Large time asymptotics of growth models on space-like paths II: PNG and parallel TASEP

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Abstract

We consider the polynuclear growth (PNG) model in 1+1 dimension with flat initial condition and no extra constraints. The joint distributions of surface height at finitely many points at a fixed time moment are given as marginals of a signed determinantal point process. The long time scaling limit of the surface height is shown to coincide with the Airy_1 process. This result holds more generally for the observation points located along any space-like path in the space-time plane. We also obtain the corresponding results for the discrete time TASEP (totally asymmetric simple exclusion process) with parallel update.

1 Introduction

The main focus of this work is a stochastic growth model in $1+1$ dimensions, called the polynuclear growth (PNG) model. It belongs to the KPZ (Kardar-Parisi-Zhang [19]) universality class and it can be described as follows (see Figure 1). At time t, the surface is described by an integer-valued height function $x \mapsto h(x,t) \in \mathbb{Z}$, $x \in \mathbb{R}, t \in \mathbb{R}_+$. It thus consists of up-steps (⌐) and down-steps (¬). The dynamics has a deterministic and a stochastic part:

(a) up- (down-) steps move to the left (right) with unit speed and disappear upon colliding,

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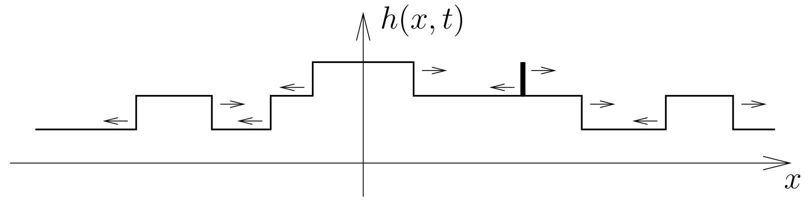

Figure 1: Illustration of the PNG height and its dynamics. The bold vertical piece is a nucleation. The arrows indicate the movements of the steps. A Java animation of the PNG dynamics is available at [9].

(b) pairs of up- and down- steps (nucleations) are randomly added on the surface with some given intensity.

The up- and down-steps of the nucleations then spread out with unit speed according to (a). The PNG model can be interpreted in several different ways, see [11] for a review.

On a macroscopic scale the surface of the PNG model grows deterministically, i.e., $\lim_{t\to\infty} t^{-1}h(\xi t,t) = H(\xi)$ is a non-random function. However, on a mesoscopic scale fluctuations grow in time. This is called roughening in statistical physics and extensive numerical studies have been made [3]. Since the PNG model is in the KPZ universality class, the fluctuation of the surface height is expected to live on a $t^{1/3}$ scale and non-trivial correlations are to be seen on a $t^{2/3}$ scale. Therefore, to have an interesting large time limit, we have to rescale the surface height as

$$\frac{h(ut^{2/3},t) - tH(ut^{-1/3})}{t^{1/3}}. \tag{1.1}$$

One of the initial conditions most natural and used for numerical simulations for PNG is the flat initial condition, i.e., $h(x,0) = 0$ for all $x \in \mathbb{R}$. We consider nucleations occurring with translation-invariant intensity. In other words, the nucleation events form a Poisson process with constant intensity in the space-time upper half-plane. We refer to the PNG model with such initial condition as *flat PNG*. In this case, by mapping the flat PNG to a point-to-line last passage directed percolation model it was proven [2, 21, 22] that the one-point distribution is, in the $t \to \infty$ limit, the GOE Tracy-Widom distribution F_1, first discovered in random matrix theory [27]. However, no information on joint height distributions at several points has been previously known.

New Results. The main results of this paper are precisely the computation and asymptotic analysis of these joint distributions. In particular, we prove the convergence of the height rescaled as in (1.1) to the Airy_1 process in the $t \to \infty$ limit (see Section 2.2 for a definition of the process). The Airy_1 process has been discovered in the context of the asymmetric exclusion process [5–7, 24]. Our result, stated in Theorem 6, is obtained by first determining an expression for the joint distributions for finite time t (Proposition 4) and then taking the appropriate scaling limit.

Proposition 4 is actually just a particular case of Theorem 5, where we determine joint distributions along any space-like paths (as in Minkowski diagram). Space-like paths are described later in details, for now one can keep in mind the special case of fixed time t. The scaling limit is analyzed at this level of generality, thus Theorem 6 holds for any space-like paths. In contrast to previous works on the subject, our approach does not rely on the so-called RSK correspondence (RSK for Robinson-Schensted-Knuth), which was successfully applied for corner growth models, but does not seem to be well suited for the flat growth.

Our real interest is the flat PNG model. However, one of the new key ingredient in our solution is a precise connection with the totally asymmetric simple exclusion process (TASEP) in discrete time with parallel update. In fact, to get the results for the flat PNG, we first consider a discrete time version of it, the Gates-Westcott dynamics [12, 23]. This model is closely related to the TASEP in discrete time with parallel update and alternating initial conditions. The corresponding results for the TASEP are Theorem 1 for the joint distributions along space-like paths, and Theorem 3 for the convergence to the Airy_1 process in the scaling limit. For the TASEP, the extreme situations of space-like paths are positions of different particles at a fixed time and positions of a fixed particle at different time moments (tagged particle). The space-like extension for TASEP is based on the previous paper [4].

Previous works on PNG. Another type of initial conditions for the PNG model has been analyzed before. It is the corner growth geometry, where nucleations occur only inside the cone $\{|x| \leq t\}$. The limit shape H is a semi-circle, and the model is called *PNG droplet*. In this geometry, the limit process has been obtained in [23]; it is known as Airy_2 process (previously called simply Airy process). The approach uses an extension to a multilayer model (inherited from the RSK construction), see [16, 23]. The multilayer method was also used in other related models [8, 13, 14, 17, 18, 26]. Also, for the flat PNG it was used to connect the associated point process at a single position and the point process of GOE eigenvalues [10]. Results on the behavior for the PNG droplet along space-like paths can be found in [8]. For a very brief description of the previously known results on TASEP fluctuations see the introductions of [4, 25].

Outline

In section 2, we introduce our models and state the results. In section 3, we give an expression of the transition probability of the discrete TASEP as a marginal of a determinantal signed point process. In section 4 the Fredholm determinant expression for the joint distributions is obtained. The argument substantially relies on the algebraic techniques of [4]. In section 5, we consider the scaling limit of the parallel TASEP. In section 6, the continuous time PNG model is considered. In section 7, we consider the scaling limit for the continous PNG model.

Acknowledgments

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2 Models and results

We start from the discrete time TASEP with parallel update. Then we will make the connection with a discrete version of the PNG, from which the continuous time PNG is obtained.

2.1 Discrete time TASEP with parallel update

We consider discrete time TASEP with parallel update and alternating initial conditions, i.e., particle i has initial position $x_i(0) = -2i$, $i \in \mathbb{Z}$. At each time step, each particle hops to its right neighbor site with probability $p = 1 - q$ provided that the site is empty. The particle positions at time t is denoted by $x_i(t), i \in \mathbb{Z}$.

The dynamics of a particle depends only on particles on its right. This fact allows us to determine the joint distributions of particle positions also for different times, but restricted to "space-like paths". To define what we mean with "space-like paths", we consider a sequence of couples (n_i, t_i), where n_i is the number of the particle and t_i is the time when this particle is observed.

On such couples we define a partial order $\prec$, given by

$$(n_i,t_i)\prec(n_j,t_j) \text{ if } n_j\geq n_i, t_j\leq t_i, \text{ and the two couples are not identical.} \tag{2.1}$$

A space-like path is a sequence of ordered couples, namely,

$$\mathcal{S}=\{(n_k,t_k),k=1,2,\ldots|(n_k,t_k)\prec(n_{k+1},t_{k+1})\}. \tag{2.2}$$

The reason of the name "space-like" will be clear in the large time limit, where everything becomes continuous. Then space-like is the same concept as in the Minkowski diagram. The border cases for space-like paths are fixed time ($t_i\equiv t,\forall i$) and fixed particle number ($n_i\equiv n,\forall i$). The next theorem proved in section 4 concerns the joint distributions of particle positions.

