Scaling Limit for Brownian Motions with One-sided Collisions

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Abstract

We consider Brownian motions with one-sided collisions, meaning that each particle is reflected at its right neighbour. For a finite number of particles a Schütz-type formula is derived for the transition probability. We investigate an infinite system with periodic initial configuration, *i.e.*, particles are located at the integer lattice at time zero. The joint distribution of the positions of a finite subset of particles is expressed as a Fredholm determinant with a kernel defining a signed determinantal point process. In the appropriate large time scaling limit, the fluctuations in the particle positions are described by the Airy_1 process.

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1 Introduction

A widely studied model of interacting Brownian motions is governed by the coupled stochastic differential equations

$$dx_j = \big(V'(x_{j+1}-x_j) - V'(x_j - x_{j-1})\big)dt + \sqrt{2}dB_j(t)\,, \tag{1.1}$$

$j = 1,\dots,N$, written here for the case where particles diffuse in one dimension. Hence $x_j(t) \in \mathbb{R}$ and $\{B_j(t), j = 1,\dots,N\}$ is a collection of N independent standard Brownian motions. The boundary terms $V'(x_{N+1} - x_N)$ and $V'(x_1 - x_0)$ are to be set equal to 0. The solutions to (1.1) define a reversible diffusion process in $\mathbb{R}^N$ with respect to the stationary measure

$$\exp\Big(-\sum_{j=1}^{N-1} V(x_{j+1}-x_j)\Big)\prod_{j=1}^{N} dx_j\,. \tag{1.2}$$

Particle j interacts with both, right and left, neighboring particles with labels $j+1$ and $j-1$.

In our contribution we will study the case where the interaction is only with the right neighbor. Hence, including an adjustment of the noise strength,

$$dx_j = V'(x_{j+1}-x_j)dt + dB_j(t)\,, \tag{1.3}$$

$j = 1,\dots,N$. Somewhat unexpectedly, the measure (1.2) is still stationary. Of course, now the diffusion process is no longer reversible. As to be discussed this modification will change dramatically the large scale properties of the dynamics.

A special case is the exponential potential $e^{-\beta x}$, $\beta > 0$, which is related to quantum Toda chains, Gelfand-Tsetlin patterns and other structures from quantum integrable systems [6, 25]. Our focus is the hard collision limit, $\beta \to \infty$. Then the positions will be ordered as $x_N \le \ldots \le x_1$. Hence the diffusion process $x(t)$ has the Weyl chamber $\mathbb{W}_N = \{x \,|\, x_N \le \ldots \le x_1\}$ as state space. Away from $\partial\mathbb{W}_N$, $x(t)$ is simply N-dimensional Brownian motion. The interactions are point-like and particle $j+1$ is reflected from particle j. These are the one-sided collisions of the title. As a rare circumstance, for every N this diffusion process possesses an explicit Schütz-type formula for its transition probability [29,34]. For the particular initial condition $x(0) = 0$, it follows from the Schütz-type formula that $x_N(t)$ has the same distribution as the largest eigenvalue of a $N \times N$ GUE random matrix. Even stronger, the process $t \to -x_N(t)$ has the same law as the top line of N-particle Dyson's Brownian motion starting at 0 [4, 31]. It then follows that

$$\lim_{t\to\infty} \frac{1}{\sigma t^{1/3}}\big(x_{\lfloor at\rfloor}(t) - \mu t\big) = \xi_{\mathrm{GUE}}\,. \tag{1.4}$$

Here $\lfloor\cdot\rfloor$ denotes integer part. The coefficients σ, μ depend on $a > 0$ and $\xi_{\rm GUE}$ is a Tracy-Widom distributed random variable. One can also consider the particle label $\lfloor at + rt^{2/3} \rfloor$. Then in (1.4) one has a stochastic process in r and it converges to the Airy_2 process [18]. Alternatively, one could consider the label $\lfloor at \rfloor$, but different times $t + rt^{2/3}$, resulting in the same limit process [35]. This can also be derived from the fixed time result using the slow decorrelations along characteristics [16, 17].

In our contribution we will investigate the equally spaced initial condition $x_j(0) = -j$, $j \in \mathbb{Z}$. Our main result is that the limit (1.4) still holds provided $\xi_{\rm GUE}$ is replaced by $\xi_{\rm GOE}$, *i.e.*, the Tracy-Widom distribution for a Gaussian Orthogonal Ensemble. Also the Airy_2 process will have to be replaced by the Airy_1 process, see Theorem 2.4.

The limit (1.4) can also be studied for the reversible process governed by (1.1). In this case other methods are available, listed under the heading of non-equilibrium hydrodynamic fluctuation theory [10], which work for a large class of potentials V. Then $t^{1/3}$ would have to be replaced by $t^{1/4}$ and $\xi_{\rm GUE}$ by a Gaussian random variable. In this case the hard collision limit corresponds to independent Brownian particles with the order of particle labels maintained. The $t^{1/4}$ behavior is a famous result by T.E. Harris [19]. For non-reversible diffusion processes, as in (1.3), one is still limited to a very special choice of V. But it is expected that the result holds in greater generality for a large class of potentials.

For the one-sided collision limit the solution to (1.3) can be represented as a last passage problem, which has the same structure as directed polymers at zero temperature [25, 26]. Also, (1.3) can be viewed as a particular discretization of the KPZ equation [23]. While these links help to come up with convincing conjectures, our proof uses disjoint methods by relying on the special structure of the transition probability. The same structure is familiar from the TASEP with periodic initial conditions as has been investigated in [8, 9, 27]. Some constructions developed there carry over directly to our case. But novel steps are needed, like the bi-orthogonalization in our set-up. Also the Lambert function apparently has not made its appearance before.

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2 Model and main results

One way to define a Brownian motion, $x(t)$, starting from $x(0) \in \mathbb{R}$ and being reflected at some continuous function $f(t)$ with $f(0) < x(0)$ is via the Skorokhod representation [1, 30]

$$\begin{aligned} x(t) &= x(0) + B(t) - \min\bigl\{0, \inf_{0\le s\le t}(x(0)+B(s)-f(s))\bigr\} \\ &= \max\bigl\{x(0)+B(t), \sup_{0\le s\le t}(f(s)+B(t)-B(s))\bigr\}, \end{aligned} \tag{2.1}$$

where B is a standard Brownian motion starting at 0.

Let B_k, $k \in \mathbb{Z}$, be independent standard Brownian motions starting at 0 and define the random variables

$$Y_{k,m}(t) = \sup_{0\le s_{k+1}\le\ldots\le s_m\le t} \sum_{i=k}^{m} (B_i(s_{i+1}) - B_i(s_i)) \tag{2.2}$$

with the convention $s_k = 0$ and $s_{m+1} = t$.

Then, iterating the Skorokhod representation, we can define N Brownian motions, $x_1, \ldots, x_N$, starting at positions $x_1(0) \ge x_2(0) \ge \ldots \ge x_N(0)$, such that the Brownian motion x_k is reflected at the trajectory of Brownian motion x_{k-1} according to

$$x_m(t) = -\max_{1\le k\le m}\{Y_{k,m}(t) - x_k(0)\}, \quad 1 \le m \le N. \tag{2.3}$$

This is a Brownian motion in the N-dimensional Weyl chamber with $\pi/4$ oblique reflections [20, 22, 33]. Equivalently we visualize the dynamics as N Brownian particles in $\mathbb{R}$ interacting through one-sided collisions. The process $\{-x_1(t), \ldots, -x_N(t)\}$ can be also interpreted as the zero-temperature O'Connell-Yor semi-directed polymer model [25,26] modified by assigning the extra weights $-x_1(0), x_1(0) - x_2(0), \ldots, x_{N-1}(0) - x_N(0)$ at time 0.

An equivalent description is given by

$$\begin{aligned} x_1(t) &= x_1(0) + B_1(t), \\ x_m(t) &= x_m(0) + B_m(t) - L_{x_{m-1}-x_m}(t), \quad m = 2, \ldots, N, \end{aligned} \tag{2.4}$$

where $L_{X-Y}(t)$ is twice the semimartingale local time at zero of $X(t) - Y(t)$. This point of view is used in [34], where Warren obtained a formula for the transition density of the system with N Brownian motions (Proposition 8 of [34], reported as Proposition 4.1 below). His result will be the starting point for our analysis.

In this contribution we consider the case of infinitely many Brownian particles starting from fixed equally spaced positions, which w.l.o.g. we set it to be 1. This system is obtained as a limit of the following system of finitely many Brownian particles. Let us denote by

$$x_m^{(M)}(t) = -\max_{k\in[-M+1,m]}\{Y_{k,m}(t)+k\}, \tag{2.5}$$

for $m \in [-M+1, M]$. This defines the system of $2M$ reflected Brownian particles starting at time zero from $x_m^{(M)}(0) = -m$. The $M\to\infty$ limit of this process is well defined in the sense that the trajectories of finitely many of them converge in uniform norm over any finite time interval (see Section 3 for the proof).

Proposition 2.1. *Let us define*

$$x_m(t) = -\max_{k\leq m}\{Y_{k,m}(t)+k\}. \tag{2.6}$$

Then, for any $T>0$,

$$\lim_{M\to\infty}\sup_{t\in[0,T]}|x_m^{(M)}(t)-x_m(t)| = 0, \quad \text{a.s.} \tag{2.7}$$

as well as

$$\sup_{t\in[0,T]}|x_m(t)|<\infty, \quad \text{a.s.} \tag{2.8}$$

As a first main result, we provide an expression for the joint distribution at fixed time t.

Proposition 2.2. *Consider the initial condition with infinitely many Brownian motions, indexed by $k\in\mathbb{Z}$, starting at positions $x_k(0)=-k$. Then, for any finite subset S of $\mathbb{Z}$, it holds*

$$\mathbb{P}\left(\bigcap_{k\in S}\{x_k(t)\geq a_k\}\right) = \det(\mathbb{1}-P_aK_t^{\rm flat}P_a)_{L^2(\mathbb{R}\times S)}, \tag{2.9}$$

where $P_a(x,k)=\mathbb{1}_{(-\infty,a_k)}(x)$ and the kernel $K_t^{\rm flat}$ is given by

$$\begin{aligned} K_t^{\rm flat}(x_1,n_1;x_2,n_2) &= -\frac{(x_1-x_2)^{n_2-n_1-1}}{(n_2-n_1-1)!}\mathbb{1}(x_1\geq x_2)\mathbb{1}(n_2>n_1) \\ &+\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_-}\mathrm{d}z\frac{e^{tz^2/2}e^{-zx_1}(-z)^{n_1}}{e^{t\varphi(z)^2/2}e^{-\varphi(z)x_2}(-\varphi(z))^{n_2}}. \end{aligned} \tag{2.10}$$

Here Γ_- is any path going from $\infty e^{-\theta\mathrm{i}}$ to $\infty e^{\theta\mathrm{i}}$ with $\theta\in[\pi/2,3\pi/4)$, crossing the real axis to the left of -1, and such that the function

$$\varphi(z) = L_0(ze^z) \tag{2.11}$$

is continuous and bounded. Here L_0 is the Lambert-W function, i.e., the principal solution for w in $z=we^w$, see Figure 1.

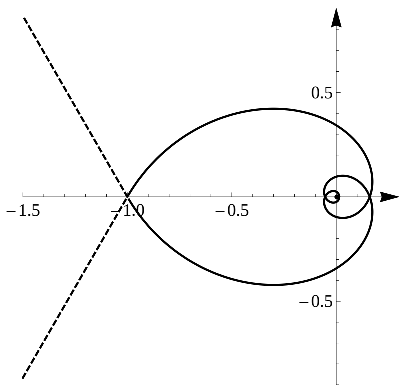

Figure 1: (Dashed line) A possible choice for the contour Γ_- and (solid line) its image by φ.

Interesting and quite unexpected is the appearance of the Lambert function, defined as the multivalued inverse of the function $z \mapsto ze^z$. It has a branch structure similar to the logarithm, but slightly more complicated. The Lambert function is of use in many different areas like combinatorics, exponential towers, delay-differential equations [11] and several problems from physics [2,15,21]. This function has been studied in detail, *e.g.* see [3,12,14], with [13] the standard reference. However, the specific behaviour needed for our asymptotic analysis does not seem to be covered in the literature.

Equal time limit process

As second main result of our contribution we provide a characterization of the law for the positions of the interacting Brownian motions in the large time limit. Due to the asymmetric reflections, the particles have an average velocity -1. For large time t the KPZ scaling theory suggests the positional fluctuations relative to the characteristic to be of order $t^{1/3}$. Nontrivial correlations between particles occur if the particle indices are of order $t^{2/3}$ apart from each other. Therefore, to describe the Brownian particles close to the origin at time t, we consider the scaling of the labels as

$$n(r,t) = \lfloor -t + 2^{5/3}t^{2/3}r \rfloor \tag{2.12}$$

and we define the rescaled process as

$$r \mapsto X_t(r) = -\frac{x_{n(r,t)}(t) + 2^{5/3}t^{2/3}r}{(2t)^{1/3}}. \tag{2.13}$$

The limit object is the Airy_1 process, which is defined as follows.

