{1}{t^{(1-\nu)/3}} X_t^k. \quad (3.23)$$

Then $\frac{1}{t^{(1-\nu)/3}} X_t^k$ vanishes as $t \rightarrow \infty$. This fact together with Lemma 3.6 and (3.17) implies

$$\left(\tilde{L}_{\mathcal{L}^+ \rightarrow E_k^+}^{\text{resc}}\right)_{k=1, \dots, m} \Rightarrow F_{\text{GUE}}\left(\frac{\min_k\{s_k - u_k(1 + 1/\sqrt{1+\beta})\}}{\sigma}\right) \quad (3.24)$$

as $t \rightarrow \infty$.

Let now $\varepsilon > 0$ and take $R > 0$ such that $\mathbb{P}(\cap_{k=1}^m \{|X_t^k| \leq R\}) \geq 1 - \varepsilon$ and define $A_R = \cap_{k=1}^m \{|X_t^k| \leq R\}$. In particular it holds

$$\begin{aligned} & |\mathbb{P}(\cap_{k=1}^m \{\max\{\tilde{X}_n^2(k), \tilde{Y}_n^2(k)\} \leq s_k\}) \\ & - \mathbb{P}(\cap_{k=1}^m \{\max\{\tilde{X}_n^2(k), \tilde{Y}_n^2(k)\} \leq s_k\} \cap A_R)| \leq \varepsilon. \end{aligned} \quad (3.25)$$

We then have

$$\mathbb{P}\left(\bigcap_{k=1}^m \left(\{\tilde{L}_{\mathcal{L}^+ \rightarrow E_k^+}^{\text{resc}} + t^{(\nu-1)/3} R \leq s_k\} \cap \{\tilde{L}_{\mathcal{L}^- \rightarrow P_k}^{\text{resc}} \leq s_k\}\right)\right) - \varepsilon \quad (3.26)$$

$$\leq \mathbb{P}\left(\bigcap_{k=1}^m \left(\{\tilde{L}_{\mathcal{L}^+ \rightarrow E_k^+}^{\text{resc}} + t^{(\nu-1)/3} X_t^k \leq s_k\} \cap A_R \cap \{\tilde{L}_{\mathcal{L}^- \rightarrow P_k}^{\text{resc}} \leq s_k\}\right)\right) \quad (3.27)$$

$$\leq \mathbb{P}\left(\bigcap_{k=1}^m \left(\{\tilde{L}_{\mathcal{L}^+ \rightarrow E_k^+}^{\text{resc}} - t^{(\nu-1)/3} R \leq s_k\} \cap A_R \cap \{\tilde{L}_{\mathcal{L}^- \rightarrow P_k}^{\text{resc}} \leq s_k\}\right)\right) \quad (3.28)$$

$$\leq \mathbb{P}\left(\bigcap_{k=1}^m \left(\{\tilde{L}_{\mathcal{L}^+ \rightarrow E_k^+}^{\text{resc}} - t^{(\nu-1)/3} R \leq s_k\} \cap \{\tilde{L}_{\mathcal{L}^- \rightarrow P_k}^{\text{resc}} \leq s_k\}\right)\right) \quad (3.29)$$

Now $\left(\tilde{L}_{\mathcal{L}^+ \rightarrow E_k^+}^{\text{resc}}\right)_{k=1, \dots, m}$ and $\left(\tilde{L}_{\mathcal{L}^- \rightarrow P_k}^{\text{resc}}\right)_{k=1, \dots, m}$ are independent random variables, since the x coordinate of all E_k^+ is smaller than the x coordinate of D_γ^m for $\gamma = 1 - t^{\nu-1}(1 + \beta/2)$ and t large enough. Hence, by (3.17) and (3.24) there is a t_0 such that for $t > t_0$

$$F(s_1, \dots, s_m) - 2\varepsilon \leq (3.26) \leq (3.27) \leq (3.29) \leq F(s_1, \dots, s_m) + \varepsilon. \quad (3.30)$$

Hence applying (3.25) to (3.27) yields

$$|\mathbb{P}(\bigcap_{k=1}^m \{\max\{\tilde{X}_n^2(k), \tilde{Y}_n^2(k)\} \leq s_k\}) - F(s_1, \dots, s_m)| \leq 3\varepsilon \quad (3.31)$$

for t large enough, thus proving (3.22). \square

3.2 Proof of Proposition 3.3

The no-crossing result (3.8) can be deduced from the validity of Assumption 3 established in [26] by a soft argument, see also Figure 4. To prove (3.9), we use an extension of the strategy used in [26], which is based on moderate deviation estimates for LPP times, given in the following two propositions.