Theorem 1. *Let particle with label i start at $x_i(0)=-2i$, $i\in\mathbb{Z}$. Consider a space-like path $\mathcal{S}$. For any given m, the joint distribution of the positions of the first m points in $\mathcal{S}$ is given by*

$$\mathbb{P}\Big(\bigcap_{k=1}^{m}\{x_{n_k}(t_k)\geq a_k\}\Big)=\det(\mathbb{1}-\chi_a^{(-)}K\chi_a^{(-)})_{\ell^2(\{(n_1,t_1),\ldots,(n_m,t_m)\}\times\mathbb{Z})} \tag{2.3}$$

where $\chi_a^{(-)}((n_k,t_k),x)=\mathbb{1}(x<a_k)$. The kernel K_t is given by

$$\begin{aligned} K((n_1,t_1),x_1;(n_2,t_2),x_2) &= -\phi^{((n_1,t_1),(n_2,t_2))}(x_1,x_2)\mathbb{1}_{[(n_1,t_1)\prec(n_2,t_2)]}\\ &+ \widetilde{K}((n_1,t_1),x_1;(n_2,t_2),x_2), \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} &\widetilde{K}((n_1,t_1),x_1;(n_2,t_2),x_2)\\ &= \frac{-1}{2\pi\mathrm{i}}\oint_{\Gamma_0}\mathrm{d}z\frac{(1+z)^{x_2+n_1+n_2}}{(-z)^{x_1+n_1+n_2+1}}\frac{(1-p)^{t_1-2n_1-x_1}}{(1+pz)^{t_1+t_2+1-(x_1+n_1+n_2)}}, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} &\phi^{((n_1,t_1),(n_2,t_2))}(x_1,x_2)\\ &= \frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_{-1}}\mathrm{d}w\frac{(1+pw)^{t_1-t_2}}{(1+w)^{x_1-x_2+1}}\left(\frac{-w}{(1+w)(1+pw)}\right)^{n_1-n_2} \end{aligned} \tag{2.6}$$

where Γ_0 (resp. Γ_{-1}) is any simple loop, anticlockwise oriented, with 0 (resp. -1) being the unique pole of the integrand inside the contour.

Remark. In the limit $p\to0$ under the time scaling by p^{-1} the discrete time TASEP converges to the continuous time TASEP, and Theorem 1 turns into a special case of Proposition 3.6 of [4], where a more general continuous time model called PushASEP was considered.

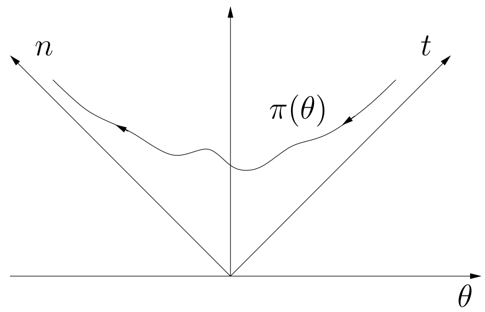

Figure 2: An example of a space-like path $\pi(\theta)$. Its slope is, in absolute value, at most 1.

2.2 Airy$_1$ process and scaling limit

Starting from Theorem 1 we can analyze large time limits. The limit process is the so-called Airy$_1$ process introduced in [6, 24], which we recall here.

Definition 2 (The Airy$_1$ process)**.** *Define the extended kernel,*

$$\begin{aligned} K_{\mathcal{A}_1}(\tau_1,\xi_1;\tau_2,\xi_2) &= -\frac{1}{\sqrt{4\pi(\tau_2-\tau_1)}}\exp\left(-\frac{(\xi_2-\xi_1)^2}{4(\tau_2-\tau_1)}\right)\mathbb{1}(\tau_2>\tau_1) \\ &+\mathrm{Ai}(\xi_1+\xi_2+(\tau_2-\tau_1)^2)\exp\left((\tau_2-\tau_1)(\xi_1+\xi_2)+\frac{2}{3}(\tau_2-\tau_1)^3\right). \end{aligned} \tag{2.7}$$

The Airy$_1$ process, $\mathcal{A}_1$, *is the process with m-point joint distributions at $\tau_1<\tau_2<\ldots<\tau_m$ given by the Fredholm determinant,*

$$\mathbb{P}\left(\bigcap_{k=1}^m\{\mathcal{A}_1(\tau_k)\le s_k\}\right)=\det(\mathbb{1}-\chi_s K_{\mathcal{A}_1}\chi_s)_{L^2(\{\tau_1,\ldots,\tau_m\}\times\mathbb{R})}, \tag{2.8}$$

where $\chi_s(\tau_k,x)=\mathbb{1}(x>s_k)$.

Theorem 1 allows us to analyze joint distributions of particle positions for situations spanning between fixed time and fixed particle number (the tagged particle problem). One way to parametrize such situations is via a space-like path. We thus consider an arbitrary smooth function π satisfying

$$|\pi'(\theta)|\le 1 \text{ and } \pi(\theta)+\theta>0, \tag{2.9}$$

see Figure 2. The requirement $\pi(\theta)+\theta>0$ reflects $t>0$. Then, we choose couples of (t,n) on $\{((\pi(\theta)+\theta)T,(\pi(\theta)-\theta)T),\theta\in\mathbb{R}\}$, where T is a large parameter. The case of fixed time, say $t=T$, is obtained by setting

$\pi(\theta) = 1-\theta$, while fixed particle number, say $n = \alpha T$, by $\pi(\theta) = \alpha + \theta$ with some constant α.

From KPZ scaling exponents [19], we expect to see a nontrivial limit if we consider positions at distance of order $T^{2/3}$. Thus, the focus on the region around θT is given by $\theta T - uT^{2/3}$, i.e., setting $\theta - uT^{-1/3}$ instead of θ and, by series expansions, we scale time and particle number as

$$\begin{aligned} t(u) &= \lfloor (\pi(\theta)+\theta)T - (\pi'(\theta)+1)uT^{2/3} + \tfrac{1}{2}\pi''(\theta)u^2T^{1/3} \rfloor, \\ n(u) &= \lfloor (\pi(\theta)-\theta)T + (1-\pi'(\theta))uT^{2/3} + \tfrac{1}{2}\pi''(\theta)u^2T^{1/3} \rfloor. \end{aligned} \tag{2.10}$$

The KPZ fluctuation exponent is 1/3, thus we expect to see fluctuations of particle positions on a scale of order $T^{1/3}$. Therefore, we define the rescaled process Ξ_T by

$$u \mapsto \Xi_T(u) = \frac{x_{n(u)}(t(u)) - (-2n(u) + \mathbf{v}t(u))}{-T^{1/3}}. \tag{2.11}$$

Here the mean speed of particles, $\mathbf{v}$, is determined to be $\mathbf{v} = 1-\sqrt{q}$ from the subsequent asymptotic analysis but can be known beforehand from the stationary measure for density 1/2 [15]. This process has a limit as $T \to \infty$ given in terms of the Airy$_1$ process. In section 5 we prove

Theorem 3. *Let Ξ_T be the rescaled process as in (2.11). Then*

$$\lim_{T\to\infty} \Xi_T(u) = \kappa_{\rm v}\mathcal{A}_1(u/\kappa_{\rm h}), \tag{2.12}$$

in the sense of finite dimensional distributions. The vertical (fluctuations) and horizontal (correlations) scaling coefficients are given by

$$\kappa_{\rm v} = (\pi(\theta)+\theta)^{1/3}(1-q)^{1/3}q^{1/6}, \tag{2.13}$$

$$\kappa_{\rm h} = \frac{(\pi(\theta)+\theta)^{2/3}(1-q)^{2/3}q^{-1/6}}{(\pi'(\theta)+1)(1-\sqrt{q})/2 + 1 - \pi'(\theta)}. \tag{2.14}$$

Remark. A similar result for the PushASEP with alternating initial condition has been proved in Theorem 2.2 of [4].

2.3 TASEP and growth models

As mentioned in the introduction, the discrete TASEP with parallel update is related to a surface growth model from which the polynuclear growth model in continuous time can be obtained as a limit. Let $t \geq 0$ and $x \in \mathbb{R}$ denote the time and the one-dimensional space coordinate respectively, and let $h_t(x)$

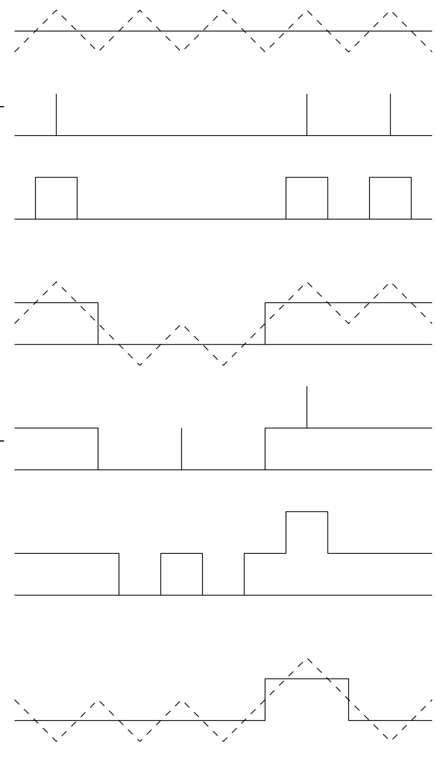

Figure 3: A surface growth model. For half-odd integer times this is equivalent to the discretized Gates-Westcott dynamics and for integer times to the discrete TASEP.

be the height of the surface at time t and at position x. Let us introduce a dynamics of $h_t(x)$ as follows. Initially, at time $t = 0$, the surface is flat; $h_0(x) = 0$, for all $x \in \mathbb{R}$. Right after each integer time ($t = 0+, 1+, 2+, \ldots$), there could occur a nucleation with width 0 and height 1 with probability q ($0 < q < 1$) independently at each integer position x such that $t+x+h_t(x)$ is even. Each nucleation is regarded as consisting of an upstep and a downstep and each upstep (resp. downstep) moves to the left (resp. right) with unit speed. This is a deterministic part of the evolution. When an upstep and a downstep collide, they merge together. See the solid line in Figure 3 for an example until $t = 2$. The dynamics of the growth model, if we focus only on half-odd times ($t = \frac{1}{2}, \frac{3}{2}, \ldots$), is the same as one considered in [23], i.e., a discretized version of the Gates-Westcott dynamics [12]. It is known that in an appropriate $q \to 0$ limit this growth model reduces to the standard continuous time PNG model [23].