Definition 2.3. *Let* $B_0(x,y) = \mathrm{Ai}(x+y)$*, with* Ai *the standard Airy function,* Δ *the one-dimensional Laplacian, and the kernel* $K_{\mathcal{A}_1}$ *defined by*

$$K_{\mathcal{A}_1}(s_1, r_1; s_2, r_2) = -\left(e^{(r_2-r_1)\Delta}\right)(s_1, s_2)\mathbb{1}(r_2 > r_1) + \left(e^{-u_1\Delta} B_0 e^{u_2\Delta}\right)(s_1, s_2). \tag{2.14}$$

The Airy$_1$ process, $\mathcal{A}_1$, *is the process with* m*-point joint distributions at* $r_1 < r_2 < \ldots < r_m$ *given by the Fredholm determinant*

$$\mathbb{P}\left(\bigcap_{k=1}^{m} \{\mathcal{A}_1(r_k) \leq s_k\}\right) = \det\left(\mathbb{1} - \chi_s K_{\mathcal{A}_1} \chi_s\right)_{L^2(\{r_1,\ldots,r_m\}\times\mathbb{R})}, \tag{2.15}$$

where $\chi_s(r_k, x) = \mathbb{1}(x > s_k)$*.*

Our second main result is the convergence of X_t to the Airy$_1$ process.

Theorem 2.4. *In the large time limit,* X_t *converges to the Airy*$_1$ *process,*

$$\lim_{t\to\infty} X_t(r) = \mathcal{A}_1(r), \tag{2.16}$$

in the sense of finite-dimensional distributions.

Proposition 2.2 is proved in Section 4 and Theorem 2.4 in Section 5. Properties of the Lambert function are collected in Appendix A.

Tagged particle limit process

The rescaled process at fixed time is not the only one in which the Airy$_1$ process appears. It is also the case for the joint distributions of the positions of a tagged Brownian motion at different times. More precisely, consider the Brownian motion that started at the origin at time 0. Define its rescaled position by

$$\tau \mapsto X_T^{\mathrm{tagged}}(\tau) := -\frac{x_0(T + \tau 2^{5/3} T^{2/3}) + (T + \tau 2^{5/3} T^{2/3})}{(2T)^{1/3}}. \tag{2.17}$$

This rescaled process converges to the Airy$_1$ process.

Theorem 2.5. *In the large time limit,*

$$\lim_{T\to\infty} X_T^{\mathrm{tagged}}(\tau) = \mathcal{A}_1(\tau), \tag{2.18}$$

in the sense of finite-dimensional distributions.

This theorem is proven in Section 6. It is a special case of the more general statement of Theorem 6.1 in Section 6. The result is based from the fixed time result, Theorem 2.4, and a slow decorrelation result, Proposition 6.2. The latter says that along special space-time directions the decorrelation happens over a macroscopic time span.

3 Limit to infinitely many Brownian particles

In this section we prove Proposition 2.1. Given standard independent Brownian motions $B_{-M+1}, \ldots, B_M$ we define as in (2.2)-(2.5),

$$Y_{k,m}(t) = \sup_{0 \le s_{k+1} \le \cdots \le s_m \le t} \sum_{i=k}^{m} (B_i(s_{i+1}) - B_i(s_i)), \tag{3.1}$$

and

$$\begin{aligned} x_m^{(M)}(t) &= -\max_{-M+1 \le k \le m} \{Y_{k,m}(t) + k\} \\ x_m(t) &= -\max_{k \le m} \{Y_{k,m}(t) + k\}. \end{aligned} \tag{3.2}$$

For the proof of Proposition 2.1 we use following concentration inequality result.

Proposition 3.1 (Proposition 2.1 of [24])**.** *For each $T > 0$ there exists a constant $C > 0$ such that for all $k < m$, $\delta > 0$,*

$$\mathbb{P}\left(\frac{Y_{k,m}(T)}{2\sqrt{(m-k+1)T}} \ge 1 + \delta\right) \le Ce^{-(m-k+1)\delta^{3/2}/C}. \tag{3.3}$$

Proof of Proposition 2.1. Let

$$A_M := \{Y_{-M,m}(T) - M \ge -M/2\} \cup \{Y_{m,m}(T) + m \le -M/2\}. \tag{3.4}$$

We can deduce exponential decay of $\mathbb{P}(A_M)$ in M from combining the Gaussian tail of $Y_{m,m}$ with Proposition 3.1, using $\delta = 1$ and elementary inequalities. In particular $\sum_{M=1}^{\infty} \mathbb{P}(A_M) < \infty$, so by Borel-Cantelli, A_M occurs only finitely many times almost surely. This means, that a.s. there exists a M', such that for all $M \ge M'$:

$$\begin{aligned} Y_{-M,m}(T) - M &< -M/2 \qquad \text{and} \\ Y_{m,m}(T) + m &> -M/2 \end{aligned} \tag{3.5}$$

Consequently, $Y_{m,m}(T) + m > Y_{-M,m}(T) - M$ for all $M \ge M'$ and therefore

$$x_m(T) = x_m^{(M')}(T), \quad \text{a.s.} \tag{3.6}$$

It remains to show uniformity over the time interval $[0, T]$. The above argument implies that almost surely for any $t \in [0, T]$ there exists a finite M_t such that $x_m(t) = x_m^{(M_t)}(t)$. Lemma 3.2 below implies that for any $t \in [0, T]$, it holds $x_m(t) = x_m^{(M')}(t)$. This settles the convergence.

Finally we show that $\sup_{t\in[0,T]}|x_m^{(M')}(t)| < \infty$. This follows from the bound

$$|Y_{k,m}(t)| \le \sum_{i=k}^{m}\left(\sup_{0\le s\le t} B_i(s) - \inf_{0\le s\le t} B_i(s)\right) < \infty. \tag{3.7}$$

□

Lemma 3.2. *Consider $0 \le t_1 \le t_2$ and m, M_{t_1}, M_{t_2} such that*

$$x_m(t_i) = x_m^{(M_{t_i})}(t_i), \quad \textit{for } i = 1, 2. \tag{3.8}$$

Then

$$x_m(t_1) = x_m^{(M_{t_2})}(t_1). \tag{3.9}$$

Proof. Define

$$S_M^m(a,b) = \left\{\mathbf{s} \in \mathbb{R}^{M+m+1} |\, a = s_{-M+1} \le \cdots \le s_m \le s_{m+1} = b\right\}. \tag{3.10}$$

Notice that the definition of $Y_{k,m}$ contains a supremum of a continuous function over the compact set $S_k^m(0,t)$, ensuring the existence of a maximizing vector $\mathbf{s}$.

Another representation of $x_m^{(M)}(t)$ is

$$\begin{aligned} x_m^{(M)}(t) &= M - \sup_{\mathbf{s}\in S_M^m(0,t)} \sum_{k=-M+1}^{m} I_k \\ I_k &= B_k(s_{k+1}) - B_k(s_k) + \delta_{0,s_k}. \end{aligned} \tag{3.11}$$

Notice that in (3.8) we can replace M_{t_i} by $M = \max\{M_{t_1}, M_{t_2}\}$. The condition (3.9) is equivalent to the existence of a $\mathbf{s}^* \in S_M^m(0,t_1)$ such that $\sum_{k=-M+1}^{m} I_k$ is maximal and $s^*_{-M_{t_2}+1} = 0$.

Let $\mathbf{s}^{(i)} \in S_M^m(0,t_i)$ be a maximizer of $\sum_{k=-M+1}^{m} I_k$. If $s_k^{(1)} \le s_k^{(2)}$ for all k, then also $s^{(1)}_{-M_{t_2}+1} \le s^{(2)}_{-M_{t_2}+1} = 0$, by (3.8), and the choice $\mathbf{s}^* = \mathbf{s}^{(1)}$ finishes the proof.

Otherwise let k^* be the maximal k such that $s_k^{(1)} > s_k^{(2)}$. There exists τ with

$$s_{k^*}^{(2)} \le s_{k^*}^{(1)} \le \tau \le s_{k^*+1}^{(1)} \le s_{k^*+1}^{(2)}. \tag{3.12}$$

This allows the following decomposition:

$$\begin{aligned} x_m^{(M_{t_i})}(t_i) = x_m^{(M)}(t_i) &= M - \sup_{\mathbf{s}\in S_M^m(0,t_i)} \sum_{k=-M+1}^{m} I_k \\ &= M - \sup_{\mathbf{s}\in S_M^{k^*}(0,\tau)} \sum_{k=-M+1}^{k^*} I_k - \sup_{\mathbf{s}\in S_{-k^*+1}^{m}(\tau,t_i)} \sum_{k=k^*}^{m} I_k. \end{aligned} \tag{3.13}$$

Now the supremum over $S_M^{k^*}(0,\tau)$ is attained by both vectors $(s_{-M+1}^{(i)},\ldots,s_{k^*}^{(i)},\tau)$. Consequently, $\sum_{k=-M}^m I_k$ is maximized also by

$$\mathbf{s}^* = (s_{-M+1}^{(2)},\ldots,s_{k^*}^{(2)},s_{k^*+1}^{(1)},\ldots,s_{m+1}^{(1)}), \tag{3.14}$$

satisfying $s^*_{-M_{t_2}+1} = s^{(2)}_{-M_{t_2}+1} = 0$. □

4 Determinantal structure of joint distributions

Let us denote by $\mathsf{x}_1(t) > \mathsf{x}_2(t) > \ldots > \mathsf{x}_N(t)$ the positions of the N Brownian motions as defined in Section 2. Their joint distribution has a density, denoted by $G(x,t|\mathsf{x}(0))$,

$$\mathbb{P}\left(\bigcap_{k=1}^N \{\mathsf{x}_k(t)\in \mathrm{d}x_k\}\middle|\mathsf{x}_1(0),\ldots,\mathsf{x}_N(0)\right) = G(x,t|\mathsf{x}(0))\prod_{k=1}^N \mathrm{d}x_k. \tag{4.1}$$

Warren [34] proves an explicit formula for G.

Proposition 4.1 (Proposition 8 of [34])**.** *The joint density of the positions of the reflected Brownian motions at time t starting from positions $\mathsf{x}_k(0)$, $k=1,\ldots,N$, is given by*

$$G(x,t|\mathsf{x}(0)) = \det(F_{i-j}(x_{N+1-i}-\mathsf{x}_{N+1-j}(0),t))_{1\leq i,j\leq N} \tag{4.2}$$

with

$$F_k(x,t) = \frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}+\delta}\mathrm{d}z\frac{e^{tz^2/2}e^{-zx}}{z^k} \tag{4.3}$$

for any $\delta>0$.

Proof. Note that $X_k^k(t)$ in [34] corresponds to $-\mathsf{x}_k(t)$ in this paper. Hence the spatial coordinates are reversed. In Proposition 8 of [34] it is shown that

$$G(x,t|\mathsf{x}(0)) = \det(P_t^{(i-j)}(x_j-\mathsf{x}_i(0)))_{1\leq i,j\leq N} \tag{4.4}$$

with

$$\begin{aligned} P_t^{(0)}(x) &= \frac{1}{\sqrt{2\pi t}}e^{-x^2/(2t)}, \\ P_t^{(-n)}(x) &= (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}x^n}P_t^{(0)}(x), \quad n\geq 1, \\ P_t^{(n)}(x) &= \int_x^\infty \mathrm{d}y\frac{(y-x)^{n-1}}{(n-1)!}P_t^{(0)}(y), \quad n\geq 1. \end{aligned} \tag{4.5}$$

Using the identity

$$\frac{1}{\sqrt{2\pi t}}e^{-x^2/(2t)} = \frac{1}{2\pi \mathrm{i}}\int_{\mathrm{i}\mathbb{R}+\delta} \mathrm{d}z\, e^{tz^2/2}e^{-zx} \tag{4.6}$$

(that holds for any δ) we have $P_t^{(0)}(x) = F_0(x,t)$. Also, we immediately get $P_t^{(-n)}(x) = F_{-n}(x,t)$ for $n \geq 1$. Further, for $\delta > 0$ we have

$$\begin{aligned} P_t^{(n)}(x) &= \frac{1}{2\pi \mathrm{i}}\int_{\mathrm{i}\mathbb{R}+\delta} \mathrm{d}z\, e^{tz^2/2}\int_x^\infty \mathrm{d}y \frac{(y-x)^{n-1}}{(n-1)!}e^{-zy} \\ &= \frac{1}{2\pi \mathrm{i}}\int_{\mathrm{i}\mathbb{R}+\delta} \mathrm{d}z\, \frac{e^{tz^2/2}e^{-zx}}{z^n} = F_n(x,t). \end{aligned} \tag{4.7}$$

for all $n \geq 1$. Thus $G(x,t|\mathsf{x}(0)) = \det(F_{i-j}(x_j - \mathsf{x}_i(0),t))_{1\leq i,j\leq N}$ and the change of indices $(i,j) \to (N+1-j, N+1-i)$ gives us (4.2). □

Equation (4.2) appeared in previously in [28] too. A joint distribution of the same form as in Proposition 4.1 occurs also in the study of the totally asymmetric simple exclusion process (TASEP) [29] (reported as Lemma 3.1 in [9]). Following the approach of [9] for TASEP, we can show that the joint distributions of the positions of the Brownian particles can be expressed as a Fredholm determinant for a given correlation kernel.

Using as a starting point Proposition 4.1 we prove the result for finitely many Brownian particles starting at $\{-N, -N+1, \ldots, -1\}$.