Proposition 3.10 (Proposition 4.2 in [26]). *Let $0 < \eta < \infty$. Then for given $\ell_0 > 0$ and $s_0 \in \mathbb{R}$, there exist constants $C, c > 0$ only dependent on ℓ_0, s_0 such that for all $\ell \geq \ell_0$ and $s \geq s_0$ we have*

$$\mathbb{P}\left(L_{(0,0) \rightarrow ([\eta\ell], [\ell])} > \mu_{\text{pp}}\ell + \ell^{1/3}s\right) \leq C \exp(-cs), \quad (3.32)$$

where $\mu_{\text{pp}} = (1 + \sqrt{\eta})^2$.

Proposition 3.11 (Proposition 4.3 in [26]). *Let $0 < \eta < \infty$ and $\mu_{\text{pp}} = (1 + \sqrt{\eta})^2$. There exist positive constants s_0, ℓ_0, C, c such that for $s \leq -s_0, \ell \geq \ell_0$,*

$$\mathbb{P}(L_{(0,0) \rightarrow (\lfloor \eta \ell \rfloor, \lfloor \ell \rfloor)} \leq \mu_{\text{pp}} \ell + s \ell^{1/3}) \leq C \exp(-c|s|^{3/2}). \quad (3.33)$$

The following proposition shows that if $\pi_{\mathcal{L}^- \rightarrow P_k}^{\max}$ passes through a point γP_l , then $L_{\mathcal{L}^- \rightarrow P_k}$ is unlikely to be larger than the leading order of $L_{\mathcal{L}^- \rightarrow \gamma P_l} + L_{\gamma P_l \rightarrow P_k}$ plus a $\mathcal{O}(t^{1/3})$ term.

Proposition 3.12. *Let $r_1, r_2 \in \mathbb{R}, \nu \in (1/3, 1)$ and $\varepsilon = ct^{\nu-1}, c > 0$. Define points $D_\gamma(r_1) = (\gamma t + \gamma r_1 t^{1/3}, \gamma t)$ for $\gamma \in [0, 1 - t^{\nu-1}(1 + \beta/2)]$ and $P(r_2) = (t + r_2 t^{1/3}, t)$. Define*

$$E_{r_1, r_2, \gamma} = \{L_{\mathcal{L}^- \rightarrow D_\gamma(r_1)} \leq (\mu_\gamma + \varepsilon/2)t\} \cap \{L_{D_\gamma(r_1) \rightarrow P(r_2)} \leq (\mu_{\text{pp}, \gamma} + \varepsilon/2)t\}. \quad (3.34)$$

where $\mu_{\text{pp}, \gamma} = 4(1 - \gamma), \mu_\gamma = 2\gamma + \beta + 2\sqrt{\gamma(\gamma + \beta)}$. Then for some constants $\tilde{c}, \tilde{C} > 0$

$$\mathbb{P}\left(\bigcup_{0 \leq \gamma \leq 1 - t^{\nu-1}(1 + \beta/2)} \Omega \setminus E_{r_1, r_2, \gamma}\right) \leq \tilde{C} e^{-\tilde{c} t^{\nu-1/3}}. \quad (3.35)$$