To see the connection to the discrete TASEP, let us focus on integer

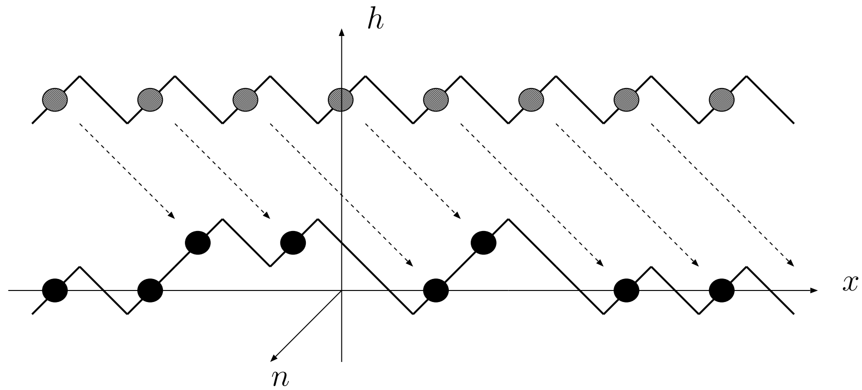

Figure 4: Surface height and TASEP particle positions. An example for $t = 4$.

times ($t = 0, 1, 2, \ldots$) and positions ($x \in \mathbb{Z}$) from now on and represent the surface as consisting of elementary upward slopes ╱ and downward slopes ╲ as indicated by dashed lines in Figure 3. At $t = 0$, even (resp. odd) x's are taken to be the center of the upward (resp. downward) slopes. Then the dynamics of the surface is described as follows: At each time step the surface grows upward by unit height deterministically and then each local maximum (╱╲) of slope turns into a local minimum (╲╱) independently with probability $p \equiv 1 - q$. If we interpret an upward (resp. a downward) slope as a site occupied by a particle (resp. an empty site), this is equivalent to the discrete time TASEP with parallel update under the alternating initial condition.

The relation between the surface height $h_t(x)$ and the position of the TASEP particle is given by

$$h_t(x) \leq H \Leftrightarrow x_{\lfloor \frac{t-x-H}{2} \rfloor}(t) \geq x \tag{2.15}$$

and is understood as follows. On the plot of the surface at some fixed time t, draw also the initial surface at $h = t$. See Figure 4 for an expample. Then, from the correspondence between the growth model and the TASEP, the surface at time t can be regarded as the particle positions. In this plot particles move along the down-right direction as indicated. The left hand side of (2.15) is equivalent to the condition that the TASEP particle corresponding to $(x, h = t - H)$ has already reached x. Since the axis of the particle number n is in the down-left direction, the value of n corresponding to $(x, h = t - H)$ is $\lfloor (t - H - x)/2 \rfloor$. This consideration results in the relation (2.15). From the relation (2.15) the joint distributions of the height of the growth model is readily obtained through

$$\mathbb{P}\Big(\bigcap_{i=1}^{m}\{h_{t_i}(x_i)\leq H_i\}\Big)=\mathbb{P}\Big(\bigcap_{i=1}^{m}\{x_{n_i}(t_i)\geq x_i\}\Big), \tag{2.16}$$

combined with Theorem 1.

When $q\to 0$, the TASEP particles move almost deterministically and the surface $h_t(x)$ grows slowly, when a particle decides not to jump (with probability q). The continuous time PNG model is obtained by taking $q\to 0$ while setting space and time units to $\sqrt{q}/2$ (the 2 is chosen to have nucleations with intensity 2 like in [23]). Denote by $\mathbf{x}$ and $\mathbf{t}$ the position and time variables in the continuous time PNG model. The PNG height function $h^{\rm PNG}(\mathbf{x},\mathbf{t})$ is then obtained by the limit

$$h^{\rm PNG}(\mathbf{x},\mathbf{t})=\lim_{q\to 0}h_{2\mathbf{t}/\sqrt{q}}(-2\mathbf{x}/\sqrt{q}). \tag{2.17}$$

Here the minus sign on the right hand side is put for a convenience. The results below do not depend on this sign because of the symmetry of the model in consideration. The joint distribution of the surface height at time $\mathbf{t}$ is given as follows.

Proposition 4. *Consider m space positions $\mathbf{x}_1<\mathbf{x}_2<\ldots<\mathbf{x}_m$. Then, the joint distribution at time $\mathbf{t}$ of the heights $h^{\rm PNG}(\mathbf{x}_k,\mathbf{t})$, $k=1,\ldots,m$, is given by*

$$\mathbb{P}\Big(\bigcap_{k=1}^{m}\big\{h^{\rm PNG}(\mathbf{x}_k,\mathbf{t})\leq H_k\big\}\Big)=\det(\mathbb{1}-\chi_H K_{\mathbf{t}}^{\rm PNG}\chi_H)_{\ell^2(\{\mathbf{x}_1,\ldots,\mathbf{x}_m\}\times\mathbb{Z})} \tag{2.18}$$

where the kernel is given by

$$\begin{aligned}&K_{\mathbf{t}}^{\rm PNG}(\mathbf{x}_1,h_1;\mathbf{x}_2,h_2)=-I_{|h_1-h_2|}\left(2(\mathbf{x}_2-\mathbf{x}_1)\right)\mathbb{1}(\mathbf{x}_2>\mathbf{x}_1)\\ &+\left(\frac{2\mathbf{t}+\mathbf{x}_2-\mathbf{x}_1}{2\mathbf{t}-\mathbf{x}_2+\mathbf{x}_1}\right)^{(h_1+h_2)/2}J_{h_1+h_2}\left(2\sqrt{4\mathbf{t}^2-(\mathbf{x}_2-\mathbf{x}_1)^2}\right)\mathbb{1}(2\mathbf{t}\geq|\mathbf{x}_2-\mathbf{x}_1|)\end{aligned} \tag{2.19}$$

where $I_n(x)$ and $J_n(x)$ are the modified Bessel functions and the Bessel functions, see e.g. [1].

The last indicator function is obvious if one thinks about the PNG model. In fact, the height at position $\mathbf{x}$ at time $\mathbf{t}$ depends on events lying in the backward light cone of $(\mathbf{x},\mathbf{t})$ on $\mathbb{R}\times\mathbb{R}_+$. Thus, when $|\mathbf{x}_2-\mathbf{x}_1|>2\mathbf{t}$, the backwards light cones of $(\mathbf{x}_1,\mathbf{t})$ and $(\mathbf{x}_2,\mathbf{t})$ do not intersect in $\mathbb{R}\times\mathbb{R}_+$, which implies that the two height functions are independent. The Fredholm determinant then splits into blocks.

The result of Proposition 4 is actually a specialization of a more general situation which follows from the TASEP correspondence. In the TASEP, the space-like paths π we had for particle numbers and times become the paths

$$(\mathbf{x},\mathbf{t}) = \big(\pi(\theta) - 3\theta, \theta + \pi(\theta)\big). \tag{2.20}$$

The condition $|\pi'(\theta)| \leq 1$ implies that $\partial\mathbf{t}/\partial\mathbf{x} \in [-1,0]$, i.e., these are space-like paths as in special relativity oriented into the past. By the symmetry of the problem, one can consider also space-like paths locally oriented into the future, just looking at the process in the other direction.

Denote by γ such a path on $\mathbb{R}\times\mathbb{R}_+^*$, i.e., $(\mathbf{x}, \mathbf{t} = \gamma(\mathbf{x}))$, then θ and $\pi(\theta)$ are given by the relations

$$\theta = (\gamma(\mathbf{x}) - \mathbf{x})/4, \quad \pi(\theta) = (3\gamma(\mathbf{x}) + \mathbf{x})/4, \tag{2.21}$$

and the joint distributions of the surface height along the path γ are expressed as in Theorem 5. This is proved in section 6.