Proposition 4.2. *Consider the initial condition* $\mathsf{x}_k(0) = -k$ *for* $k = 1, \ldots, N$. *Then, for any subset* S *of* $\{1, 2, \ldots, N\}$, *it holds*

$$\mathbb{P}\left(\bigcap_{k\in S}\{\mathsf{x}_k(t) \geq a_k\}\right) = \det(\mathbb{1} - P_a K_t P_a)_{L^2(\mathbb{R}\times S)}, \tag{4.8}$$

where $P_a(x,k) = \mathbb{1}_{(-\infty,a_k)}(x)$ *and the kernel* K_t *is given by*

$$K_t(x_1,n_1;x_2,n_2) = -\phi^{(n_1,n_2)}(x_1,x_2) + \sum_{k=1}^{n_2}\Psi^{n_1}_{n_1-k}(x_1)\Phi^{n_2}_{n_2-k}(x_2). \tag{4.9}$$

Here

$$\begin{aligned} \phi^{(n_1,n_2)}(x_1,x_2) &= \frac{(x_1-x_2)^{n_2-n_1-1}}{(n_2-n_1-1)!}\mathbb{1}(x_1\geq x_2)\mathbb{1}(n_2>n_1), \\ \Psi^n_{n-k}(x) &= \frac{(-1)^{n-k}}{2\pi \mathrm{i}}\int_{\mathrm{i}\mathbb{R}-\delta}\mathrm{d}z e^{tz^2/2}e^{-z(x+k)}z^{n-k}, \\ \Phi^n_{n-\ell}(x) &= \frac{(-1)^{n-\ell}}{2\pi \mathrm{i}}\oint_{\Gamma_0}\mathrm{d}w \frac{e^{w(x+\ell)}}{e^{tw^2/2}w^{n-\ell}}\frac{1+w}{w}, \end{aligned} \tag{4.10}$$

for $\delta > 0$.

Proof. The proof is very similar to the one in [9], except that now space is continuous. We report in Appendix B the relevant results from [9]. The straightforward but key identity is

$$F_{n+1}(x,t) = \int_x^\infty \mathrm{d}y\, F_n(y,t). \tag{4.11}$$

Let us denote by $x_1^k := x_k$, $k = 1,\ldots,N$. The k-th row of the determinant of (4.2) is given by

$$\left[F_{k-1}(x_1^{N+1-k} - x_N(0),t) \quad \cdots \quad F_{k-N}(x_1^{N+1-k} - x_1(0),t)\right]. \tag{4.12}$$

Using repeatedly the identity (4.11) we can rewrite this row as the $(k-1)$-th fold integral

$$\int_{x_1^{N+1-k}}^\infty \mathrm{d}x_2^{N+2-k} \ldots \int_{x_{k-1}^{N-1}}^\infty \mathrm{d}x_k^N \left[F_0(x_k^N - x_N(0),t) \quad \cdots \quad F_{-N+1}(x_k^N - x_1(0),t)\right]. \tag{4.13}$$

We do this replacement to each row $k \geq 2$ and by multi-linearity of the determinant we get

$$G(x,t|x(0)) = \int_{\mathcal{D}'} \det\left[F_{-j+1}(x_i^N - x_{N+1-j}(0),t)\right]_{1\leq i,j\leq N} \prod_{2\leq k\leq n\leq N} \mathrm{d}x_k^n, \tag{4.14}$$

where the set $\mathcal{D}'$ is given by

$$\mathcal{D}' = \{x_k^n \in \mathbb{R}, 2 \leq k \leq n \leq N | x_k^n \geq x_{k-1}^{n-1}\}. \tag{4.15}$$

Then, using the antisymmetry in the variables $x_1^N,\ldots,x_N^N$ of the determinant in (4.14) we can reduce the integration over $\mathcal{D}$ (see Appendix B, Lemma B.1) defined by

$$\mathcal{D} = \{x_k^n \in \mathbb{R}, 2 \leq k \leq n \leq N | x_k^n > x_k^{n+1}, x_k^n \geq x_{k-1}^{n-1}\}. \tag{4.16}$$

The next step is to encode the constraint of the integration over $\mathcal{D}$ into a formula and then consider the measure over $\{x_k^n, 1 \leq k \leq n \leq N\}$, which turns out to have determinantal correlations functions. At this point the allowed configuration are such that $x_k^n \leq x_{k+1}^n$. For a while, we still consider ordered configurations at each level, i.e., with $x_1^n \leq x_2^n \leq \ldots \leq x_n^n$ for $1 \leq n \leq N$. Let us set

$$\widetilde{\mathcal{D}} = \{x_k^n \in \mathbb{R}, 1 \leq k \leq n \leq N | x_k^n > x_k^{n+1}, x_k^n \geq x_{k-1}^{n-1}\}. \tag{4.17}$$

Defining $\phi(x,y)=\mathbb{1}(x>y)$, it is easy to verify that

$$\prod_{n=1}^{N-1}\det\left[\phi(x_i^n,x_j^{n+1})\right]_{1\leq i,j\leq n+1}=\begin{cases}1, & \text{if } \{x_k^n, 1\leq k\leq n\leq N\}\in\widetilde{\mathcal{D}},\\ 0, & \text{otherwise,}\end{cases} \tag{4.18}$$

where x_{n+1}^n are "virtual" variables and $\phi(x_{n+1}^n,x):=1$. We also set

$$\Psi_{N-k}^N(x):=(-1)^{N-k}F_{-N+k}(x-x_k(0),t), \tag{4.19}$$

for $k\in\{1,\ldots,N\}$. Then, (4.14) can be obtained as a marginal of the measure

$$\frac{1}{Z_N}\prod_{n=1}^{N-1}\det\left[\phi(x_i^n,x_j^{n+1})\right]_{1\leq i,j\leq n+1}\det\left[\Psi_{N-i}^N(x_j^N)\right]_{1\leq i,j\leq N} \tag{4.20}$$

for some constant Z_N. Notice that the measure (4.20) is symmetric in the x_k^n's since by permuting two of them (at the same level n) one gets twice a factor -1. Thus, we relax the constraint of ordered configurations at each level. The only effect is to modify the normalization constant Z_N.

It is known from Lemma 3.4 of [9] (see Appendix B, Lemma B.2) that a (signed) measure of the form (4.20) has determinantal correlation functions and the correlation kernel is given as follow. Let us set

$$\phi^{(n_1,n_2)}(x,y)=\begin{cases}\phi^{(*(n_2-n_1))}(x,y), & \text{if } n_1<n_2,\\ 0, & \text{if } n_1\geq n_2,\end{cases} \tag{4.21}$$

and

$$\Psi_{n-k}^n(x):=(\phi^{(n,N)}*\Psi_{N-k}^N)(x),\quad \text{for } 1\leq k\leq N. \tag{4.22}$$

Assume that we have found families $\{\Phi_0^n(x),\ldots,\Phi_{n-1}^n(x)\}$ such that $\Phi_k^n(x)$ is a polynomial of degree k and they satisfy the biorthogonal relation

$$\int_{\mathbb{R}}\mathrm{d}x\,\Psi_{n-k}^n(x)\Phi_{n-\ell}^n(x)=\delta_{k,\ell},\quad 1\leq k,\ell\leq n. \tag{4.23}$$

Then, the measure (4.20) has correlation kernel given by

$$K_t(n_1,x_1;n_2,x_2)=-\phi^{(n_1,n_2)}(x_1,x_2)+\sum_{k=1}^{n_2}\Psi_{n_1-k}^{n_1}(x_1)\Phi_{n_2-k}^{n_2}(x_2). \tag{4.24}$$

Notice that in (4.14) only F_k with $k\leq 0$ arises. In this case, compare with (4.3), every sign of δ can be used, so that by defining Ψ_{N-k}^N above we decide to use the integration path over $\mathrm{i}\mathbb{R}-\delta$, so that

$$\Psi_{N-k}^N(x)=\frac{(-1)^{N-k}}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}-\delta}\mathrm{d}z e^{tz^2/2}e^{-z(x-x_k(0))}z^{N-k}, \tag{4.25}$$

for any $\delta > 0$. A simple computation gives (now we use $x_k(0) = -k$)

$$\Psi^n_{n-k}(x) = \frac{(-1)^{n-k}}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}-\delta} \mathrm{d}z e^{tz^2/2} e^{-z(x+k)} z^{n-k}. \tag{4.26}$$

With a little bit of experience, it is not hard to find the biorthogonal functions. They are given by

$$\Phi^n_{n-\ell}(x) = \frac{(-1)^{n-\ell}}{2\pi \mathrm{i}} \oint_{\Gamma_0} dw \frac{e^{w(x+\ell)}}{e^{tw^2/2} w^{n-\ell}} \frac{1+w}{w}. \tag{4.27}$$

Remark that the choice of the sign of δ in the definition of Ψ^N_{N-k} above is irrelevant for the biorthogonalization, since there is no pole at $z = 0$ for $k = 1, \ldots, n$. Indeed, (4.23) can be written as

$$\int_{\mathbb{R}_-} \mathrm{d}x\, \Psi^n_{n-k}(x) \Phi^n_{n-\ell}(x) + \int_{\mathbb{R}_+} \mathrm{d}x\, \Psi^n_{n-k}(x) \Phi^n_{n-\ell}(x). \tag{4.28}$$

For the first term, we choose $\delta > 0$ and the path Γ_0 for w satisfying $\mathrm{Re}(z - w) < 0$. Then, we can take the integral over x inside and we obtain

$$\int_{\mathbb{R}_-} \mathrm{d}x\, \Psi^n_{n-k}(x) \Phi^n_{n-\ell}(x) = -\frac{(-1)^{k-\ell}}{(2\pi \mathrm{i})^2} \int_{\mathrm{i}\mathbb{R}-\delta} \mathrm{d}z \oint_{\Gamma_0} \mathrm{d}w\, \frac{e^{tz^2/2} z^{n-k} e^{-zk}}{e^{tw^2/2} w^{n-\ell} e^{-w\ell}} \frac{1+w}{w(z-w)}. \tag{4.29}$$

For the second term, we choose $\delta < 0$ and the path Γ_0 for w satisfying $\mathrm{Re}(z - w) > 0$. Then, we can take the integral over x inside and we obtain the same expression up to a minus sign. The net result of (4.28) is a residue at $z = w$, which is given by

$$\frac{(-1)^{k-\ell}}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}w\, \frac{1+w}{w} (we^w)^{\ell-k} = \frac{(-1)^{k-\ell}}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}W\, W^{\ell-k-1} = \delta_{k,\ell}, \tag{4.30}$$

where we made the change of variables $W = we^w$. Finally, a simple computation gives,

$$\phi^{(n_1,n_2)}(x_1, x_2) = \frac{(x_1 - x_2)^{n_2-n_1-1}}{(n_2 - n_1 - 1)!} \mathbb{1}(x_1 \geq x_2) \mathbb{1}(n_2 > n_1), \tag{4.31}$$

which has also the integral representations

$$\begin{aligned} \phi^{(n_1,n_2)}(x_1, x_2) &= \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{i}\mathbb{R}-\delta} \mathrm{d}z\, \frac{e^{-z(x_1-x_2)}}{(-z)^{n_2-n_1}} \\ &= \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}w\, \frac{e^{w(x_1-x_2)}}{w^{n_2-n_1}} \mathbb{1}(x_1 \geq x_2) \mathbb{1}(n_2 > n_1). \end{aligned} \tag{4.32}$$

□

Remark 4.3. $\Phi^{n_2}_{n_2-k}(x)=0$ for $k>n_2$ since the pole at $w=0$ in (4.27) vanishes. Therefore we can extend the sum over k to ∞. If we choose the integration paths such that $|we^w|<|ze^z|$, then we can take the sum into the integrals and perform the (geometric) sum explicitly, with the result

$$\sum_{k=1}^{n_2}\Psi^{n_1}_{n_1-k}(x_1)\Phi^{n_2}_{n_2-k}(x_2)=\frac{1}{(2\pi\mathrm{i})^2}\int_{\mathrm{i}\mathbb{R}-\delta}\mathrm{d}z\oint_{\Gamma_0}\mathrm{d}w\frac{e^{tz^2/2}e^{-zx_1}(-z)^{n_1}}{e^{tw^2/2}e^{-wx_2}(-w)^{n_2}}\frac{(1+w)e^w}{ze^z-we^w}. \tag{4.33}$$

A possible choice of the paths such that $|we^w|<|ze^z|$ is satisfied is the following: $w=e^{\mathrm{i}\theta}/4$ with $\theta\in[-\pi,\pi)$ and $z=-1+\mathrm{i}y$ with $y\in\mathbb{R}$.

Remark 4.4. It is possible to reformulate K_t^{flat} in a slightly different way. By doing the change of variables $w=\varphi(z)$, we get $\frac{\mathrm{d}z}{\mathrm{d}w}=\frac{(1+w)e^w}{(1+z)e^z}$. Let us denote by $z_k(w)$, $k\in\mathbb{Z}$, the solutions of

$$ze^z=we^w \tag{4.34}$$

with the trivial one indexed by $z_0(w)=w$. Then,

$$\begin{aligned}K_t^{\mathrm{flat}}(x_1,n_1;x_2,n_2)&=-\frac{(x_1-x_2)^{n_2-n_1-1}}{(n_2-n_1-1)!}\mathbb{1}(x_1\geq x_2)\mathbb{1}(n_2>n_1)\\&+\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_0}\mathrm{d}w\frac{e^{tz_k(w)^2/2}e^{-z_k(w)x_1}(-z_k(w))^{n_1}}{e^{tw^2/2}e^{-wx_2}(-w)^{n_2}}\frac{(1+w)e^w}{(1+z_k(w))e^{z_k(w)}}.\end{aligned} \tag{4.35}$$

Remark 4.5. The form of the kernel (4.35) can be also derived by considering the low density totally asymmetric simple exclusion process (TASEP). One considers the initial condition of particles starting every d position, *i.e.*, with density $\rho=1/d$. The kernel for this system is given in [8], see Theorem 2.1, where one should however replace $(1+pu_i(v))^t/(1+pv)^t$ by $e^{u_i(v)t}/e^{vt}$ since in [8] a discrete time model was considered. Then taking the $d\to\infty$ limit, with space and time rescaled diffusively, one recovers (4.35).