Proof. We denote by μ_1 the constant μ_{pp} from Proposition 3.2 for

$$L_{\mathcal{L}^- \rightarrow D_\gamma(r_1)} \stackrel{d}{=} L_{(0,0) \rightarrow (\ell(1 + \beta/\gamma + \gamma^{-2/3} r_1 \ell^{-2/3}), \ell)} =: L_{(0,0) \rightarrow (\eta_1 \ell, \ell)} \quad (3.36)$$

with $\ell = \gamma t$ and by μ_2 the constant μ_{pp} from Proposition 3.2 for

$$L_{D_\gamma(r_1) \rightarrow P(r_2)} \stackrel{d}{=} L_{(0,0) \rightarrow (\tilde{\ell} + (r_2 - r_1 \gamma)(\tilde{\ell}/(1 - \gamma))^{1/3}, \tilde{\ell})} =: L_{(0,0) \rightarrow (\eta_2 \tilde{\ell}, \tilde{\ell})} \quad (3.37)$$

with $\tilde{\ell} = (1 - \gamma)t$. A simple computations gives

$$\begin{aligned} \mu_1 \ell &= \gamma t \left(1 + \sqrt{1 + \beta/\gamma + r_2 t^{-2/3}}\right)^2 = (\mu_\gamma + \mathcal{O}(t^{-2/3}))t, \\ \mu_2 \tilde{\ell} &= (1 - \gamma)t \left(1 + \sqrt{1 + (r_2 - r_1 \gamma)t^{-2/3}/(1 - \gamma)}\right)^2 = (\mu_{\text{pp}, \gamma} + \mathcal{O}(t^{-2/3}))t. \end{aligned} \quad (3.38)$$

Since $\nu > 1/3$ we thus have for some $d, C_1, c_1 > 0$ and t large enough

$$\begin{aligned} \mathbb{P}((E_{r_1, r_2, \gamma})^c) &\leq \mathbb{P}(L_{(0,0) \rightarrow (\eta_1 \ell, \ell)} > \mu_1 \ell + d(\ell/\gamma)^\nu) \\ &\quad + \mathbb{P}(L_{(0,0) \rightarrow (\eta_2 \tilde{\ell}, \tilde{\ell})} > \mu_2 \tilde{\ell} + d(\tilde{\ell}/(1 - \gamma))^\nu) \\ &\leq C_1 e^{-c_1 t^{\nu-1/3}}, \end{aligned} \quad (3.39)$$

where for the last inequality we used Proposition 3.10. Since there are only $\mathcal{O}(t)$ many events $(E_{r_1, r_2, \gamma})^c$, (3.35) follows. \square

The next proposition shows that the leading order of $L_{\mathcal{L}^- \rightarrow \gamma P_l} + L_{\gamma P_l \rightarrow P_k}$ is $\mathcal{O}(t^\nu)$ smaller than the one of $L_{\mathcal{L}^- \rightarrow P_l}$.

Proposition 3.13. *Let $\beta > 0$, $\nu \in (1/3, 1)$ and $\gamma \in [0, 1 - t^{\nu-1}(1 + \beta/2)]$. Then, for t large enough, we have with $\varepsilon = Ct^{\nu-1}$*

$$\frac{(\mu_{\text{pp},\gamma} + \mu_\gamma + \varepsilon - \mu)t}{t^{1/3}} \leq -Ct^{\nu-1/3}, \quad (3.40)$$

where $\mu_{\text{pp},\gamma} = 4(1 - \gamma)$, $\mu_\gamma = 2\gamma + \beta + 2\sqrt{\gamma(\gamma + \beta)}$, $\mu = (1 + \sqrt{1 + \beta})^2$ and $C = C(\beta) = -\frac{1}{2}(2 - 2\sqrt{1 + \beta} + \frac{\beta}{\sqrt{1 + \beta}}) > 0$.

Proof. For $Z(\gamma) = \mu_\gamma + \mu_{\text{pp},\gamma}$ we have $Z(0) < Z(1)$ and $Z'(\gamma) \neq 0$, thus Z is monotonely increasing in $[0, 1]$. Let $\gamma_0 = 1 - t^{\nu-1}(1 + \beta/2)$. Then

$$\begin{aligned} Z(\gamma_0) &= 4 + \beta + 2\gamma_0(-1 + \sqrt{1 + \beta/\gamma_0}) \\ &= 4 + \beta + 2(1 - t^{\nu-1}(1 + \beta/2)) \left(-1 + \sqrt{1 + \beta} + \frac{t^{\nu-1}\beta(1 + \beta/2)}{2\sqrt{1 + \beta}} + \mathcal{O}(t^{2\nu-2}) \right) \\ &= 2 + \beta + 2\sqrt{1 + \beta} + t^{\nu-1}(1 + \beta/2) \left(-2\sqrt{1 + \beta} + 2 + \frac{\beta}{\sqrt{1 + \beta}} \right) + \mathcal{O}(t^{2\nu-2}) \\ &= \mu - 2(1 + \beta/2)Ct^{\nu-1} + \mathcal{O}(t^{2\nu-2}). \end{aligned} \quad (3.41)$$