Theorem 5. *Let us denote by $\mathbf{t}_k = \gamma(\mathbf{x}_k)$. Then, the joint distributions of $h^{\rm PNG}(\mathbf{x}_k,\mathbf{t}_k)$, $k=1,\ldots,m$, is given by*

$$\mathbb{P}\Big(\bigcap_{k=1}^m \big\{h^{\rm PNG}(\mathbf{x}_k,\mathbf{t}_k) \leq H_k\big\}\Big) = \det(\mathbb{1} - \chi_H K^{\rm PNG}\chi_H)_{\ell^2(\{(\mathbf{x}_1,\mathbf{t}_1),\ldots,(\mathbf{x}_m,\mathbf{t}_m)\}\times\mathbb{Z})} \tag{2.22}$$

where the kernel is given by

$$\begin{aligned} K^{\rm PNG}((\mathbf{x}_1,\mathbf{t}_1),h_1;(\mathbf{x}_2,\mathbf{t}_2),h_2) &= -\left(\frac{\mathbf{x}_2-\mathbf{x}_1+\mathbf{t}_1-\mathbf{t}_2}{\mathbf{x}_2-\mathbf{x}_1-\mathbf{t}_1+\mathbf{t}_2}\right)^{|h_1-h_2|/2} \\ &\times I_{|h_1-h_2|}\left(2\sqrt{(\mathbf{x}_2-\mathbf{x}_1)^2-(\mathbf{t}_2-\mathbf{t}_1)^2}\right)\mathbb{1}_{\{(\mathbf{t}_1+\mathbf{x}_1,\mathbf{t}_1)\prec(\mathbf{t}_2+\mathbf{x}_2,\mathbf{t}_2)\}} \\ &+ \left(\frac{(\mathbf{t}_1+\mathbf{t}_2)+(\mathbf{x}_2-\mathbf{x}_1)}{(\mathbf{t}_1+\mathbf{t}_2)-(\mathbf{x}_2-\mathbf{x}_1)}\right)^{(h_2+h_1)/2} J_{h_1+h_2}\left(2\sqrt{(\mathbf{t}_1+\mathbf{t}_2)^2-(\mathbf{x}_2-\mathbf{x}_1)^2}\right) \\ &\times \mathbb{1}_{\{\mathbf{t}_1+\mathbf{t}_2\geq|\mathbf{x}_1-\mathbf{x}_2|)\}} \end{aligned} \tag{2.23}$$

where $I_n(x)$ and $J_n(x)$ are the modified Bessel functions and the Bessel functions. The first term is present only when $\mathbf{x}_2-\mathbf{x}_1 \geq \mathbf{t}_1-\mathbf{t}_2 > 0$ or $\mathbf{x}_2-\mathbf{x}_1 > \mathbf{t}_1-\mathbf{t}_2 \geq 0$ due to (2.1).

In the first term, for $\mathbf{x}_2 > \mathbf{x}_1$, the condition $\mathbf{x}_2-\mathbf{x}_1 \geq \mathbf{t}_1-\mathbf{t}_2$ is satisfied for $\mathbf{t}_k = \gamma(\mathbf{x}_k)$. Also, notice that when $\mathbf{x}_2-\mathbf{x}_1 \to \mathbf{t}_1-\mathbf{t}_2$, the first term of the kernel goes to $(2(\mathbf{x}_2-\mathbf{x}_1))^{|h_1-h_2|}/(|h_1-h_2|)!$ since $I_{|n|}(x) = \frac{1}{|n|!}\left(\frac{x}{2}\right)^{|n|} + \mathcal{O}(x^{|n|+1})$ for small x.

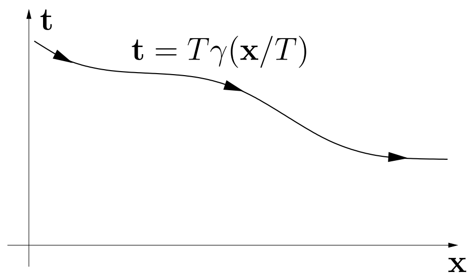

Figure 5: A space-time path γ for continuous time PNG. T is proportional to the PNG time $\mathbf{t}$.

2.4 Scaling limit for the continuous PNG model

The last result of this paper is the large time behavior of the flat PNG. The large parameter denoted by T is proportional to time $\mathbf{t}$. Using the function γ, we consider $\mathbf{t} = T\gamma(\mathbf{x}/T)$, see Figure 5.

Since the system is translation invariant, we focus around the origin, i.e., we look at the PNG height at

$$\begin{cases} \mathbf{x}(u) = uT^{2/3}, \\ \mathbf{t}(u) = \gamma(0)T + \gamma'(0)uT^{2/3} + \frac{1}{2}\gamma''(0)u^2T^{1/3}. \end{cases} \tag{2.24}$$

The surface height grows with the speed equal to 2. Thus, for large time $\mathbf{t}$, the macroscopic height will be close to $2\mathbf{t}$. Fluctuations live on a $T^{1/3}$ scale. Consequently, we define the rescaled height process $h_T^{\rm PNG}$ by

$$u \mapsto h_T^{\rm PNG}(u) = \frac{h^{\rm PNG}(\mathbf{x}(u), \mathbf{t}(u)) - 2\mathbf{t}(u)}{T^{1/3}}. \tag{2.25}$$

The large T (thus large time too) behavior of $h_T^{\rm PNG}$ is given in terms of the Airy_1 process as stated below.

Theorem 6. *Let $h_T^{\rm PNG}$ be the rescaled process as in (2.25). Then, in the limit of large T, we have*

$$\lim_{T\to\infty} h_T^{\rm PNG}(u) = S_v \mathcal{A}_1(u/S_h), \tag{2.26}$$

in the sense of finite dimensional distributions. The scaling coefficients S_v and S_h are given by

$$S_v = (2\gamma(0))^{1/3}, \quad S_h = (2\gamma(0))^{2/3} = S_v^2. \tag{2.27}$$

For $\gamma(x) = 1$, i.e., fixed time, this was conjectured to hold in [6]. The proof of this theorem is given in section 6.

3 Transition probability for the finite system

Let $G(x_1,\dots,x_N;t)$ denote the transition probability of the parallel TASEP with N particles starting at $t=0$ at positions $y_N<\dots<y_1$. This is the probability that the N particles starting from positions $y_N<\dots<y_1$ at $t=0$ are at positions $x_N<\dots<x_1$ at t.

Consider a determinantal signed point process on the set $\underline{x}=\{x_i^n, 1\le i\le n\le N\}$ by setting the measure

$$W_N(\underline{x})=\left(\prod_{n=1}^{N-1}\det(\phi^\sharp(x_i^n,x_{j+1}^{n+1}))_{0\le i,j\le n}\right)\det(F_{-i+1}(x_j^N-y_{N+1-i},t+1-i))_{1\le i,j\le N} \tag{3.1}$$

where

$$\phi^\sharp(x,y)=\begin{cases}1, & y\ge x,\\ p, & y=x-1,\\ 0, & y\le x-2,\end{cases} \tag{3.2}$$

the function $F_n(x,t)$ defined by

$$F_{-n}(x,t)=\frac{1}{2\pi \mathrm{i}}\oint_{\Gamma_{0,-1}}\mathrm{d}w\frac{w^n}{(1+w)^{n+x+1}}(1+(1-q)w)^t, \tag{3.3}$$

with the paths $\Gamma_{0,-1}$ being any simple loops anticlockwise oriented including 0, -1 and no other poles and we used the convention, $x_0^n=-\infty$.

The following proposition states that the one time transition probability of the TASEP is given as a marginal of the signed measure (3.1).

Proposition 7. *Let us set $x_1^n=x_n, n=1,\dots,N$. Then*

$$G(x_1,\dots,x_N;t)=\sum_{\mathcal{D}}W_N(\underline{x}) \tag{3.4}$$

where summation is over the variables in the set,

$$\mathcal{D}=\{x_i^n, 2\le i\le n\le N|x_i^n>x_{i-1}^n\} \tag{3.5}$$

varying over $\mathbb{Z}$.

Note that $W_N(\underline{x})$ is actually symmetric with respect to permutations of variables with same upper index, so the ordering in (3.5) is used for singling out the minimal $x_1^n=\min\{x_i^n, i=1,\dots,n\}$.

Remark. Similar representations for the transition probability of continuous time TASEP, discrete time TASEP with sequential update and PushASEP have been obtained in [4–6].

In the different parts of the proof of Proposition 7, we will use several properties of the function F_n, which are listed below.

Lemma 8.

$$F_{n+1}(x,t) = \sum_{y=x}^{\infty} F_n(y,t), \tag{3.6}$$

$$F_n(x,t+1) = qF_n(x,t) + (1-q)F_n(x-1,t) \tag{3.7}$$

$$= F_n(x,t) + (1-q)F_{n-1}(x-1,t), \tag{3.8}$$

$$(\phi^\sharp * F_n)(x,t) = F_{n+1}(x,t+1), \tag{3.9}$$

$$F_{-n}(x,-n) = 0 \quad \text{for } x < -n, n > 0, \tag{3.10}$$

$$F_n(x,n) = 0 \quad \text{for } x > n, n > 0, \tag{3.11}$$

$$F_n(n,n) = (1-q)^n, n \geq 0, \tag{3.12}$$

$$F_{-n}(-n,-n) = 1/(-q)^n, n \geq 0. \tag{3.13}$$

Here "$*$" represents the convolution: $(\phi^\sharp * f)(x) = \sum_y \phi^\sharp(x,y)f(y)$.

Proof of Lemma 8. These are proven by using the definition (3.2) and (3.3). □

The first step in the proof of Proposition 7 is the following Lemma.