Proof of Proposition 2.2. The idea is to consider the finite system, replace x_i by x_i-M and n_i by n_i+M, and then take the $M\to\infty$ limit. The part of the kernel which should survive the limit, is the M-independent part. The reformulation of Remark 4.3 can be used but it is not the best for our purpose. Instead, notice that the path used in Ψ^n_{n-k} does not have necessarily be vertical. We can take any path passing to the left of 0 and such that it asymptotically have an angle between in $(\pi/4,3\pi/4)$. In that case, the

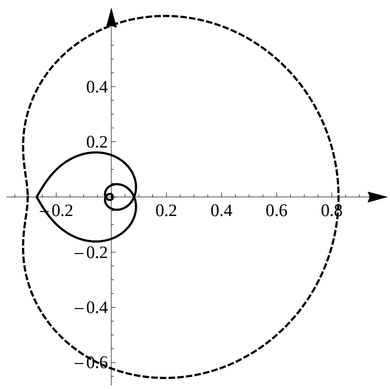

Figure 2: (Dashed line) Image of $w \mapsto we^w$ for $w \in \Gamma_0$ and (solid line) of $z \mapsto ze^z$ for $z \in \Gamma_-$ (which has infinitely many small loops around the origin).

quadratic term in z is strong enough to ensure convergence of the integral. Thus, we choose the path for

$$z \in \Gamma_- := \{-2 + e^{2\pi \mathrm{i}/3\,\mathrm{sgn}(y)}|y|, y \in \mathbb{R}\}, \tag{4.36}$$

and

$$w \in \Gamma_0 := \{e^{\mathrm{i}\theta}, \theta \in [-\pi, \pi)\}, \tag{4.37}$$

see Figure 2.

Computing the finite sum over k leads to

$$\begin{aligned}\sum_{k=1}^{n_2} \Psi^{n_1}_{n_1-k}(x_1)\Phi^{n_2}_{n_2-k}(x_2) =& \frac{1}{(2\pi\mathrm{i})^2}\int_{\Gamma_-}\mathrm{d}z \oint_{\Gamma_0}\mathrm{d}w \frac{e^{tz^2/2}e^{-zx_1}(-z)^{n_1}}{e^{tw^2/2}e^{-wx_2}(-w)^{n_2}} \\ &\times \frac{(1+w)e^w}{ze^z - we^w}\left(1 - \left(\frac{we^w}{ze^z}\right)^{n_2}\right).\end{aligned} \tag{4.38}$$

If we do the change of variables $x_i \to x_i - M$ and $n_i \to n_i + M$, then (4.38) becomes

$$\begin{aligned}&\frac{1}{(2\pi\mathrm{i})^2}\int_{\Gamma_-}\mathrm{d}z \oint_{\Gamma_0}\mathrm{d}w \frac{e^{tz^2/2}e^{-zx_1}(-z)^{n_1}}{e^{tw^2/2}e^{-wx_2}(-w)^{n_2}}\frac{(1+w)e^w}{ze^z - we^w}e^{M(z-w)}(z/w)^M \\ &-\frac{1}{(2\pi\mathrm{i})^2}\int_{\Gamma_-}\mathrm{d}z \oint_{\Gamma_0}\mathrm{d}w \frac{e^{tz^2/2}e^{-zx_1}(-z)^{n_1}}{e^{tw^2/2}e^{-wx_2}(-w)^{n_2}}\frac{(1+w)e^w}{ze^z - we^w}\left(\frac{we^w}{ze^z}\right)^{n_2}.\end{aligned} \tag{4.39}$$

Denote by $K_t^{(1)}$ the first term in (4.39) and by $K_t^{(2)}$ the second term. $K_t^{(2)}$ is independent of M.

By Proposition 4.2 we have

$$\mathbb{P}\left(\bigcap_{k\in S}\{x_k^{(M)}(t)\geq a_k\}\right)=\det(\mathbb{1}-P_aK_{t,M}P_a)_{L^2(\mathbb{R}\times S)}, \tag{4.40}$$

where

$$\begin{aligned}K_{t,M}(x_1,n_1;x_2,n_2)=&-\phi^{(n_1,n_2)}(x_1,x_2)+K_t^{(1)}(x_1,n_1;x_2,n_2)\\&+K_t^{(2)}(x_1,n_1;x_2,n_2).\end{aligned} \tag{4.41}$$

By Proposition 2.1 it follows

$$\lim_{M\to\infty}\mathbb{P}\left(\bigcap_{k\in S}\{x_k^{(M)}(t)\geq a_k\}\right)=\mathbb{P}\left(\bigcap_{k\in S}\{x_k(t)\geq a_k\}\right). \tag{4.42}$$

Therefore to conclude the proof we need to show that

$$\lim_{M\to\infty}\det(\mathbb{1}-P_aK_{t,M}P_a)_{L^2(\mathbb{R}\times S)}=\det(\mathbb{1}-P_aK_t^{\rm flat}P_a)_{L^2(\mathbb{R}\times S)}. \tag{4.43}$$

It is easy to verify that

$$\left|e^{M(z-w)}(z/w)^M\right|\leq e^{M\left(-1-\frac12|y|+\frac12\ln(4+2|y|+y^2)\right)}\leq e^{-M/4}, \tag{4.44}$$

and to get the bounds

$$\begin{aligned}|K^{(1)}(x_1,n_1;x_2,n_2)|&\leq Ce^{-M/4}e^{(2x_1-x_2)}=Ce^{-M/4}e^{3(x_2-x_1)/2}e^{(x_1+x_2)/2},\\|K^{(2)}(x_1,n_1;x_2,n_2)|&\leq Ce^{(2x_1-x_2)}=Ce^{3(x_2-x_1)/2}e^{(x_1+x_2)/2}.\end{aligned} \tag{4.45}$$

for some constant C uniform for x_1,x_2 bounded from above. Using the second integral representation in (4.32) we get

$$\begin{aligned}|\phi^{(n_1,n_2)}(x_1,x_2)|&\leq Ce^{(x_1-x_2)}\mathbb{1}(x_1\geq x_2)\mathbb{1}(n_2>n_1)\\&\leq Ce^{3(x_2-x_1)/2}e^{(x_1+x_2)/2}e^{-|x_1-x_2|/2}\mathbb{1}(n_2>n_1).\end{aligned} \tag{4.46}$$

With these estimates one can show that the Fredholm determinant series expansion is uniformly integrable/summable in M. Dominated convergence allow us to take the $M\to\infty$ inside the Fredholm series. The details are exactly as in the proof of Proposition 3.6 of [7]. This gives

$$\lim_{M\to\infty}\det(\mathbb{1}-P_aK_{t,M}P_a)_{L^2(\mathbb{R}\times S)}=\det(\mathbb{1}-P_a\widetilde{K}_t^{\rm flat}P_a)_{L^2(\mathbb{R}\times S)}, \tag{4.47}$$

where

$$\widetilde{K}_t^{\rm flat}(x_1,n_1;x_2,n_2)=-\phi^{(n_1,n_2)}(x_1,x_2)+K^{(2)}(x_1,n_1;x_2,n_2). \tag{4.48}$$

It remains to verify that $\widetilde{K}_t^{\rm flat} = K_t^{\rm flat}$. Since $K^{(2)}(x_1,n_1;x_2,n_2)$ is given by

$$-\frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0}\mathrm{d}w\int_{\Gamma_-}\mathrm{d}z\frac{e^{tz^2/2}e^{-zx_1}(-z)^{n_1}}{e^{tw^2/2}e^{-wx_2}(-w)^{n_2}}\frac{(1+w)e^w}{ze^z-we^w}\left(\frac{we^w}{ze^z}\right)^{n_2}, \tag{4.49}$$

the pole at $w=0$ is not present. Let us do the change of variables $W=we^w$, *i.e.*,

$$w=w(W)=L_0(W),\quad \text{where } L_0 \text{ is the LambertW function.} \tag{4.50}$$

By the choice of the integration contours, the path for W is still a simple loop around the origin and it contains the image of ze^z, see Figure 2. Therefore,

$$\begin{aligned}(4.49) &= -\frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0}\mathrm{d}W\int_{\Gamma_-}\mathrm{d}z\frac{e^{tz^2/2}e^{-zx_1}(-z)^{n_1}}{e^{tw(W)^2/2}e^{-w(W)x_2}(-w(W))^{n_2}}\frac{(W/ze^z)^{n_2}}{ze^z-W}\\ &= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_-}\mathrm{d}z\frac{e^{tz^2/2}e^{-zx_1}(-z)^{n_1}}{e^{t\varphi(z)^2/2}e^{-\varphi(z)x_2}(-\varphi(z))^{n_2}}\end{aligned} \tag{4.51}$$

where $\varphi(z)=L_0(ze^z)$. At this point, the path Γ_- can be deformed to a generic path as in the Proposition. The convergence is ensured by the term $e^{tz^2/2}$. $\square$

5 Asymptotic analysis

5.1 Proof of Theorem 2.4

To ensure convergence of the Fredholm determinants one needs a pointwise limit as well as integrable bounds of the kernel. The structure of the proof follows the approach of [8]. However, due to the presence of the Lambert function, the search of a steep descent path is more involved than in previous works.

We will use an explicit expression of the Airy$_1$ kernel defined in (2.14)

$$\begin{aligned}K_{\mathcal{A}_1}(s_1,r_1;s_2,r_2) &= -\frac{1}{\sqrt{4\pi(r_2-r_1)}}\exp\left(-\frac{(s_2-s_1)^2}{4(r_2-r_1)}\right)\mathbb{1}(r_2>r_1)\\ &+\mathrm{Ai}\left(s_1+s_2+(r_2-r_1)^2\right)\exp\left((r_2-r_1)(s_1+s_2)+\frac{2}{3}(r_2-r_1)^3\right).\end{aligned} \tag{5.1}$$

The scaling limit (2.13) amounts to setting

$$\begin{aligned}n_i &= -t+2^{5/3}t^{2/3}r_i,\\ x_i &= -2^{5/3}t^{2/3}r_i-(2t)^{1/3}s_i,\quad i=1,2.\end{aligned} \tag{5.2}$$

Finally, we consider a conjugated version of the kernel $K_t^{\rm flat}$ of Proposition 2.2,

$$K^{\rm conj}(x_1,n_1;x_2,n_2)=e^{x_2-x_1}K_t^{\rm flat}(x_1,n_1;x_2,n_2), \tag{5.3}$$

which decomposes as

$$K^{\rm conj}(x_1,n_1;x_2,n_2)=-e^{x_2-x_1}\phi^{(n_1,n_2)}(x_1,x_2)+K_0^{\rm conj}(x_1,n_1;x_2,n_2). \tag{5.4}$$

Proposition 5.1 (Uniform convergence on compact sets)**.** *Consider $r_1,r_2\in\mathbb{R}$ as well as $L,\tilde L>0$ fixed. Then, with x_i, n_i defined by (5.2), the kernel converges as*

$$\lim_{t\to\infty}(2t)^{1/3}K^{\rm conj}(x_1,n_1;x_2,n_2)=K_{\mathcal{A}_1}(s_1,r_1;s_2,r_2) \tag{5.5}$$

uniformly for $(s_1,s_2)\in[-L,\tilde L]^2$.

Corollary 5.2. *Consider $r_1,r_2\in\mathbb{R}$ fixed. For any fixed $L,\tilde L>0$ there exists t_0 such that for $t>t_0$ the bound*

$$\left|(2t)^{1/3}K^{\rm conj}(x_1,n_1;x_2,n_2)\right|\leq {\rm const}_{L,\tilde L} \tag{5.6}$$

holds for all $s_1,s_2\in[-L,\tilde L]$.

Proposition 5.3 (Large deviations)**.** *For any $L>0$ there exist $\tilde L>0$ and $t_0>0$ such that the estimate*

$$\left|(2t)^{1/3}K_0^{\rm conj}(x_1,n_1;x_2,n_2)\right|\leq e^{-(s_1+s_2)} \tag{5.7}$$

holds for any $t>t_0$ and $(s_1,s_2)\in[-L,\infty)^2\setminus[-L,\tilde L]^2$.

Proposition 5.4. *For any fixed $r_2-r_1>0$ there exist* $\mathrm{const}_1>0$ *and $t_0>0$ such that the bound*

$$\left|(2t)^{1/3}e^{x_2-x_1}\phi^{(n_1,n_2)}(x_1,x_2)\right|\leq {\rm const}_1 e^{-|s_2-s_1|} \tag{5.8}$$

holds for any $t>t_0$ and $s_1,s_2\in\mathbb{R}$.

With these estimates one proves Theorem 2.4.

Proof of Theorem 2.4. Given the previous bounds, the proof is identical to the proof of Theorem 2.5 in [8]. In our case moderate and large deviations are merged into the single Proposition 5.3. The constants appearing in [8] specialize to $\kappa=2^{1/3}$ and $\mu=-2^{5/3}$ in our setting. □

Now let us prove the convergence of the kernel.