In particular, $(Z(\gamma_0) + \varepsilon - \mu)t^{2/3} \leq -Ct^{\nu-1/3}$ for t large enough. \square

We can now proceed to the proof of Proposition 3.3.

Proof of Proposition 3.3. Define the events

$$F_k^{m,+} = \bigcap_{\gamma \in [0, 1 - t^{\nu-1}(1 + \beta/2)]} \{D_\gamma^m \notin \pi_{\mathcal{L}^+ \rightarrow E_k^+}^{\max}\}. \quad (3.42)$$

The fact that $\mathbb{P}(F_m^{m,+})$ converges to 1 as $t \rightarrow \infty$ is precisely the content of Assumption 3 in our LPP model (3.2) with $\eta = 1 + u_m t^{-2/3}$, which was checked in [26], Section 4.3, Proof of Corollary 2.4. Now we have $F_m^{m,+} \subseteq F_k^{m,+}$ for $k = 1, \dots, m-1$, since if $\pi_{\mathcal{L}^+ \rightarrow E_k^+}^{\max}$ has to branch from $\pi_{\mathcal{L}^+ \rightarrow E_m^+}^{\max}$ to contain a point D_γ^m , $\pi_{\mathcal{L}^+ \rightarrow E_k^+}^{\max}$ has to cross $\pi_{\mathcal{L}^+ \rightarrow E_m^+}^{\max}$ again to reach E_k^+ , which is impossible. Consequently,

$$\lim_{t \rightarrow \infty} \mathbb{P} \left(\bigcup_{k=1}^m (F_k^{m,+})^c \right) = 0, \quad (3.43)$$

which is exactly (3.8). Next define the events

$$I_{k,m,\gamma} = \{D_\gamma^m \in \pi_{\mathcal{L}^- \rightarrow P_k}^{\max}\} \quad \text{and} \quad F_k^{m,-} = \bigcap_{\gamma \in [0, 1 - t^{\nu-1}(1 + \beta/2)]} (I_{k,m,\gamma})^c. \quad (3.44)$$

Now we consider the events $E_{r_1, r_2, \gamma}$ from Proposition 3.12 with $r_1 = u_m$ and $r_2 = u_k$ and take $\varepsilon = Ct^{\nu-1}$ with $C = C(\beta)$ from Proposition 3.13. We then bound

$$\mathbb{P}(I_{k, m, \gamma}) \leq \mathbb{P}\left(I_{k, m, \gamma} \cap \bigcap_{\gamma} E_{u_m, u_k, \gamma}\right) + \mathbb{P}\left(\bigcup_{\gamma} (E_{u_m, u_k, \gamma})^c\right). \quad (3.45)$$

By Proposition 3.12, we have $\mathbb{P}\left(\bigcup_{\gamma} (E_{u_m, u_k, \gamma})^c\right) \leq \tilde{C}e^{-\tilde{c}t^{\nu-1/3}}$. Now for $\omega \in I_{k, m, \gamma} \cap \bigcap_{\gamma} E_{u_m, u_k, \gamma}$ we have

$$L_{\mathcal{L} \rightarrow P_k}(\omega) \leq (\mu_{\text{pp}, \gamma} + \mu_{\gamma} + \varepsilon)t. \quad (3.46)$$

Consequently, by Proposition 3.13 we have

$$I_{k, m, \gamma} \cap \bigcap_{\gamma} E_{r_1, r_2, \gamma} \subseteq \{L_{\mathcal{L} \rightarrow P_k} \leq \mu t - Ct^{\nu-1/3}\}. \quad (3.47)$$