Lemma 9. *Let us set*

$$\phi^\sharp_\nu(x,y) = \begin{cases} \nu^{y-x}, & y \geq x, \\ 1-q, & y = x-1, \\ 0, & y \leq x-2 \end{cases} \tag{3.14}$$

and $\phi^\sharp_\nu(-\infty, y) = \nu^y$. *Then, for any antisymmetric function* $f(b_1,\ldots,b_n)$,

$$\sum_{\substack{b_n > \ldots > b_1 > b_0 \\ b_0:\text{fixed}}} \det(\phi^\sharp_\nu(a_i,b_j))_{0\leq i,j\leq n} \cdot f(b_1,\ldots,b_n)$$

$$= g_\nu(a_1,b_0) \sum_{\substack{b_n > \ldots > b_1 > b_0 \\ b_0:\text{fixed}}} \det(\phi^\sharp_\nu(a_i,b_j))_{1\leq i,j\leq n} \cdot f(b_1,\ldots,b_n) \tag{3.15}$$

where $a_n > \ldots > a_1, a_0 = -\infty$ *and*

$$g_\nu(a,b) = \begin{cases} 0, & b \geq a, \\ \nu^b(1-(1-q)\nu), & b = a-1, \\ \nu^b, & b \leq a-2. \end{cases} \tag{3.16}$$

Proof of Lemma 9. From the antisymmetry of f and of the determinant, (3.15) is equivalent to

$$\sum_{\substack{b_1,\dots,b_n>b_0\\ b_0:\text{fixed}}} \det(\phi^\sharp_\nu(a_i,b_j))_{0\le i,j\le n}\cdot f(b_1,\dots,b_n)$$
$$= g_\nu(a_1,b_0)\sum_{\substack{b_1,\dots,b_n>b_0\\ b_0:\text{fixed}}} \det(\phi^\sharp_\nu(a_i,b_j))_{1\le i,j\le n}\cdot f(b_1,\dots,b_n). \tag{3.17}$$

Since a basis of the antisymmetric functions is made of the antisymmetric delta functions and the relation to prove is linear in f, it is enough to consider

$$f(b_1,\dots,b_n)=\begin{cases}(-1)^\sigma, & \text{if } (b_1,\dots,b_n)=(b_{\sigma_1},\dots,b_{\sigma_n}) \text{ for some } \sigma\in S_n,\\ 0, & \text{otherwise}\end{cases} \tag{3.18}$$

for fixed $b_1,\dots,b_n > b_0$. Here S_n is the group of all permutations of $\{1,\dots,n\}$. For this special choice of f, the left hand side of (3.17) is $n!$ times the single determinant,

$$\det\begin{bmatrix}\nu^{b_0} & \nu^{b_1} & \dots & \nu^{b_n}\\ \phi^\sharp_\nu(a_1,b_0) & \phi^\sharp_\nu(a_1,b_1) & \dots & \phi^\sharp_\nu(a_1,b_n)\\ \vdots & \vdots & & \vdots\\ \phi^\sharp_\nu(a_n,b_0) & \phi^\sharp_\nu(a_n,b_1) & \dots & \phi^\sharp_\nu(a_n,b_n)\end{bmatrix}. \tag{3.19}$$

We have the following three cases.

(a) $a_1\le b_0$: the second row gives $(\nu^{b_0-a_1},\dots,\nu^{b_n-a_1})$ which is proportional to the first row. Therefore in this case the LHS is zero.

(b) $a_1=b_0+1$: The second row is $(1-q,\nu^{b_1-a_1},\dots,\nu^{b_n-a_1})$. Subtracting ν^{a_1} times the second row from the first row one obtains

$$\nu^{b_0}(1-(1-q)\nu)\cdot\det(\phi^\sharp_\nu(a_i,b_j))_{1\le i,j\le n}. \tag{3.20}$$

(c) $a_1>b_0+1$: The first column is $(\nu^{b_0},0,\dots,0)^t$. Thus the determinant is $\nu^{b_0}\cdot\det(\phi^\sharp_\nu(a_i,b_j))_{1\le i,j\le n}$.

The result in each case agrees with $1/n!$ times the RHS of (3.17) and hence the lemma is proved. □

Let $\mathcal{N}(x_1,\dots,x_N)$ denote the number of j's s.t. $x_j-x_{j+1}=1, j=1,\dots,N-1$. Using the above lemma with $\nu=1$ in which case $\phi^\sharp_\nu$ reduces to $\phi^\sharp$, we have the following result.

Lemma 10. *With $x_1^n = x_n, n = 1, \dots, N,$ one has*

$$\sum_{\mathcal{D}} W_N(\underline{x}) = q^{\mathcal{N}(x_1,\dots,x_N)} \det[F_{j-i}(x_{N-j+1} - y_{N-i+1}, t+j-i)]_{1\le i,j\le N} =: \widetilde{G}(x_1,\dots,x_N;t). \tag{3.21}$$

Proof of Lemma 10. For simplicity, we denote

$$f_i(x) = F_{-i+1}(x - y_{N-i+1}, t-i+1), \tag{3.22}$$

for $i = 1, \dots, N$. From the definitions (3.1), the LHS of (3.21) writes

$$\sum_{\substack{x_n^n > x_{n-1}^n > \dots > x_1^n \\ x_1^n:\text{fixed}, 1\le n\le N}} \left(\prod_{n=1}^{N-1} \det(\phi^\sharp(x_i^n, x_{j+1}^{n+1}))_{0\le i,j\le n} \right) \det(f_i(x_j^N))_{1\le i,j\le N}. \tag{3.23}$$

Applying Lemma 9 with $\nu = 1, n = N-1$, $a_i = x_i^{N-1}$, $i = 1, \dots, N-1$, $b_i = x_{i+1}^N$, $i = 0, \dots, N-1$ and

$$f(b_1, \dots, b_n) = \det(f_i(x_j^N))_{1\le i,j\le N}, \tag{3.24}$$

we obtain

$$(3.23) = g_1(x_1^{N-1}, x_1^N) \cdot \sum_{\substack{x_n^n > x_{n-1}^n > \dots > x_1^n \\ x_1^n:\text{fixed}, 1\le n\le N-1}} \left(\prod_{n=1}^{N-2} \det(\phi^\sharp(x_i^n, x_{j+1}^{n+1}))_{0\le i,j\le n} \right) \times \sum_{\substack{x_N^N > x_{N-1}^N > \dots > x_1^N \\ x_1^N:\text{fixed}}} \det(\phi^\sharp(x_i^{N-1}, x_{j+1}^N))_{1\le i,j\le N-1} \cdot \det(f_i(x_j^N))_{1\le i,j\le N}. \tag{3.25}$$

Heine's identity,

$$\frac{1}{n!} \sum_{x_1,\dots,x_n} \det(\phi_i(x_j))_{1\le i,j\le n} \det(\psi_i(x_j))_{1\le i,j\le n} = \det\left[\phi_i * \psi_j\right]_{1\le i,j\le n}, \tag{3.26}$$

allows us to rewrite the last summation in (3.25) as

$$\det \begin{bmatrix} f_1(x_1^N) & (\phi^\sharp * f_1)(x_1^{N-1}) & \dots & (\phi^\sharp * f_1)(x_{N-1}^{N-1}) \\ \vdots & \vdots & & \vdots \\ f_N(x_1^N) & (\phi^\sharp * f_N)(x_1^{N-1}) & \dots & (\phi^\sharp * f_N)(x_{N-1}^{N-1}) \end{bmatrix}. \tag{3.27}$$

We repeat the procedure up to a total of $j-1$ times in column j and we get

$$(3.25) = \left(\prod_{n=1}^{N-1} g_1(x_1^n, x_1^{n+1})\right) \det[\overbrace{\phi^\sharp * \ldots * \phi^\sharp}^{j-1} * f_i(x_1^{N-j+1})]_{1\le i,j\le N}. \quad (3.28)$$

The proof of the lemma is finished using (3.22), (3.9) and $\prod_{n=1}^{N-1} g_1(x_1^n, x_1^{n+1}) = q^{\mathcal{N}(x_1,\ldots,x_N)}$. ☐

Proof of Proposition 7. We need to prove

$$G(x_1,\ldots,x_N;t) = \widetilde{G}(x_1,\ldots,x_N;t). \quad (3.29)$$

This statement was also proved in [20] by the Bethe ansatz techniques. Our proof is by induction in t. We start by showing that the initial conditions agree, i.e., $\widetilde{G}(x_1,\ldots,x_N;0) = G(x_1,\ldots,x_N;0)$, that is,

$$q^{\mathcal{N}(x_1^1,\ldots,x_1^N)} \cdot \det[F_{j-i}(x_{N-j+1} - y_{N-i+1}, j-i)]_{1\le i,j\le N} = \prod_{n=1}^{N} \delta_{x_n,y_n}. \quad (3.30)$$

We first show that LHS of (3.30) is zero if $x_N \neq y_N$. If $x_N \le y_N - 1$, since $y_{N-i+1} \ge y_N + i - 1$, one has $x_N - y_{N-i+1} < -i+1, i = 1,\ldots,N$. Then, from (3.10) we have $F_{1-i}(x_N - y_{N-i+1}, 1-i) = 0$, i.e., the first column of LHS of (3.30) is zero. Similarly, if $x_N \ge y_N + 1$, since $x_{N-j+1} \ge x_N + j - 1$, one has $x_{N-j+1} - y_N > j-1, j = 1,\ldots,N$. Then, from (3.11) we have $F_{j-1}(x_{N-j+1} - y_N, j-1) = 0$, i.e., the first row of LHS of (3.30) is zero. This agrees with RHS of (3.30) also being zero if $x_N \neq y_N$.