Proof of Proposition 5.1. We start with the first part of the conjugated kernel (5.4) in its integral representation (4.32),

$$(2t)^{1/3}e^{x_2-x_1}\phi^{(n_1,n_2)}(x_1,x_2)=\frac{(2t)^{1/3}}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}-\delta}\mathrm{d}z\,\frac{e^{(z+1)(x_2-x_1)}}{(-z)^{n_2-n_1}}. \tag{5.9}$$

Setting $\delta=1$ and using the change of variables $z=-1+(2t)^{-1/3}\zeta$ as well as the shorthand $r=r_2-r_1$ and $s=s_2-s_1$, we have

$$(5.9)=\frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}}\mathrm{d}\zeta\,\frac{e^{(2t)^{-1/3}\zeta(x_2-x_1)}}{(1-(2t)^{-1/3}\zeta)^{n_2-n_1}}=\frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}}\mathrm{d}\zeta\,e^{-s\zeta}f_t(\zeta,r) \tag{5.10}$$

with

$$f_t(\zeta,r)=\frac{e^{-2^{4/3}t^{1/3}r\zeta}}{(1-(2t)^{-1/3}\zeta)^{2^{5/3}t^{2/3}r}}=e^{-2^{4/3}t^{1/3}r\zeta-2^{5/3}t^{2/3}r\log(1-(2t)^{-1/3}\zeta)}. \tag{5.11}$$

Since this integral is 0 for $r\leq 0$ we can assume $r>0$ from now on. The function $f_t(\zeta,r)$ satisfies the pointwise limit $\lim_{t\to\infty}f_t(\zeta,r)=e^{r\zeta^2}$. Applying Bernoulli's inequality, we arrive at the t-independent integrable bound

$$\begin{aligned}|f_t(\zeta,r)|&=|1-(2t)^{-1/3}\zeta|^{-2^{5/3}t^{2/3}r}=\left(1+(2t)^{-2/3}|\zeta|^2\right)^{-2^{2/3}t^{2/3}r}\\&\leq(1+r|\zeta|^2)^{-1}.\end{aligned} \tag{5.12}$$

Thus by dominated convergence

$$\left|\frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}}\mathrm{d}\zeta\,\left(e^{-s\zeta}f_t(\zeta,r)-e^{-s\zeta+r\zeta^2}\right)\right|\leq\frac{1}{2\pi}\int_{\mathrm{i}\mathbb{R}}|\mathrm{d}\zeta|\,\left|f_t(\zeta,r)-e^{r\zeta^2}\right|\xrightarrow{t\to\infty}0. \tag{5.13}$$

This implies that the convergence of the integral is uniform in s. The limit is easily identified as

$$\begin{aligned}\lim_{t\to\infty}-(2t)^{1/3}e^{x_2-x_1}\phi^{(n_1,n_2)}(x_1,x_2)&=-\frac{1}{2\pi\mathrm{i}}\int_{\mathrm{i}\mathbb{R}}\mathrm{d}\zeta\,e^{-s\zeta+r\zeta^2}\mathbb{1}(r>0)\\&=-\frac{1}{\sqrt{4\pi r}}e^{-s^2/4r}\mathbb{1}(r>0),\end{aligned} \tag{5.14}$$

which is the first part of the kernel (5.1).

Now we turn to the main part of the kernel,

$$K_0^{\mathrm{conj}}(x_1,n_1;x_2,n_2)=\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_-}\mathrm{d}z\frac{e^{tz^2/2}e^{-(z+1)x_1}(-z)^{n_1}}{e^{t\varphi(z)^2/2}e^{-(\varphi(z)+1)x_2}(-\varphi(z))^{n_2}}. \tag{5.15}$$

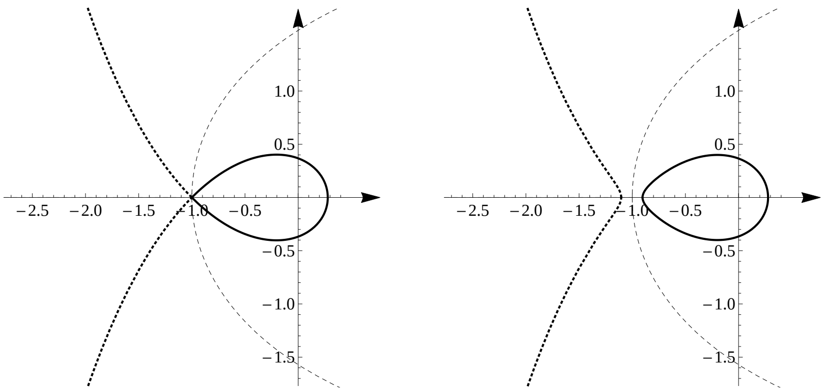

Figure 3: (Dotted line) The contour Γ^ρ and (solid line) its image under φ for (left picture) $\rho = 0$ and (right picture) some small positive ρ. The dashed lines separate the ranges of the principal branch 0 (right) and the branches 1 (top left) and -1 (bottom left).

Inserting the scaling (5.2) and using the identity $z/\varphi(z) = e^{\varphi(z)-z}$ we define the functions

$$\begin{aligned} f_3(z) &= \frac{1}{2}\left(z^2 + 2z - \varphi(z)^2 - 2\varphi(z)\right) \\ f_2(z) &= 2^{5/3}\Big(r_1\left[z + 1 + \log(-z)\right] - r_2\left[\varphi(z) + 1 + \log(-\varphi(z))\right]\Big) \\ f_1(z) &= 2^{1/3}\left(s_1(z+1) - s_2(\varphi(z)+1)\right), \end{aligned} \tag{5.16}$$

which transforms the kernel to

$$K_0^{\mathrm{conj}}(x_1, n_1; x_2, n_2) = \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_-} \mathrm{d}z \exp\Big(t f_3(z) + t^{2/3} f_2(z) + t^{1/3} f_1(z)\Big). \tag{5.17}$$

Define for $0 \leq \rho < 1$ a contour by

$$\Gamma^\rho = \left\{ L_{\lfloor\tau\rfloor}\left(-(1-\rho)e^{2\pi\mathrm{i}\tau - 1}\right), \tau \in \mathbb{R} \setminus [0,1) \right\} \tag{5.18}$$

with $L_k(z)$ being the k-th branch of the Lambert W function. We specify the contour Γ_- by $\Gamma := \Gamma^0$, which is shown in Figure 3, along with a ρ-deformed version, which will be used later in the asymptotic analysis. Lemma A.1 ensures that this contour is an admissible choice. By Lemma A.2 (with $\rho = 0$), Γ is a steep descent curve for the function f_3 with maximum real part 0 at $z = -1$ and strictly negative everywhere else. We can therefore restrict the contour to $\Gamma_\delta = \{z \in \Gamma, |z + 1| < \delta\}$ by making an error which is exponentially small in t, uniformly for $s_i \in [-L, \tilde{L}]$.

By (4.22) in [13] the Lambert W function can be expanded around the branching point $-e^{-1}$ as

$$L_0(z) = -1 + p - \frac{1}{3}p^2 + \frac{11}{72}p^3 + \dots, \tag{5.19}$$

with $p(z) = \sqrt{2(ez+1)}$. Inserting the Taylor series of ze^z provides the expansion of φ and hence of the functions f_i in the neighbourhood of $z=-1$,

$$\begin{aligned}
\varphi(-1+\zeta) &= -1 - \zeta - \frac{2}{3}\zeta^2 + \mathcal{O}(\zeta^3), \\
f_3(-1+\zeta) &= -\frac{2}{3}\zeta^3 + \mathcal{O}(\zeta^4), \\
f_2(-1+\zeta) &= 2^{2/3} r\zeta^2 + \mathcal{O}(\zeta^3), \\
f_1(-1+\zeta) &= 2^{1/3}(s_1+s_2)\zeta + \mathcal{O}(\zeta^2).
\end{aligned} \tag{5.20}$$

The $\mathcal{O}$-terms should be understood as uniform in s_i for $s_i \in [-L, \tilde{L}]$ and with r_i fixed. Let $\tilde{f}_i(\zeta)$ be the expression $f_i(\zeta)$ omitting the error term. Define also

$$F(\zeta) = \exp\left(tf_3(-1+\zeta) + t^{2/3} f_2(-1+\zeta) + t^{1/3} f_1(-1+\zeta)\right)$$

and the corresponding version $\tilde{F}(\zeta)$ without the errors. Let further $\widehat{\Gamma}_\delta = \{z+1;\, z \in \Gamma_\delta\}$. Using the inequality $|e^x - 1| \le |x|\, e^{|x|}$, the error made by integrating over $\tilde{F}$ instead of F can be estimated as

$$\begin{aligned}
&\left| \frac{(2t)^{1/3}}{2\pi \mathrm{i}} \int_{\widehat{\Gamma}_\delta} d\zeta \, \left(F(\zeta) - \tilde{F}(\zeta)\right) \right| \\
&\le \frac{(2t)^{1/3}}{2\pi} \int_{\widehat{\Gamma}_\delta} d\zeta \, |\tilde{F}(\zeta)| e^{\mathcal{O}\left(\zeta^4 t + \zeta^3 t^{2/3} + \zeta^2 t^{1/3} + \zeta\right)} \mathcal{O}\left(\zeta^4 t + \zeta^3 t^{2/3} + \zeta^2 t^{1/3} + \zeta\right) \\
&\le \frac{(2t)^{1/3}}{2\pi} \int_{\widehat{\Gamma}_\delta} d\zeta \, \left| e^{t\tilde{f}_3(\zeta-1)(1+\chi_3) + t^{2/3}\tilde{f}_2(\zeta-1)(1+\chi_2) + t^{1/3}\tilde{f}_1(\zeta-1)(1+\chi_1)} \right| \\
&\qquad \times \mathcal{O}\left(\zeta^4 t + \zeta^3 t^{2/3} + \zeta^2 t^{1/3} + \zeta\right),
\end{aligned} \tag{5.21}$$

where χ_1, χ_2, χ_3 are constants, which can be made as small as desired for δ small enough. Since the contour $\widehat{\Gamma}_\delta$ is close to $\{|\zeta|\, e^{\frac{3}{4}\mathrm{i}\pi \operatorname{sgn}(\zeta)}, \zeta \in (-\delta, \delta)\}$ the leading term in the exponential, $t\tilde{f}_3(\zeta-1)(1+\chi_3) = -\frac{2}{3}\zeta^3(1+\chi_3)t$ has negative real part and therefore ensures the integral to stay bounded for $t \to \infty$. By the change of variables $\zeta = t^{-1/3}\xi$ the $t^{1/3}$ prefactor cancels and the remaining $\mathcal{O}$-terms imply that the overall error is $\mathcal{O}(t^{-1/3})$.

The final step is to evaluate $\frac{(2t)^{1/3}}{2\pi\mathrm{i}}\int_{\widehat{\Gamma}_\delta} d\zeta\, \tilde{F}(\zeta)$. The change of variables $\zeta = -(2t)^{-1/3}\xi$ converts the contour of integration to $\eta_t = \{-(2t)^{1/3}\zeta, \zeta \in \widehat{\Gamma}_\delta\}$, and hence

$$\begin{aligned}\frac{(2t)^{1/3}}{2\pi\mathrm{i}}\int_{\widehat{\Gamma}_\delta} d\zeta\, \tilde{F}(\zeta) &= \frac{(2t)^{1/3}}{2\pi\mathrm{i}}\int_{\widehat{\Gamma}_\delta} d\zeta e^{-\frac{2}{3}t\zeta^3 + r(2t)^{2/3}\zeta^2 + (s_1+s_2)(2t)^{1/3}\zeta} \\ &= \frac{-1}{2\pi\mathrm{i}}\int_{\eta_t} d\xi e^{\frac{\xi^3}{3} + r\xi^2 - (s_1+s_2)\xi}.\end{aligned} \tag{5.22}$$

For $t \to \infty$ the contour η_t converges to $\eta_\infty = \{|\xi|\, e^{\frac{\mathrm{i}\pi}{4}\operatorname{sgn}(\xi)}, \xi \in \mathbb{R}\}$. Since there are no poles in the relevant region with the cubic term guaranteeing convergence, we can change η_∞ to the usual Airy contour $\eta = \{|\xi|\, e^{\frac{\mathrm{i}\pi}{3}\operatorname{sgn}(\xi)}, \xi \in \mathbb{R}\}$, so that

$$\begin{aligned}\lim_{t\to\infty}\frac{(2t)^{1/3}}{2\pi\mathrm{i}}\int_{\widehat{\Gamma}_\delta} d\zeta\, \tilde{F}(\zeta) &= \frac{-1}{2\pi\mathrm{i}}\int_{\eta} d\xi e^{\frac{\xi^3}{3} + r\xi^2 - (s_1+s_2)\xi} \\ &= \mathrm{Ai}\left(s_1 + s_2 + r^2\right) e^{\frac{2}{3}r^3 + (s_1+s_2)r}.\end{aligned} \tag{5.23}$$

□

5.2 Kernel bounds

Bound on the main part of the kernel

Proof of Proposition 5.3. The result for $(s_1, s_2) \in [-L, \tilde{L}]^2$ follows from the estimates in the proof of Proposition 5.1. Thus, let us consider the region $(s_1, s_2) \in [-L, \infty]^2 \setminus [-L, \tilde{L}]^2$, so the inequality $s_1 + s_2 \geq \tilde{L} - L \geq 0$ holds. Define also nonnegative variables $\tilde{s}_i = s_i + L$. Since $\tilde{s}_i$ are not anymore bounded from above, we slightly redefine our functions f by decomposing $f_1 = f_{11} + f_{12}$,

$$\begin{aligned} f_3(z) &= \frac{1}{2}\left(z^2 + 2z - \varphi(z)^2 - 2\varphi(z)\right), \\ f_2(z) &= 2^{5/3}\Big(r_1\left[z + 1 + \log(-z)\right] - r_2\left[\varphi(z) + 1 + \log(-\varphi(z))\right]\Big), \\ f_{11}(z) &= 2^{1/3}\left(\tilde{s}_1(z+1) - \tilde{s}_2(\varphi(z)+1)\right), \\ f_{12}(z) &= 2^{1/3}L\left(\varphi(z) - z\right). \end{aligned} \tag{5.24}$$

Using the shorthand $G(z) = tf_3(z) + t^{2/3}f_2(z) + t^{1/3}\left(f_{11}(z) + f_{12}(z)\right)$ the kernel that we want to bound attains the form

$$(2t)^{1/3}K_0^{\mathrm{conj}}(x_1, n_1; x_2, n_2) = \frac{(2t)^{1/3}}{2\pi\mathrm{i}}\int_\Gamma \mathrm{d}z\, e^{G(z)}. \tag{5.25}$$