Using Proposition 3.11, we thus see for t large enough and some constants $\tilde{C}_1, \tilde{c}_1 > 0$

$$\mathbb{P}\left(I_{k, m, \gamma} \cap \bigcap_{\gamma} E_{r_1, r_2, \gamma}\right) \leq \tilde{C}_1 e^{-\tilde{c}_1 t^{\nu-1/3}}. \quad (3.48)$$

Putting all together, we obtain

$$\mathbb{P}(I_{k, m, \gamma}) \leq \tilde{C}_2 e^{-\tilde{c}_2 t^{\nu-1/3}} \quad (3.49)$$

for some constants $\tilde{C}_2, \tilde{c}_2 > 0$. Since there are only $\mathcal{O}(t)$ many events $I_{k, m, \gamma}$, this implies that

$$\mathbb{P}(F_k^{m, -}) \geq 1 - \mathcal{O}(t)\tilde{C}_2 e^{-\tilde{c}_2 t^{\nu-1/3}} \quad (3.50)$$

which converges to 1 as $t \rightarrow \infty$, for all $k = 1, \dots, m$, proving (3.9). \square

References

- [1] J. Baik, G. Ben Arous, and S. Péché, *Phase transition of the largest eigenvalue for non-null complex sample covariance matrices*, Ann. Probab. **33** (2006), 1643–1697.
- [2] J. Baik, P.L. Ferrari, and S. Péché, *Limit process of stationary TASEP near the characteristic line*, Comm. Pure Appl. Math. **63** (2010), 1017–1070.
- [3] J. Baik, P.L. Ferrari, and S. Péché, *Convergence of the two-point function of the stationary TASEP*, Singular Phenomena and Scaling in Mathematical Models, Springer, 2014, pp. 91–110.

- [4] M. Balázs, E. Cator, and T. Seppäläinen, *Cube root fluctuations for the corner growth model associated to the exclusion process*, Electron. J. Probab. **11** (2006), 1094–1132.
- [5] G. Ben Arous and I. Corwin, *Current fluctuations for TASEP: a proof of the Prähofer-Spohn conjecture*, Ann. Probab. **39** (2011), 104–138.
- [6] A. Borodin, P.L. Ferrari, M. Prähofer, and T. Sasamoto, *Fluctuation Properties of the TASEP with Periodic Initial Configuration*, J. Stat. Phys. **129** (2007), 1055–1080.
- [7] A. Borodin, P.L. Ferrari, and T. Sasamoto, *Transition between Airy₁ and Airy₂ processes and TASEP fluctuations*, Comm. Pure Appl. Math. **61** (2008), 1603–1629.
- [8] A. Borodin and V. Gorin, *Lectures on integrable probability*, arXiv:1212.3351 (2012).
- [9] P.J. Burke, *The output of a queuing system*, Operations Res. **4** (1956), 699–704.
- [10] E. Cator and L.P.R. Pimentel, *On the local fluctuations of last-passage percolation models*, Stochastic Processes and their Applications **125** (2015), 538–551.
- [11] C.F. Coletti, P.A. Ferrari, and L.P.R. Pimentel, *The variance of the shock in the HAD process*, arXiv:0801.2526 (2008).
- [12] C.F. Coletti and L.P.R. Pimentel, *On the collision between two PNG droplets*, J. Stat. Phys. **126** (2007), 1145–1164.
- [13] I. Corwin, *The Kardar-Parisi-Zhang equation and universality class*, Random Matrices: Theory Appl. **01** (2012), 1130001.
- [14] I. Corwin, P.L. Ferrari, and S. Péché, *Limit processes for TASEP with shocks and rarefaction fans*, Comm. Pure Appl. Math (2010), 1017–1070.
- [15] I. Corwin, P.L. Ferrari, and S. Péché, *Universality of slow decorrelation in KPZ models*, Ann. Inst. H. Poincaré Probab. Statist. **48** (2012), 134–150.
- [16] I. Corwin and A. Hammond, *Brownian Gibbs property for Airy line ensembles*, Inventiones mathematicae **195** (2013), 441–508.