Now let us assume $x_N = y_N$. There are two cases.

(a) $y_{N-1} > y_N + 1$. In this case, since $x_N - y_{N-i+1} = y_N - y_{N-i+1} < -i+1, i = 2,\ldots,N$, one has $F_{1-i}(x_N - y_{N-i+1}, 1-i) = 0, i = 2,\ldots,N$. Then the first column of LHS of (3.30) is $(1,0,\ldots,0)^t$ and hence the determinant is equal to $\det[F_{j-i}(x_{N-j+1} - y_{N-i+1}, j-i)]_{2\le i,j\le N}$.

(b) $y_{N-1} = y_N + 1$. First let us see that LHS of (3.30) is zero when $x_{N-1} \neq y_{N-1}$. We have $x_{N-1} \ge x_N + 1 = y_N + 1 = y_{N-1}$. If $x_{N-1} \ge y_{N-1} + 1$, we have $x_{N-j+1} - y_N \ge x_{N-1} + j - 2 - (y_{N-1} - 1) \ge j$, for $j = 2,\ldots,N$, and $x_{N-j+1} - y_{N-1} \ge j - 1$, for $j = 2,\ldots,N$. Then the first and the second row of LHS of (3.30) are both of the form, $(*,0,\ldots,0)$ where $*$ represents an arbitrary number and hence the determinant is zero. Hence LHS of (3.30) is zero if $x_{N-1} \neq y_{N-1}$. On the other hand, when $x_{N-1} = y_{N-1}$, the upper-left 2×2 submatrix of the determinant is

$$\begin{bmatrix} F_0(0,0) & F_1(1,1) \\ F_{-1}(-1,-1) & F_0(0,0) \end{bmatrix} = \begin{bmatrix} 1 & 1-q \\ -1/q & 1 \end{bmatrix}, \quad (3.31)$$

whose determinant is $1/q$.

Repeating the same procedure, at each step one has either case (a) or (b). The final result is that $y_k = x_k$, for $k = 1, \ldots, N$, otherwise the determinant in LHS of (3.30) is zero. Moreover, when $y_k = x_k$, $k = 1, \ldots, N$, denote by $n_1, n_1 + n_2, \ldots, n_1 + \ldots + n_\ell$ the values of j such that $x_{j-1} - x_j > 1$. Then LHS of (3.30) is equal to $\prod_{m=1}^{\ell} D_{n_m}$ with

$$D_n = \det\left[F_{j-i}(j-i, j-i)\right]_{1 \leq i,j \leq n} \tag{3.32}$$

Finally using (3.12), (3.13), we obtain an explicit form of the matrix. To compute its determinant it is enough to develop along the first row. The determinant of the $(1,1)$ minor is D_{n-1}, while the one of the $(1,2)$ minor is $(-1/q)D_{n-1}$ because the minor is the same as the $(1,1)$ minor except the first column is multiplied by $-1/q$. All the other minors have determinant zero, because the first two column are linearly dependent. Thus, $D_n = 1 \cdot D_{n-1} - (1-q)/(-q) D_{n-1}$, and since $D_1 = 1$, it follows that

$$D_n = \frac{1}{q^{n-1}}. \tag{3.33}$$

This ends the part of the proof concerning initial conditions.

Next we prove that (3.29) holds for $t+1$ if it does for t. Since this is true for $t = 0$, by induction it will be true for all $t \in \mathbb{N}$. G satisfies the TASEP dynamics, thus

$$\begin{aligned} & G(x_1, \ldots, x_N; t+1) \\ = & \sum_z G(z_1, \ldots, z_N, t) w(z, x) = \sum_z \widetilde{G}(z_1, \ldots, z_N, t) w(z, x) \\ = & \sum_z w(z, x) q^{\mathcal{N}(z_1, \ldots, z_N)} \det[F_{j-i}(z_{N-j+1} - y_{N-i+1}, t + j - i)]_{1 \leq i,j \leq N}. \end{aligned} \tag{3.34}$$

Here

$$w(z, x) = \prod_{n=1}^{N} v_n, \quad v_n = \begin{cases} 1, & z_n = z_{n-1} - 1, x_n = z_n, \\ q, & z_n < z_{n-1} - 1, x_n = z_n, \\ 1-q, & z_n < z_{n-1} - 1, x_n = z_n + 1, \end{cases} \tag{3.35}$$

and in the second equality we have used the assumption of the induction. We rewrite $\widetilde{G}(x_1, \ldots, x_N; t+1)$ using (3.7) and (3.8) as follows. For k from 1 to N:

(a) if $x_k = x_{k+1} + 1$, then we use (3.8) to the $N+1-k$th column. Then, the new term with the $(1-q)$ factor in front cancels out because it is proportional

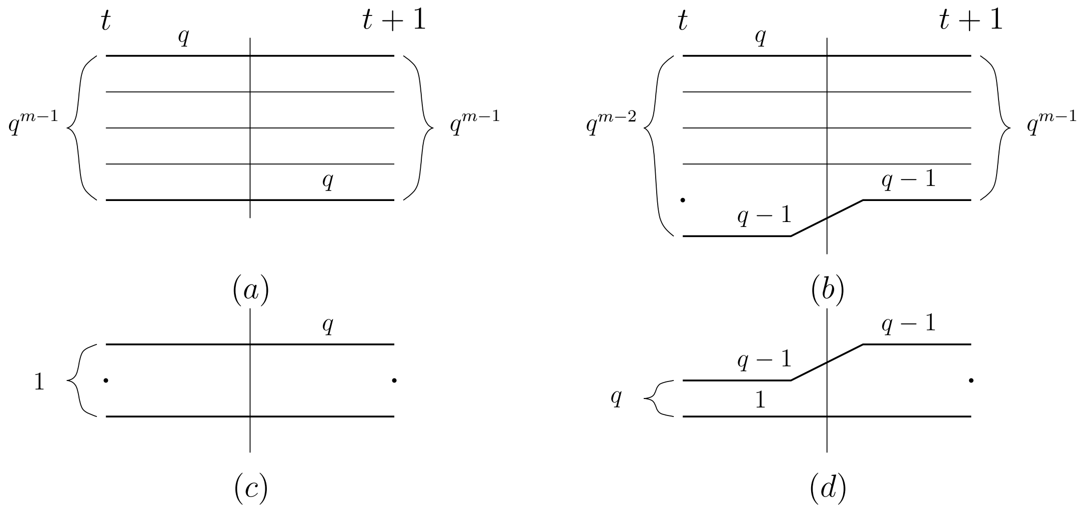

Figure 6: Graphical representation of (3.38). The dots represents empty places, while a line leaving/arriving to a point is an occupied position. In (a) and (b), on the left (resp. right) we indicate the weights different from 1 of LHS (resp. RHS) of (3.38). In (c) and (d) the bottom and top lines of two blocks at distance 2 at time $t+1$ are represented, for the cases corresponding to (a) and (b) for the top block.

to its left column of the determinant.

(b) if $x_k > x_{k+1}+1$, then we just use (3.7).

With these replacements we get

$$\begin{aligned} &\widetilde{G}(x_1,\ldots,x_N;t+1) \\ =\ &\sum_z \tilde{w}(z,x) q^{\mathcal{N}(x_1,\ldots,x_N)} \det[F_{j-i}(z_{N-j+1}-y_{N-i+1},t+j-i)]_{1\leq i,j\leq N}, \end{aligned} \tag{3.36}$$

where

$$\tilde{w}(z,x)=\prod_{n=1}^N \tilde{v}_n, \quad \tilde{v}_n = \begin{cases} 1, & x_n = x_{n+1}+1, z_n = x_n, \\ q, & x_n > x_{n+1}+1, z_n = x_n, \\ 1-q, & x_n > x_{n+1}+1, z_n = x_n - 1. \end{cases} \tag{3.37}$$

Comparing (3.34) and (3.36), it is enough to show

$$q^{\mathcal{N}(z_1,\ldots,z_N)} w(z,x) = q^{\mathcal{N}(x_1,\ldots,x_N)} \tilde{w}(z,x). \tag{3.38}$$

This indeed holds and can be seen by checking case by case. We illustrate it using Figure 6. First consider a block of particles, say m of them at time

$t+1$. There are two possibility of reaching this situations in one time step, as indicated in Figure 6 (a) and (b). The products of all the weights on the right and on the left are the same, i.e., (3.38) holds for a single block of particles. If two blocks of particles at time $t+1$ are at distance at least 2, they are independent during one time step. We just have to check that (3.38) holds for two blocks at distance 2 at time $t+1$. Case (a) is illustrated in (c) and the weights are unchanged for both blocks. Case (b) is illustrated in (d). This time, the q on the top line of the second block becomes a 1, but this is compensated by an extra factor q on the left. $\square$