We deform the contour Γ to $\Gamma^\rho = \{L_{\lfloor\tau\rfloor}(-e^{2\pi\mathrm{i}\tau-1}(1-\rho)), \tau \in \mathbb{R}\setminus[0,1)\}$, where ρ is given by

$$\rho = 2^{-5/3}\min\left\{t^{-2/3}(s_1+s_2), \varepsilon\right\} \tag{5.26}$$

for some small $\varepsilon > 0$ to be chosen in the following. The point where Γ^ρ crosses the real line is given by $z_0 = -1-\sqrt{2\rho}+\mathcal{O}(\rho)$ according to Lemma A.1. We also decompose the kernel as

$$(2t)^{1/3}K_0^{\mathrm{conj}}(x_1,n_1;x_2,n_2) = e^{G(z_0)}\frac{(2t)^{1/3}}{2\pi\mathrm{i}}\int_{\Gamma^\rho}\mathrm{d}z\, e^{G(z)-G(z_0)}. \tag{5.27}$$

For estimating the first factor one uses of the fact that $\rho < \varepsilon$ and applies the Taylor approximation,

$$\begin{aligned}
f_3(-1+\zeta) &= -\frac{2}{3}\zeta^3 + \mathcal{O}(\zeta^4),\\
f_2(-1+\zeta) &= 2^{2/3}r\zeta^2 + \mathcal{O}(\zeta^3),\\
f_{11}(-1+\zeta) &= 2^{1/3}(\tilde{s}_1+\tilde{s}_2)\zeta + \mathcal{O}(\zeta^2),\\
f_{12}(-1+\zeta) &= -2^{4/3}L\zeta + \mathcal{O}(\zeta^2).
\end{aligned} \tag{5.28}$$

Inserting $\zeta = -\sqrt{2\rho}+\mathcal{O}(\rho)$ and using the two inequalities for ρ coming from (5.26) we can bound the arguments of the exponential as

$$\begin{aligned}
\mathrm{Re}(tf_3(-1+\zeta)) &\le \frac{2}{3}t(2\rho)^{3/2}\left(1+\mathcal{O}(\sqrt{\rho})\right) \le \frac{1}{3}(s_1+s_2)^{3/2}\left(1+\mathcal{O}(\sqrt{\varepsilon})\right),\\
\mathrm{Re}(t^{2/3}f_2(-1+\zeta)) &\le |r|(s_1+s_2)\left(1+\mathcal{O}(\sqrt{\varepsilon})\right),\\
\mathrm{Re}(t^{1/3}f_{11}(-1+\zeta)) &\le -(\tilde{s}_1+\tilde{s}_2)(s_1+s_2)^{1/2}\left(1+\mathcal{O}(\sqrt{\varepsilon})\right)\\
&\le -(s_1+s_2)^{3/2}\left(1+\mathcal{O}(\sqrt{\varepsilon})\right),\\
\mathrm{Re}(t^{1/3}f_{12}(-1+\zeta)) &\le 2L(s_1+s_2)^{1/2}\left(1+\mathcal{O}(\sqrt{\varepsilon})\right).
\end{aligned} \tag{5.29}$$

Now choose first ε such that the f_{11} term dominates the f_3 term. Then choose $\tilde{L}$ such that the $(s_1+s_2)^{3/2}$-terms dominate all other terms, leading to the bound

$$\left|e^{G(z_0)}\right| \le e^{-\mathrm{const}_2(s_1+s_2)^{3/2}} \tag{5.30}$$

for some $\mathrm{const}_2 > 0$.

The remaining task is to show boundedness of the integral $(2t)^{1/3}\int_{\Gamma^\rho}\mathrm{d}z\, e^{G(z)-G(z_0)}$. At first we notice that by Lemma A.1 the terms $z+1$ and $-(\varphi(z)+1)$ attain their maximum real part at z_0, so $\tilde{s}_i \ge 0$ results in

$$\mathrm{Re}\left(f_{11}(z) - f_{11}(z_0)\right) \le 0 \tag{5.31}$$

along Γ^ρ. This leads to the estimate

$$\left|\int_{\Gamma^\rho} \mathrm{d}z\, e^{G(z)-G(z_0)}\right| \leq \int_{\Gamma^\rho} |\mathrm{d}z| \left|e^{t\hat{f}_3(z)+t^{2/3}\hat{f}_2(z)+t^{1/3}\hat{f}_{12}(z)}\right|, \tag{5.32}$$

where $\hat{f}_i(z) = f_i(z) - f_i(z_0)$. Notice that in the integral on the right hand side the variables s_i no longer appear. Integrability is ensured by Lemmas A.1 and A.2 claim 5 respectively. As Γ^ρ is a steep descent path for $\hat{f}_3$ by Lemma A.2, we can restrict the contour to a δ-neighbourhood of the critical point, $\Gamma^\rho_\delta = \{z \in \Gamma^\rho, |z - z_0| < \delta\}$, at the expense of an error of order $\mathcal{O}(e^{-\mathrm{const}_\delta t})$.

Since the contour Γ^ρ_δ approaches a straight vertical line, we can set $z = z_0 + \mathrm{i}\xi$ and expand for small ξ as

$$\begin{aligned}
\mathrm{Re}(\hat{f}_3(z_0+\mathrm{i}\xi)) &= -2\sqrt{2\rho}\,\xi^2\,(1+\mathcal{O}(\xi))\,(1+\mathcal{O}(\sqrt{\rho}))\,,\\
\mathrm{Re}(\hat{f}_2(z_0+\mathrm{i}\xi)) &= -2^{2/3} r\xi^2\,(1+\mathcal{O}(\xi))\,(1+\mathcal{O}(\sqrt{\rho}))\,,\\
\mathrm{Re}(\hat{f}_{12}(z_0+\mathrm{i}\xi)) &= \frac{1}{3}2^{4/3}L\xi^2\,(1+\mathcal{O}(\xi))\,(1+\mathcal{O}(\sqrt{\rho}))\,.
\end{aligned} \tag{5.33}$$

By choosing δ and ε small enough there are some constants χ_3, χ_2, χ_1 close to 1 such that

$$\begin{aligned}
\int_{\Gamma^\rho_\delta} \mathrm{d}z\, \left|e^{G(z)-G(z_0)}\right| &\leq \int_{-\delta}^{\delta} \mathrm{d}\xi\, e^{\xi^2\left(-\chi_3 2\sqrt{2\rho}t - \chi_2 2^{2/3} r t^{2/3} + \chi_1 \frac{2^{4/3}}{3} L t^{1/3}\right)}\\
&= \int_{-\delta}^{\delta} \mathrm{d}\xi\, e^{\eta t^{2/3}\xi^2} \leq t^{-1/3}\sqrt{\frac{\pi}{\eta}},
\end{aligned} \tag{5.34}$$

where

$$\eta = 2\sqrt{2\rho}t^{1/3}\chi_3 + 2^{2/3} r\chi_2 - \frac{2^{4/3}}{3} L t^{-1/3}\chi_1. \tag{5.35}$$

Since $\sqrt{2\rho}t^{1/3} \geq 2^{1/3}\min\left\{\sqrt{\tilde{L}-L}, \sqrt{\varepsilon}t^{1/3}\right\}$, the first term dominates the other two for $\tilde{L}$ and t large enough. Then, η is bounded from below by some positive constant η_0. Combining (5.30) and (5.34) we finally arrive at

$$\begin{aligned}
&\left|(2t)^{1/3} K^{\mathrm{conj}}(x_1, n_1; x_2, n_2)\right|\\
&\leq \frac{(2t)^{1/3}}{2\pi} t^{-1/3}\sqrt{\frac{\pi}{\eta_0}} e^{-\mathrm{const}_2(s_1+s_2)^{3/2}}\left(1+\mathcal{O}(e^{-c(\delta)t})\right) \leq e^{-(s_1+s_2)},
\end{aligned} \tag{5.36}$$

where the last inequality holds for t and $\tilde{L}$ large enough. $\square$

Bound on ϕ

Proof of Proposition 5.4. We start with the elementary representation of ϕ given in (4.31) and insert the scaling,

$$\begin{aligned}(2t)^{1/3}e^{x_2-x_1}&\frac{(x_1-x_2)^{n_2-n_1-1}}{(n_2-n_1-1)!}\\ &=(2t)^{1/3}e^{-2^{5/3}t^{2/3}r-(2t)^{1/3}s}\frac{(2^{5/3}t^{2/3}r+(2t)^{1/3}s)^{2^{5/3}t^{2/3}r-1}}{(2^{5/3}t^{2/3}r-1)!}\\ &=(1+\mathcal{O}(t^{-2/3}))\frac{2^{1/3}}{\sqrt{2\pi r}}\frac{e^{-(2t)^{1/3}s}}{1+2^{-4/3}t^{-1/3}s/r}(1+2^{-4/3}t^{-1/3}s/r)^{2^{5/3}t^{2/3}r}.\end{aligned} \tag{5.37}$$

Since the factorial depends on r and t only, the error from the Stirling formula is uniform in s. Introducing $\tilde{s}=2^{-4/3}t^{-1/3}s/r$ and some const_3 depending only on r we have

$$|(5.37)|\le \mathrm{const}_3\, e^{-(2t)^{1/3}s}(1+\tilde{s})^{2^{5/3}t^{2/3}r-1} \tag{5.38}$$

for t large enough.

Applying the inequality $1+x\le\exp\left(x-x^2/2+x^3/3\right)$ we arrive at

$$\begin{aligned}|(5.37)|&\le \mathrm{const}_3\, e^{-(2t)^{1/3}s+\left(2^{5/3}t^{2/3}r-1\right)\left(\tilde{s}-\tilde{s}^2/2+\tilde{s}^3/3\right)}\\ &=\mathrm{const}_3\, e^{-\frac{s^2}{4r}\left(1-\frac{2}{3}\tilde{s}\right)-\left(\tilde{s}-\tilde{s}^2/2+\tilde{s}^3/3\right)}.\end{aligned} \tag{5.39}$$

In the case $|\tilde{s}|\le 1$ we now use the basic inequality

$$e^{-a^2/b}\le e^b e^{-|a|} \tag{5.40}$$

to obtain the desired bound.

Inserting the scaling into the conditions $n_2>n_1$ and $x_1\ge x_2$ appearing in (4.31) results in $\tilde{s}\ge -1$. So we are left to prove the claim for $\tilde{s}>1$.

From (5.38) one obtains

$$|(5.37)|\le \mathrm{const}_3\,\tfrac{1}{2}e^{-\tilde{s}\cdot 2^{5/3}t^{2/3}r}(1+\tilde{s})^{2^{5/3}t^{2/3}r}=\mathrm{const}_3\,\tfrac{1}{2}\left((1+\tilde{s})e^{-\tilde{s}}\right)^{2^{5/3}t^{2/3}r}. \tag{5.41}$$

The elementary estimate $(1+\tilde{s})e^{-\tilde{s}}\le e^{-\tilde{s}/4}$ finally results in

$$|(5.37)|\le \mathrm{const}_3\,\tfrac{1}{2}e^{-\frac{1}{4}\tilde{s}\cdot 2^{5/3}t^{2/3}r}=\mathrm{const}_3\,\tfrac{1}{2}e^{-\frac{1}{4}(2t)^{1/3}s}\le \mathrm{const}_1\, e^{-s} \tag{5.42}$$

for $t\ge 32$. ∎

6 Tagged particle and slow decorrelations

In this section we want to prove the following result and then use it to show Theorem 2.5.

Theorem 6.1. *Let us fix a $\nu \in [0,1)$, choose any $\theta_1, \ldots, \theta_m \in [-t^\nu, t^\nu]$, $u_1, \ldots, u_m \in \mathbb{R}$ and define the rescaled random variables*

$$X_t^{\rm resc}(u_k, \theta_k) := -\frac{x_{[-t+u_k 2^{5/3} t^{2/3}+\theta_k]}(t+\theta_k) + 2\theta_k + u_k 2^{5/3} t^{2/3}}{(2t)^{1/3}}. \tag{6.1}$$

Then, for any $s_1, \ldots, s_m \in \mathbb{R}$ fixed, it holds

$$\lim_{t\to\infty} \mathbb{P}\left(\bigcap_{k=1}^m \{X_t^{\rm resc}(u_k,\theta_k) \leq s_k\}\right) = \mathbb{P}\left(\bigcap_{k=1}^m \{\mathcal{A}_1(u_k) \leq s_k\}\right). \tag{6.2}$$

As a corollary we have Theorem 2.5.

Proof of Theorem 2.5. This follows by taking $\theta_k = \tau_k 2^{5/3} t^{2/3}$ and $u_k = -\tau_k$ in Theorem 6.1. Indeed,

$$X_t^{\rm resc}(u_k, \theta_k) = -\frac{x_{[-t]}(t + \tau_k 2^{5/3} t^{2/3}) + \tau_k 2^{5/3} t^{2/3}}{(2t)^{1/3}}, \tag{6.3}$$

which by translation invariance by an integer has the same distribution as $X_t^{\rm tagged}(\tau_k)$ (the difference due to the integer value approximation is at most $1/(2t)^{1/3}$, which is asymptotically irrelevant). $\square$

For the proof of theorem 6.1 we need the following slow-decorrelation result.