- [17] V. Dotsenko, *Two-time free energy distribution function in $(1 + 1)$ directed polymers*, J. Stat. Mech. (2013), P06017.
- [18] V. Dotsenko, *Two-time free energy distribution function in the KPZ problem*, arXiv:1507.06135 (2015).
- [19] P.A. Ferrari, *Shock fluctuations in asymmetric simple exclusion*, Probab. Theory Relat. Fields **91** (1992), 81–101.
- [20] P.A. Ferrari and L. Fontes, *Shock fluctuations in the asymmetric simple exclusion process*, Probab. Theory Relat. Fields **99** (1994), 305–319.
- [21] P.A. Ferrari and C. Kipnis, *Second class particles in the rarefaction fan*, Ann. Inst. H. Poincaré **31** (1995), 143–154.
- [22] P.A. Ferrari, J. Martin, and L.P.R. Pimentel., *A phase transition for Competition Interfaces*, Ann. Appl. Probab. **19** (2009), 281–317.
- [23] P.A. Ferrari and L.P.R. Pimentel., *Competition interfaces and second class Particles*, Ann. Probab. **33** (2005), 12351254.
- [24] P.L. Ferrari, *Slow decorrelations in KPZ growth*, J. Stat. Mech. (2008), P07022.
- [25] P.L. Ferrari, *From interacting particle systems to random matrices*, J. Stat. Mech. (2010), P10016.
- [26] P.L. Ferrari and P. Nejjar, *Anomalous Shock Fluctuations in TASEP and last-passage percolation models*, Probab. Theory Relat. Fields **61** (2015), 61–109.
- [27] P.L. Ferrari and H. Spohn, *Scaling limit for the space-time covariance of the stationary totally asymmetric simple exclusion process*, Comm. Math. Phys. **265** (2006), 1–44.
- [28] P.L. Ferrari and H. Spohn, *Random Growth Models*, The Oxford handbook of random matrix theory (G. Akemann, J. Baik, and P. Di Francesco, eds.), Oxford Univ. Press, Oxford, 2011, pp. 782–801.
- [29] P.L. Ferrari and H. Spohn, *On time correlations for KPZ growth in one dimension*, arXiv:1602.00486 (2016).
- [30] J. Gärtner and E. Presutti, *Shock fluctuations in a particle system*, Ann. Inst. H. Poincaré (A) **53** (1990), 1–14.

- [31] J. Hägg, *Local Gaussian fluctuations in the Airy and discrete PNG processes*, Ann. Probab. **36** (2008), 1059–1092.
- [32] K. Johansson, *Shape fluctuations and random matrices*, Comm. Math. Phys. **209** (2000), 437–476.
- [33] K. Johansson, *Discrete polynuclear growth and determinantal processes*, Comm. Math. Phys. **242** (2003), 277–329.
- [34] K. Johansson, *Two time distribution in Brownian directed percolation*, arXiv:1502.00941 (2015).
- [35] M. Kardar, G. Parisi, and Y.Z. Zhang, *Dynamic scaling of growing interfaces*, Phys. Rev. Lett. **56** (1986), 889–892.
- [36] T.M. Liggett, *Coupling the simple exclusion process*, Ann. Probab. **4** (1976), 339–356.
- [37] M. Prähofer and H. Spohn, *Current fluctuations for the totally asymmetric simple exclusion process*, In and out of equilibrium (V. Sidoravicius, ed.), Progress in Probability, Birkhäuser, 2002.
- [38] M. Prähofer and H. Spohn, *Scale invariance of the PNG droplet and the Airy process*, J. Stat. Phys. **108** (2002), 1071–1106.
- [39] J. Quastel, *Introduction to KPZ*, Current Developments in Mathematics (2011), 125–194.
- [40] J. Quastel and D. Remenik, *Local behavior and hitting probabilities of the Airy₁ process*, Prob. Theory Relat. Fields **157** (2013), 605–634.
- [41] J. Quastel and H. Spohn, *The one-dimensional KPZ equation and its universality class*, J. Stat. Phys. **160** (2015), 965–984.
- [42] C.A. Tracy and H. Widom, *On orthogonal and symplectic matrix ensembles*, Comm. Math. Phys. **177** (1996), 727–754.