4 Joint distributions along space-like paths

Theorem 11. *Let us consider particles starting from $y_1 > y_2 > \ldots$ and denote $x_j(t)$ the position of jth particle at time t. Take a sequence of particles and times which are space-like, i.e., a sequence of m couples $\mathcal{S} = \{(n_k,t_k), k=1,\ldots,m \,|\, (n_k,t_k) \prec (n_{k+1},t_{k+1})\}$. The joint distribution of their positions $x_{n_k}(t_k)$ is given by*

$$\mathbb{P}\Big(\bigcap_{k=1}^{m}\{x_{n_k}(t_k)\geq a_k\}\Big) = \det(\mathbb{1}-\chi_a K\chi_a)_{\ell^2(\{(n_1,t_1),\ldots,(n_m,t_m)\}\times\mathbb{Z})} \tag{4.1}$$

where $\chi_a((n_k,t_k),x)=\mathbb{1}(x<a_k)$. Here K is the extended kernel with entries

$$K((n_1,t_1),x_1;(n_2,t_2),x_2) = -\phi^{*((n_1,t_1),(n_2,t_2))}(x_1,x_2)+\sum_{k=1}^{n_2}\Psi^{n_1,t_1}_{n_1-k}(x_1)\Phi^{n_2,t_2}_{n_2-k}(x_2) \tag{4.2}$$

where

$$\begin{aligned}&\phi^{*((n_1,t_1),(n_2,t_2))}(x_1,x_2) \\ =\;& \frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_{0,-1}}\mathrm{d}w\frac{(1+pw)^{t_1-t_2}}{(1+w)^{x_1-x_2+1}}\left(\frac{w}{(1+w)(1+pw)}\right)^{n_1-n_2}\mathbb{1}_{[(n_1,t_1)\prec(n_2,t_2)]},\end{aligned} \tag{4.3}$$

the functions $\{\Psi^{n,t}_j\}_{j=0}^{n-1}$ are given by

$$\Psi^{n,t}_j(x) = \frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_{-1}}\mathrm{d}w\frac{(1+pw)^t}{(1+w)^{x-y_{n-j}+1}}\left(\frac{w}{(1+w)(1+pw)}\right)^j, \tag{4.4}$$

where the contour Γ_{-1} is any simple loop anticlockwise oriented and including -1 and no other poles. The functions $\{\Phi^{n,t}_j\}_{j=0}^{n-1}$ are characterized by the two conditions:

$$\sum_{x\in\mathbb{Z}}\Psi^{n,t}_k(x)\Phi^{n,t}_l(x)=\delta_{k,l},\quad 0\leq k,l\leq n-1, \tag{4.5}$$

and

$$\Phi_j^{n,t}(x) \text{ is a polynomial of degree } j,\ j=0,\dots,n-1.$$

Proof of Theorem 11. The statement is the analogue of Proposition 3.1 of [4]. The proof is also quite similar. We start with the analog of Theorem 4.1 of [4].

Proposition 12. *Set $t_0=0$ and $n_{m+1}=0$. The joint distribution of particles from Theorem 11 is a marginal of a determinantal measure, obtained by summation of the variables in the set*

$$D=\{x_k^l(t_i), 1\le k\le l, 1\le l\le n_i, 0\le i\le m\}\setminus\{x_1^{n_i}(t_i), 1\le i\le m\};$$

the range of summation for any variable in this set is $\mathbb{Z}$. Precisely,

$$\begin{aligned}
&\mathbb{P}(x_{n_i}(t_i)=x_1^{n_i}(t_i), 1\le i\le m \mid x_k(0)=y_k(0), 1\le k\le n_1)\\
&\quad=\text{const}\times\sum_D \det\left[\Psi^{n_1,t_0}_{n_1-l}(x_k^{n_1}(t_0))\right]_{1\le k,l\le n_1}\\
&\qquad\times\prod_{i=1}^m\Bigg[\det[\mathcal{T}_{t_i,t_{i-1}}(x_l^{n_i}(t_i),x_k^{n_i}(t_{i-1}))]_{1\le k,l\le n_i}\\
&\qquad\qquad\times\prod_{n=n_{i+1}+1}^{n_i}\det[\phi^\sharp(x_k^{n-1}(t_i),x_l^n(t_i))]_{1\le k,l\le n}\Bigg]
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{T}_{t_i,t_{i-1}}(x,y) &= \frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_{0,-1}}\mathrm{d}w\frac{(1+pw)^{t_i-t_{i-1}}}{(1+w)^{x-y+1}},\\
\phi^\sharp(x,y) &= \frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_{0,-1}}\mathrm{d}w\frac{1}{(1+w)^{x-y+1}}\frac{(1+pw)(1+w)}{w}
\end{aligned}$$

and $\phi^\sharp(x_n^{n-1},y)=1$.

The proof of this Proposition is word-for-word repetition of the proof of Theorem 4.1 of [4] in the case when all parameters $v_i=1$ with the following replacements:

PushASEP	Parallel TASEP
$\varphi(x,y)$	$\phi^\sharp(x,y)$
$F_n(x,a(t),b(t))$	$\widetilde{F}_n(x,t):=F_n(x,t+n)$

The role of Lemmas 4.3 and 4.4 of [4] used in the proof of Theorem 4.1 there is played by the identity

$$(\phi^\sharp * \widetilde{F}_n)(x,t)=\widetilde{F}_{n+1}(x,t),$$

cf. (3.9), and Proposition 7 above, respectively.

As the next step, we would like to apply Theorem 4.2 of [4]. This cannot be done directly because the functions (4.43) of [4] are not well-defined (the series diverge), and we need to deform our measure to guarantee the convergence of all the series involved.

Let $v_1, \ldots, v_n$ be arbitrary nonzero complex numbers. Set $(j = 1, \ldots, n)$

$$\phi_j(x,y) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_{v_j-1,-1}} \mathrm{d}w \frac{1}{(1+w)^{x-y+1}} \frac{(1+p(v_j^{-1}(1+w)-1))(1+w)}{v_j^{-1}(1+w)-1},$$

$$\widetilde{\Psi}^{n,t}_{n-j}(x) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_{-1}} \mathrm{d}w \frac{(1+pw)^t}{(1+w)^{x-y_j+1}} \prod_{k=j+1}^{n} \frac{v_k^{-1}(1+w)-1}{(1+p(v_k^{-1}(1+w)-1))(1+w)}.$$

Also set $\phi_j(x_j^{j-1}, y) = v_j^y$ (recall that x_j^{j-1} are fictitious variables). Observe that as all $v_j \to 1$ these functions converge to $\phi^\sharp$ and $\Psi^{n,t}_{n-j}$.

Consider the following deformation of the measure from Proposition 12:

$$\begin{aligned}
\text{const} \quad &\times \quad \det\left[\widetilde{\Psi}^{n_1,t_0}_{n_1-l}(x_k^{n_1}(t_0))\right]_{1\le k,l\le n_1} \\
&\times \quad \prod_{i=1}^{m} \Bigg[\det[\mathcal{T}_{t_i,t_{i-1}}(x_l^{n_i}(t_i), x_k^{n_i}(t_{i-1}))]_{1\le k,l\le n_i} \\
&\qquad \prod_{n=n_{i+1}+1}^{n_i} \det[\phi_n(x_k^{n-1}(t_i), x_l^n(t_i))]_{1\le k,l\le n} \Bigg]. \qquad (4.6)
\end{aligned}$$

Since all functions $\widetilde{\Psi}^{n,t}_j(x)$ are finitely supported, there are only finitely many sets of values of the variables $\{x_k^n\}_{n\le n_1,\, k\le n}$ for which the above weights are nonzero. This implies that all the correlation functions are analytic in the parameters $v_1, \ldots, v_n$.

Following [4], let us introduce additional notation. For any level n there is a number $c(n)$ of terms $\det[\mathcal{T}]$ in (4.6) which involve the particles with upper index n (in other words $c(n)$ is $\#\{i \,|\, n_i = n\}$). Let us denote the time moments involved in these factors by $t_0^n < \ldots < t_{c(n)}^n$. Notice that $t_0^n = t_{c(n+1)}^{n+1}$, $t_0^{n_1} = t_0$, $t_1^{n_1} = t_1$, and $t_0^0 = t_{c(0)}^0 = t_m$.

Let us now apply Theorem 4.2 of [4] to the measure (4.6). Since we are using the same notation (except here we have extra tilde over Ψ's), let us also use (4.36)–(4.40) of [4]. The computation of the matrix $[M_{k,l}]_{k,l=1}^{n_1}$ yields, cf. (4.59) in [4],

$$M_{k,l} = \sum_{y\in\mathbb{Z}} v_k^y \frac{1}{2\pi\mathrm{i}} \oint_{|w+1|=\mathrm{const}} \mathrm{d}w \frac{(1+pw)^{t^k_{c(k)}}}{(1+w)^{y-y_l+1}} \tag{4.7}$$

$$\times \prod_{j=l+1}^{n_1} \frac{v_j^{-1}(1+w)-1}{(1+p(v_j^{-1}(1+w)-1))(1+w)} \prod_{j=k+1}^{n_1} \frac{(1+p(v_j^{-1}(1+w)-1))(1+w)}{v_j^{-1}(1+w)-1}$$

with the positive oriented integration contour going around points $w=-1$ and $w=v_j-1$, $j=k+1,\ldots,n$, but not other poles of the integrand.