Proposition 6.2. *For a $\nu \in [0,1)$, let us consider $\theta \in [-t^\nu, t^\nu]$. Then, for any $\varepsilon > 0$,*

$$\lim_{t\to\infty} \mathbb{P}\left(|x_{n+\theta}(t+\theta) - x_n(t) + 2\theta| \geq \varepsilon t^{1/3}\right) = 0. \tag{6.4}$$

Proof. Without loss of generality we consider $\theta \geq 0$. For $\theta < 0$ one just have to denote $\tilde t = t + \theta$ so that $\tilde t - \theta = t$ and the proof remains valid with t replaced by $\tilde t$. Recall that by definition we have

$$x_m(t) = -\max_{k\leq m}\{Y_{k,m}(t) - x_k(0)\}, \quad 1 \leq m \leq N, \tag{6.5}$$

with $x_k(0) = -k$. We also define

$$x_m^{\rm step}(t) = -\max_{1\leq k\leq m}\{Y_{k,m}(t)\} = -Y_{1,m}(t), \quad 1 \leq m \leq N. \tag{6.6}$$

First we need an inequality, namely

$$\begin{aligned}
-x_{n+\theta}(t+\theta) &= \max_{k\le n+\theta}\{k+Y_{k,n+\theta}(t+\theta)\} \ge \max_{k\le n}\{k+Y_{k,n+\theta}(t+\theta)\}\\
&= \max_{k\le n}\{k+\sup_{0\le s_{k+1}\le\ldots\le s_{n+\theta+1}=t+\theta}\sum_{i=k}^{n+\theta}(B_i(s_{i+1})-B_i(s_i))\}\\
&\ge \max_{k\le n}\{k+\sup_{\substack{0\le s_{k+1}\le\ldots\le s_{n+\theta+1}=t+\theta\\ \text{with } s_{n+1}=t}}\sum_{i=k}^{n+\theta}(B_i(s_{i+1})-B_i(s_i))\}\\
&= -x_n(t)-\widetilde{x}_\theta^{\rm step}(\theta),
\end{aligned} \tag{6.7}$$

with

$$\widetilde{x}_\theta^{\rm step}(\theta)=\sup_{t\le s_{n+2}\le\ldots\le s_{n+\theta}\le t+\theta}\sum_{i=n+1}^{n+\theta}(B_i(s_{i+1})-B_i(s_i)). \tag{6.8}$$

Remark that $x_n(t)$ and $\widetilde{x}_\theta^{\rm step}(\theta)$ are independent and $\widetilde{x}_\theta^{\rm step}(\theta)\stackrel{d}{=}x_\theta^{\rm step}(\theta)$.

From Theorem 2.4 we have

$$\begin{aligned}
\chi_1(t) &:= -\frac{x_n(t)+n+t}{(2t)^{1/3}} \stackrel{D}{\Longrightarrow} D_1,\\
\chi_2(t) &:= -\frac{x_{n+\theta}(t+\theta)+n+t+2\theta}{(2t)^{1/3}} \stackrel{D}{\Longrightarrow} D_1,
\end{aligned} \tag{6.9}$$

with $D_1(s)=F_1(2s)$, F_1 being the GOE Tracy-Widom distribution function [32]. Further, it is known by the connection with the GUE random matrices [4, 31], that

$$-\frac{\widetilde{x}_\theta^{\rm step}(\theta)+2\theta}{(2\theta)^{1/3}} \stackrel{D}{\Longrightarrow} D_2, \tag{6.10}$$

where $D_2(s)=F_2(2^{1/3}s)$, F_2 being the GUE Tracy-Widom distribution function [31]. Therefore,

$$\chi_3(t) := -\frac{\widetilde{x}_\theta^{\rm step}(\theta)+2\theta}{(2t)^{1/3}} \stackrel{D}{\Longrightarrow} 0, \tag{6.11}$$

by (6.10) and $\theta/t\to 0$. By (6.9) and (6.11) we have $\chi_1(t)+\chi_3(t)\stackrel{D}{\Longrightarrow} D_1$. Further, (6.7) implies that

$$\chi_2(t)=\chi_1(t)+\chi_3(t)+R_t \tag{6.12}$$

for some random variable $R_t\ge 0$. Since both $\chi_2(t)$ and $\chi_1(t)+\chi_3(t)$ converges in distribution to D_1 and $R_t\ge 0$, by Lemma 4.1 of [5] (reported below) we have $R_t\to 0$ in probability as $t\to\infty$. This together with (6.11) leads to $\chi_2(t)-\chi_1(t)\to 0$ in probability, which is the rescaled version of our statement. $\square$

Lemma 6.3 (Lemma 4.1 of [5]). *Consider two sequences of random variables* $\{X_n\}$ *and* $\{\tilde{X}_n\}$ *such that for each* n, X_n *and* $\tilde{X}_n$ *are defined on the same probability space* Ω_n. *If* $X_n \geq \tilde{X}_n$ *and* $X_n \Rightarrow D$ *as well as* $\tilde{X}_n \Rightarrow D$ *then* $X_n - \tilde{X}_n$ *converges to zero in probability. Conversely if* $\tilde{X}_n \Rightarrow D$ *and* $X_n - \tilde{X}_n$ *converges to zero in probability then* $X_n \Rightarrow D$ *as well.*

Finally we come to the proof of Theorem 6.1.

Proof of theorem 6.1. Let us define the random variables

$$\Xi_k := X_t^{\mathrm{resc}}(u_k, \theta_k) - X_t^{\mathrm{resc}}(u_k, 0). \tag{6.13}$$

such that

$$\mathbb{P}\left(\bigcap_{k=1}^{m} \{X_t^{\mathrm{resc}}(u_k, \theta_k) \leq s_k\}\right) = \mathbb{P}\left(\bigcap_{k=1}^{m} \{X_t^{\mathrm{resc}}(u_k, 0) + \Xi_k \leq s_k\}\right). \tag{6.14}$$

The slow-decorrelation result (Proposition 6.2) implies $\Xi_k \to 0$ in probability as $t \to \infty$. Introducing $\varepsilon > 0$ we can use inclusion-exclusion to decompose

$$(6.14) = \mathbb{P}\left(\bigcap_{k=1}^{m} \{X_t^{\mathrm{resc}}(u_k, 0) + \Xi_k \leq s_k\} \cap \{|\Xi_k| \leq \varepsilon\}\right) + \sum_j \mathbb{P}\left(R_j\right), \tag{6.15}$$

where the sum on the right hand side is finite and each R_j satisfies $R_j \subset \{|\Xi_k| > \varepsilon\}$ for at least one k. Using the limit result from Theorem 2.4,

$$\lim_{t\to\infty} \mathbb{P}\left(\bigcap_{k=1}^{m} \{X_t^{\mathrm{resc}}(u_k, 0) \leq s_k\}\right) = \mathbb{P}\left(\bigcap_{k=1}^{m} \{\mathcal{A}_1(u_k) \leq s_k\}\right), \tag{6.16}$$

leads to

$$\begin{aligned} \limsup_{t\to\infty} \mathbb{P}\left(\bigcap_{k=1}^{m} \{X_t^{\mathrm{resc}}(u_k, \theta_k) \leq s_k\}\right) &\leq \mathbb{P}\left(\bigcap_{k=1}^{m} \{\mathcal{A}_1(u_k) \leq s_k + \varepsilon\}\right), \\ \liminf_{t\to\infty} \mathbb{P}\left(\bigcap_{k=1}^{m} \{X_t^{\mathrm{resc}}(u_k, \theta_k) \leq s_k\}\right) &\geq \mathbb{P}\left(\bigcap_{k=1}^{m} \{\mathcal{A}_1(u_k) \leq s_k - \varepsilon\}\right). \end{aligned} \tag{6.17}$$

Since the joint distribution function of the Airy$_1$ process is continuous in $s_1, \ldots, s_m$, we can take the limit $\varepsilon \to 0$ and obtain

$$\lim_{t\to\infty} \mathbb{P}\left(\bigcap_{k=1}^{m} \{X_t^{\mathrm{resc}}(u_k, \theta_k) \leq s_k\}\right) = \mathbb{P}\left(\bigcap_{k=1}^{m} \{\mathcal{A}_1(u_k) \leq s_k\}\right). \tag{6.18}$$

$\square$

A Bounds on the Lambert W function

Lemma A.1 (Path of Γ^ρ and its image under φ)**.** *For any $\rho \in [0,1)$ the contour $\Gamma^\rho = \{\gamma(\tau) = L_{\lfloor\tau\rfloor}(-e^{2\pi i\tau-1}(1-\rho)), \tau \in \mathbb{R}\setminus[0,1)\}$, with $L_k(z)$ being the k-th branch of the Lambert W function, satisfies*

1. *Γ^ρ crosses the real line at one unique $z_0 \leq -1$.*
2. *$z_0 = -1 - \sqrt{2\rho} + \mathcal{O}(\rho)$.*
3. *$\mathrm{Re}(z) < Re(z_0)$ for all $z \in \Gamma^\rho \setminus \{z_0\}$.*
4. *$\mathrm{Re}(z)$ is monotone along each part of $\Gamma^\rho \setminus \{z_0\}$.*
5. *$\left|\frac{d}{d\tau}\mathrm{Re}(\gamma(\tau))\right| \leq 3\pi$ for $|\tau| \geq 2$.*
6. *Γ^ρ has asymptotic angle $\pm\pi/2$.*

In addition,

7. *$\varphi(z)$ crosses the real line infinitely often at the two unique points $z_0^* = \varphi(z_0) \geq -1$ and $z_1^* > z_0^*$ when z moves along Γ^ρ.*
8. *$z_0^* = -1$ iff $\rho = 0$.*
9. *$\mathrm{Re}(\varphi(z)) > Re(\varphi(z_0))$ for all $z \in \Gamma^\rho$ with $\varphi(z) \neq \varphi(z_0)$.*
10. *$\mathrm{Re}(z)$ is monotone along each part of $\varphi\left(\Gamma^\rho\right) \setminus \{z_0^*, z_1^*\}$.*

Lemma A.2 (Behaviour of f_3 along Γ^ρ)**.**
The function $f_3(z) = (z+1)^2 - (\varphi(z)+1)^2$ satisfies

1. *$f_3\left(\Gamma^\rho\right)$ crosses the real line at one unique $\widehat{z_0} = f_3(z_0)$, where z_0 is given as in Lemma A.1.*
2. *$\widehat{z_0} = 0$ if $\rho = 0$.*
3. *$\mathrm{Re}(f_3(z)) < Re(f_3(z_0))$ for all $z \in \Gamma^\rho \setminus \{z_0\}$*
4. *$\mathrm{Re}(f_3(z))$ is monotone along each part of $f_3\left(\Gamma^\rho\right) \setminus \{\widehat{z_0}\}$.*
5. *$\left|\frac{d}{d\tau}\mathrm{Re}(f_3(\gamma(\tau)))\right| \geq 4\pi^2|\tau|$ for $|\tau| \geq 5$.*

Proof of Lemma A.1. Write $\gamma(\tau) = L_{\lfloor \tau \rfloor}\left(-e^{2\pi \mathrm{i}\tau - 1}(1-\rho)\right)$. The branch cut of the Lambert function is done in such a way that

$$\begin{aligned} (2k-2)\pi \le \mathrm{Im}\left(L_k(z)\right) &\le (2k+1)\pi && \text{for } k > 0, \\ -\pi \le \mathrm{Im}\left(L_k(z)\right) &\le \pi && \text{for } k = 0, \\ (2k-1)\pi \le \mathrm{Im}\left(L_k(z)\right) &\le (2k+2)\pi && \text{for } k < 0, \end{aligned} \tag{A.1}$$

see also Figure 4 of [13]. The curve $\gamma(\tau)$ changes branches every time when $\tau \in \mathbb{Z}$, but since at each jump point the function $-e^{2\pi \mathrm{i}\tau-1}(1-\rho)$ meets the line $(-\infty, 0]$, which is the location of the branch cut, the function $\gamma(\tau)$ is in fact continuous at these points, and Γ^ρ therefore connected.

The function $L_k(z)$ satisfies the differential identity ((3.2) in [13])

$$L_k'(z) = \frac{L_k(z)}{z(1+L_k(z))}. \tag{A.2}$$

By elementary calculus we therefore have

$$\gamma'(\tau) = \frac{d}{d\tau}\gamma(\tau) = 2\pi \mathrm{i}\left(1 - \frac{1}{\gamma(\tau)+1}\right). \tag{A.3}$$

From the structure of the branches one has $\lim_{\tau \searrow 1}\gamma(\tau) = \lim_{\tau \nearrow 0}\gamma(\tau) \le -1$. This limit is our z_0. The image of $-1/e$ under L_{-1}, L_0 and L_1 is -1, so $z_0 = -1$, which corresponds to $\rho = 0$.

Consider first $\tau > 1$. Since all the involved branches lie in the upper half plane, we have $\mathrm{Im}(\gamma(\tau)) > 0$. Additionally, the fact that the transformations $z \mapsto z+1$ and $z \mapsto -z^{-1}$ map the upper half plane onto itself implies by (A.3) the inequality $\mathrm{Re}(\gamma'(\tau)) < 0$. This in turn implies $\mathrm{Re}(\gamma(\tau)) \le -1$ which can be inserted in (A.3), leading to $\mathrm{Im}(\gamma'(\tau)) \ge 2\pi$. So for $\tau > 1$ the curve $\gamma(\tau)$ is moving monotone north-west in τ. Analogously we can argue that $\gamma(\tau)$ is moving monotone south-west in $|\tau|$ for $\tau < 0$.

Thereby the Claims 1, 3 and 4 are settled. To see Claim 6, we notice that for large $|\tau|$ also $|\gamma(\tau)|$ is large and the fraction in (A.3) tends to zero, resulting in $\gamma'(\tau) \to 2\pi \mathrm{i}$.

By (A.1) we have $|\mathrm{Im}(\gamma(\tau))| \ge 2\pi$ for all $|\tau| \ge 2$. Inserting this in (A.3) results in $|\gamma'(\tau)| \le 2\pi(1 + 1/2\pi) \le 3\pi$ and consequently Claim 5.

Again by [13] the series (5.19) is the expansion of L_1 or L_{-1} respectively, when inserting $p(z) = -\sqrt{2(ez+1)}$ instead of $p(z) = \sqrt{2(ez+1)}$. Which branch one gets depends on the sign of $\mathrm{Im}(z)$. Claim 2 follows.

For the corresponding statements on $\varphi(z)$ first notice the identity

$$\varphi(\gamma(\tau)) = L_0\left(\gamma(\tau)e^{\gamma(\tau)}\right) = L_0\left(-e^{2\pi \mathrm{i}\tau-1}(1-\rho)\right) = \gamma(\tau - \lfloor \tau \rfloor), \tag{A.4}$$

from which it is clear that $\varphi(\gamma(\tau))$ is periodic in τ. We can therefore reduce our considerations to $\tau \in [0, 1)$.

By (4.4) of [13], the principal branch of the Lambert W function is given by

$$\{a + \mathrm{i}b \in \mathbb{C}, a + b\cot(b) > 0 \text{ and } -\pi < b < \pi\}. \tag{A.5}$$

So regarding points of the principal branch, by

$$\operatorname{sgn}\operatorname{Im}\big((a+\mathrm{i}b)e^{a+\mathrm{i}b}\big) = \operatorname{sgn}(a\sin b + b\cos b) = \operatorname{sgn} b, \tag{A.6}$$

the function $z \mapsto ze^z$ preserves the sign of the imaginary part. But then its inverse function L_0 must do the same. Consequently, $\operatorname{Im}(\gamma(\tau)) < 0$ for $0 < \tau < 1/2$ and $\operatorname{Im}(\gamma(\tau)) > 0$ for $1/2 < \tau < 1$. In the same way as before this leads through (A.3) to $\operatorname{Re}(\gamma'(\tau)) > 0$ for $0 < \tau < 1/2$ and $\operatorname{Re}(\gamma'(\tau)) < 0$ for $1/2 < \tau < 1$. This settles Claim 7, 9 and 10 with $z_1^* = \gamma(1/2)$.

The equation $z_0^* = -1$ is equivalent to $L_0\left(-e^{-1}(1-\rho)\right) = -1$, which clearly holds for $\rho = 0$, and by injectivity in the principal branch for no other ρ. □

Proof of Lemma A.2. With $\{\tau\} = \tau - \lfloor\tau\rfloor$ being the fractional part of τ we write using (A.4)

$$f_3(\gamma(\tau)) = (\gamma(\tau)+1)^2 - (\gamma(\{\tau\})+1)^2. \tag{A.7}$$

Differentiating with respect to τ results in

$$\frac{d}{d\tau} f_3(\gamma(\tau)) = 4\pi\mathrm{i}\big(\gamma(\tau) - \gamma(\{\tau\})\big). \tag{A.8}$$

By Lemma A.1 we know that $\operatorname{Re}\big(\gamma(\tau) - \gamma(\{\tau\})\big) < 0$ which gives $\operatorname{Im}\frac{d}{d\tau} f_3(\gamma(\tau)) < 0$. The monotonicity of the imaginary part entails the uniqueness in Claim 1.

Regarding the real part, first notice that for $\tau \nearrow 0$ or $\tau \searrow 1$, $\operatorname{Im}\big(\gamma(\tau) - \gamma(\{\tau\})\big)$ tends to zero, resulting in $\operatorname{Re}\frac{d}{d\tau} f_3(\gamma(\tau)) = 0$. By differentiating a second time we arrive at

$$\frac{d^2}{d\tau^2} f_3(\gamma(\tau)) = 8\pi^2 \left(\frac{1}{\gamma(\tau)+1} - \frac{1}{\gamma(\{\tau\})+1}\right). \tag{A.9}$$

From Lemma A.1 the right hand side has negative real part. Integrating results in the desired monotonicity and therefore Claim 3 and 4.

By (A.1) we have $|\operatorname{Im}(\gamma(\tau))| \geq 2\pi(|\tau| - 2)$ for all $\tau \in \mathbb{R}$. Combining this with $|\operatorname{Im}(\gamma(\{\tau\}))| \leq \pi$ results in $|\operatorname{Im}(\gamma(\tau) - \gamma(\{\tau\}))| \geq \pi|\tau|$ for $|\tau| \geq 5$. With (A.8), Claim 5 follows.

Claim 2 is a corollary of Lemma A.1, Claim 2. □

B Correlation kernel for determinantal measures

Here are two useful results from [9]. They are written for the continuous case. The proofs are identical to the discrete case.

Lemma B.1 (See Lemma 3.3 of [9]). *Let f an antisymmetric function of $\{x_1^N,\ldots,x_N^N\}$. Then, whenever f has enough decay to make the sums finite,*

$$\int_{\mathcal{D}} f(x_1^N,\ldots,x_N^N) \prod_{2\le i\le j\le N} \mathrm{d}x_i^j = \int_{\mathcal{D}'} f(x_1^N,\ldots,x_N^N) \prod_{2\le i\le j\le N} \mathrm{d}x_i^j \tag{B.1}$$

where

$$\begin{aligned}\mathcal{D} &= \{x_i^j, 2\le i\le j\le N | x_i^j > x_i^{j+1}, x_i^j \ge x_{i-1}^{j-1}\},\\ \mathcal{D}' &= \{x_i^j, 2\le i\le j\le N | x_i^j \ge x_{i-1}^{j-1}\},\end{aligned} \tag{B.2}$$

and the positions $x_1^1 > x_1^2 > \ldots > x_1^N$ being fixed.

Lemma B.2 (See Lemma 3.4 of [9]). *Assume we have a signed measure on $\{x_i^n, n=1,\ldots,N, i=1,\ldots,n\}$ given in the form,*

$$\frac{1}{Z_N}\prod_{n=1}^{N-1}\det[\phi_n(x_i^n,x_j^{n+1})]_{1\le i,j\le n+1}\det[\Psi_{N-i}^N(x_j^N)]_{1\le i,j\le N}, \tag{B.3}$$

where x_{n+1}^n are some "virtual" variables and Z_N is a normalization constant. If $Z_N\neq 0$, then the correlation functions are determinantal.

To write down the kernel we need to introduce some notations. Define

$$\phi^{(n_1,n_2)}(x,y) = \begin{cases}(\phi_{n_1} * \cdots * \phi_{n_2-1})(x,y), & n_1 < n_2,\\ 0, & n_1\ge n_2,\end{cases}. \tag{B.4}$$

where $(a*b)(x,y) = \int_{\mathbb{R}} \mathrm{d}z\, a(x,z)b(z,y)$, and, for $1\le n<N$,

$$\Psi_{n-j}^n(x) := (\phi^{(n,N)} * \Psi_{N-j}^N)(y), \quad j=1,\ldots,N. \tag{B.5}$$

Set $\phi_0(x_1^0,x)=1$. Then the functions

$$\{(\phi_0 * \phi^{(1,n)})(x_1^0,x),\ldots,(\phi_{n-2} * \phi^{(n-1,n)})(x_{n-1}^{n-2},x),\phi_{n-1}(x_n^{n-1},x)\} \tag{B.6}$$

are linearly independent and generate the n-dimensional space V_n. Define a set of functions $\{\Phi_j^n(x), j=0,\ldots,n-1\}$ spanning V_n defined by the orthogonality relations

$$\int_{\mathbb{R}} \mathrm{d}x\,\Phi_i^n(x)\Psi_j^n(x) = \delta_{i,j} \tag{B.7}$$

for $0 \leq i, j \leq n-1$.

Further, if $\phi_n(x_{n+1}^n, x) = c_n \Phi_0^{(n+1)}(x)$*, for some* $c_n \neq 0$*,* $n = 1, \ldots, N-1$*, then the kernel takes the simple form*

$$K(n_1, x_1; n_2, x_2) = -\phi^{(n_1,n_2)}(x_1, x_2) + \sum_{k=1}^{n_2} \Psi_{n_1-k}^{n_1}(x_1) \Phi_{n_2-k}^{n_2}(x_2). \tag{B.8}$$

References

[1] R.F. Anderson and S. Orey, *Small random perturbation of dynamical systems with reflecting boundary*, Nagoya Math. J. **60** (1976), 189–216.

[2] T.C. Banwell and A. Jayakumar, *Exact analytical solution for current flow through diode with series resistance*, Electronics Letters **36** (2000), 291–292.

[3] D.A. Barry, J.-Y. Parlange, L. Li, H. Prommer, C.J. Cunningham, and F. Stagnitti, *Analytical approximations for real values of the Lambert W-function*, Mathematics and Computers in Simulation **53** (2000), 95–103.

[4] Y. Baryshnikov, *GUEs and queues*, Probab. Theory Relat. Fields **119** (2001), 256–274.

[5] G. Ben Arous and I. Corwin, *Current fluctuations for TASEP: a proof of the Prähofer-Spohn conjecture*, Ann. of Probab. **39** (2011), 104–138.

[6] A. Borodin and I. Corwin, *Macdonald processes*, Probab. Theory Relat. Fields (online first) (2013).

[7] A. Borodin and P.L. Ferrari, *Large time asymptotics of growth models on space-like paths I: PushASEP*, Electron. J. Probab. **13** (2008), 1380–1418.

[8] A. Borodin, P.L. Ferrari, and M. Prähofer, *Fluctuations in the discrete TASEP with periodic initial configurations and the Airy*$_1$ *process*, Int. Math. Res. Papers **2007** (2007), rpm002.

[9] A. Borodin, P.L. Ferrari, M. Prähofer, and T. Sasamoto, *Fluctuation Properties of the TASEP with Periodic Initial Configuration*, J. Stat. Phys. **129** (2007), 1055–1080.

[10] C.-C. Chang and H.-T. Yau, *Fluctuations of one dimensional Ginzburg-Landau models in nonequilibrium*, Comm. Math. Phys. **145** (1992), 209–234.

[11] Y. Chen and K.L. Moore, *Analytical stability bound for delayed second-order systems with repeating poles using Lambert function W*, Automatica **38** (2002), 891–895.

[12] R.M. Corless, G.H. Gonnet, D.E.G. Hare, and D.J. Jeffrey, *Lambert's W Function in Maple*, The Maple Technical Newsletter **9** (1993), 12–22.

[13] R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey, and D.E. Knuth, *On the Lambert W Function*, Advances in Computational Mathematics **5** (1996), 329–359.

[14] R.M. Corless, D.J. Jeffrey, and D.E. Knuth, *A sequence of series for the Lambert W function*, Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation, 1997, pp. 197–204.

[15] R.M. Corless, D.J. Jeffrey, and S.R. Valluri, *Some applications of the Lambert W function to physics*, Canadian Journal of Physics **78** (2000), 823–831.

[16] I. Corwin, P.L. Ferrari, and S. Péché, *Universality of slow decorrelation in KPZ models*, Ann. Inst. H. Poincaré Probab. Statist. **48** (2012), 134–150.

[17] P.L. Ferrari, *Slow decorrelations in KPZ growth*, J. Stat. Mech. (2008), P07022.

[18] P. J. Forrester and T. Nagao, *Determinantal correlations for classical projection processes*, J. Stat. Mech. (2011), P08011.

[19] T.E. Harris, *Diffusion with "collisions" between particles*, J. Appl. Probab. (1965), no. 2, 323–338.

[20] J.M. Harrison and R.J. Williams, *Multidimensional reflected Brownian motions having exponential stationary distributions*, Ann. Probab. (1987), 115–137.

[21] A. Jain and A. Kapoor, *Exact analytical solutions of the parameters of real solar cells using Lambert W-function*, Solar Energy Materials and Solar Cells **81** (2004), 269–277.

[22] I. Karatzas, S. Pal, and M. Shkolnikov, *Systems of Brownian particles with asymmetric collisions*, arXiv:1210.0259 (2012).

[23] M. Kardar, G. Parisi, and Y.Z. Zhang, *Dynamic scaling of growing interfaces*, Phys. Rev. Lett. **56** (1986), 889–892.

[24] M. Ledoux, *Deviation inequalities on largest eigenvalues*, Geometric Aspects of Functional Analysis, Lecture Notes in Mathematics, vol. 1910, Springer Berlin / Heidelberg, 2007, pp. 167–219.

[25] N. O'Connell, *Directed polymers and the quantum Toda lattice*, Ann. Probab. **40** (2012), 437–458.

[26] N. O'Connell and M. Yor, *Brownian analogues of Burke's theorem*, Stoch. Proc. Appl. **96** (2001), 285–304.

[27] T. Sasamoto, *Spatial correlations of the 1D KPZ surface on a flat substrate*, J. Phys. A **38** (2005), L549–L556.

[28] T. Sasamoto and M. Wadati, *Determinantal form solution for the derivative nonlinear Schrödinger type model*, J. Phys. Soc. Jpn **67** (1998), 784–790.

[29] G.M. Schütz, *Exact solution of the master equation for the asymmetric exclusion process*, J. Stat. Phys. **88** (1997), 427–445.

[30] A.V. Skorokhod, *Stochastic equations for diffusions in a bounded region*, Theory Probab. Appl. (1961), 264–274.

[31] C.A. Tracy and H. Widom, *Level-spacing distributions and the Airy kernel*, Comm. Math. Phys. **159** (1994), 151–174.

[32] C.A. Tracy and H. Widom, *On orthogonal and symplectic matrix ensembles*, Comm. Math. Phys. **177** (1996), 727–754.

[33] S.R.S. Varadhan and R.J. Williams, *Brownian motion in a wedge with oblique reflection*, Comm. Pure Appl. Math. (1984), 405–443.

[34] J. Warren, *Dyson's Brownian motions, intertwining and interlacing*, Electron. J. Probab. **12** (2007), 573–590.

[35] T. Weiss, *Scaling behaviour of the directed polymer model of Baryshnikov and O'Connell at zero temperature*, Bachelor Thesis, TU-München (2011).