To make sure that we can move the sum inside the integral without making the sum divergent, and to guarantee the convergence of the sum in (4.9) below, we assume that

$$|v_1| > |v_2| > \cdots > |v_{n_1}|. \tag{4.8}$$

Then we obtain $M_{k,l}=0$ for $l<k$, and $M_{k,k}\neq 0$ for any k. Thus, the matrix M is upper triangular and nondegenerate.

Following Theorem 4.2 of [4], we now need to find the linear span of

$$\Big\{(\phi_1 * \phi^{(t^1_{c(1)},t^n_a)})(x_1^0,x),\ldots,(\phi_n * \phi^{(t^n_{c(n)},t^n_a)})(x_n^{n-1},x)\Big\}.$$

We have

$$(\phi_k * \phi^{(t^k_{c(k)},t^n_a)})(x_k^{k-1},x) = \sum_{y\in\mathbb{Z}} v_k^y \frac{1}{2\pi\mathrm{i}} \oint_{|w+1|=\mathrm{const}\gg 1} \mathrm{d}w \frac{(1+pw)^{t^k_{c(k)}-t^n_a}}{(1+w)^{y-x+1}}$$

$$\times \prod_{j=k+1}^{n} \frac{(1+p(v_j^{-1}(1+w)-1))(1+w)}{v_j^{-1}(1+w)-1} = \mathrm{const}\cdot v_k^x, \tag{4.9}$$

where the constant is given by $(1+p(v_k-1))^{t^k_{c(k)}-t^n_a} v_k^{n-k-1} \prod_{j=k+1}^n \frac{v_j+p(v_k-v_j)}{v_k-v_j}$. Remark that this constant diverges if $v_j=v_k$.

Thus, the corresponding biorthogonal functions $\widetilde{\Phi}_j^{n,t}$ must be linear combinations of $v_1^x,\ldots,v_n^x$. Equivalently, $\widetilde{\Phi}_j^{n,t}$ are linear combinations of

$$f_k(x) = \frac{1}{2\pi\mathrm{i}} \oint \mathrm{d}z (1+z)^x \prod_{j=n-k}^{n} \frac{(1+p(v_j^{-1}(1+z)-1))(1+z)}{v_j^{-1}(1+z)-1}, \tag{4.10}$$

$k=0,\ldots,n-1$, with the integration contour going around the poles $z=v_j-1$, $j=n-k,\ldots,n$. Indeed, $f_k(x)$ is a linear combination of $v_{n-k}^x,\ldots,v_n^x$.

Denote by $G = [G_{k,l}]_{k,l=0}^{n-1}$ the Gram matrix

$$G_{k,l} = \sum_{x\in\mathbb{Z}} f_k(x)\widetilde{\Psi}_l^{n,t}(x). \tag{4.11}$$

Then we have

$$\widetilde{\Phi}_k^{n,t}(x) = \sum_{l=0}^{n-1}[G^{-1}]_{k,l} f_l(x). \tag{4.12}$$

Theorem 4.2 of [4] implies that the correlation functions of the measure (4.6) are determinantal, and the correlation kernel is expressed in terms of $\widetilde{\Phi}_k^{n,t}$'s, $\widetilde{\Psi}_l^{n,t}$'s, and ϕ_n's. This statement can be analytically continued in the parameters $v_1, \dots, v_n$ varying in a small enough neighborhood of 1. Indeed, the only ingredients of the obtained formula that may not be analytic are the matrix elements of G^{-1}. Since M is triangular, G is also triangular. In particular, $G_{k,k} = v_{n-k}^{y_{n-k}+2}(1+p(v_{n-k}-1))^t$. So the diagonal elements of G are rational functions in $v_1, \dots, v_n$ not equal to zero at $v_1 = v_2 = \dots = v_n = 1$. Thus, the matrix elements of G^{-1} are analytic around this point, and we can continue the result to $v_1 = \dots = v_n = 1$.

Observe that for $v_1 = \dots = v_n = 1$, $f_k(x)$ is a degree k polynomial in x. Thus, $\Phi_k^{n,t}(x)$ are the unique polynomials of degrees $\deg \Phi_k^{n,t}(x) = k$ that are biorthogonal to $\Psi_l^{n,t}$'s. ☐

Theorem 11 holds for general fixed initial conditions. We want to apply it to the alternating initial condition. For that we first have to do the orthogonalization with the result given in the next lemma.

Lemma 13. *For initial conditions $y_j = -2j$, $j = 1, \dots, n$, we have*

$$\Psi_j^{n,t}(x) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_{0,-1}} \mathrm{d}w \frac{w^j(1+pw)^{t-j}}{(1+w)^{x+2n-j+1}}, \tag{4.13}$$

and

$$\Phi_j^{n,t}(x) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mathrm{d}z \frac{(1+2z+pz^2)(1+z)^{x+2n-j-1}}{z^{j+1}(1+pz)^{t-j+1}}, \tag{4.14}$$

where, as before, $p = 1-q$. In particular, $\Phi_0^{n,t}(x) = 1$.

Proof of Lemma 13. The formula for Ψ_j^n is just obtained by substituting the initial conditions into (4.13). Now we prove that the orthonormality relation (4.5) holds. For $k = 0, \dots, n-1$, the pole at $w = 0$ in Ψ_k^n is not present, and for $x < -2n+k$, $\Psi_k^n(x) = 0$ because the residue at -1 vanishes. Thus

$$\begin{aligned}
&\sum_{x\in\mathbb{Z}} \Phi_j^{n,t}(x)\Psi_k^{n,t}(x) \\
= \; & \frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_0} \mathrm{d}z \oint_{\Gamma_{-1}} \mathrm{d}w \frac{(1+2z+pz^2)(1+z)^{2n-j-1}}{z^{j+1}(1+pz)^{t-j+1}} \frac{w^k(1+pw)^{t-k}}{(1+w)^{2n-k-1}} \\
& \times \sum_{x=-2n+k}^{\infty} \left(\frac{1+z}{1+w}\right)^x \qquad (4.15)
\end{aligned}$$

where we have the constraint on the integration paths $|1+z| < |1+w|$. The last term (the sum) equals

$$\left(\frac{1+z}{1+w}\right)^{-2n+k} \frac{1+w}{w-z}. \qquad (4.16)$$

Now the pole at $w = -1$ has disappeared and instead of it there is a simple pole at $w = z$. Thus, the integral over w is just the residue at $w = z$, leading to

$$\begin{aligned}
\sum_{x\in\mathbb{Z}} \Phi_j^{n,t}(x)\Psi_k^{n,t}(x) &= \frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_0} \mathrm{d}z \frac{1+2z+pz^2}{(1+pz)^2}\left(\frac{z(1+z)}{1+pz}\right)^{k-j-1} \\
&= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_0} \mathrm{d}u u^{k-j-1} = \delta_{j,k} \qquad (4.17)
\end{aligned}$$

where we used the change of variable $u = \frac{z(1+z)}{1+pz}$. □

Lemma 13 together with Theorem 11 leads to the kernel for the alternating initial condition.

Proposition 14. *For $y_j = -2j, j = 1, \ldots, n$, the kernel K in Theorem 11 is given by*

$$\begin{aligned}
K((n_1,t_1),x_1;(n_2,t_2),x_2) &= -\phi^{((n_1,t_1),(n_2,t_2))}(x_1,x_2)\mathbb{1}_{[(n_1,t_1)\prec(n_2,t_2)]} \\
&\quad + \widetilde{K}((n_1,t_1),x_1;(n_2,t_2),x_2) \qquad (4.18)
\end{aligned}$$

where $\phi^{((n_1,t_1),(n_2,t_2))}(x_1,x_2)$ is given by (2.6) and

$$\begin{aligned}
\widetilde{K}((n_1,t_1),x_1;(n_2,t_2),x_2) = \; & \frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_{-1}} \mathrm{d}w \oint_{\Gamma_0} \mathrm{d}z \frac{w^{n_1}(1+pw)^{t_1-n_1+1}}{(1+w)^{x_1+n_1+1}} \\
& \times \frac{(1+z)^{x_2+n_2}(1+2z+pz^2)}{z^{n_2}(1+pz)^{t_2-n_2+2}} \frac{1}{(w-z)(w+\frac{1+z}{1+pz})}. \qquad (4.19)
\end{aligned}$$

Here Γ_0 (resp Γ_{-1}) is any simple loop, anticlockwise oriented, which includes the pole at $z=0$ (resp. $w=-1$), satisfying $\{-\frac{1+z}{1+pz}, z\in\Gamma_0\}\subset\Gamma_{-1}$ and no point of Γ_0 lies inside Γ_{-1}.

Proof of Proposition 14. We substitute (4.13) and (4.14) in the kernel (4.2). Since $\Phi_j^{n,t}(x)=0$ for $j<0$, we can extend the sum over k to ∞. We can take the sum inside the integrals if the integration paths satisfy $\left|\frac{1+pw}{w(1+w)}\frac{z(1+z)}{1+pz}\right|<1$. Then we compute the geometric series and obtain

$$\sum_{k=1}^{\infty}\Psi_{n_1-k}^{n_1,t_1}(x_1)\Phi_{n_2-k}^{n_2,t_2}(x_2)=\frac{1}{(2\pi\mathrm{i})^2}\oint_{\Gamma_{0,-1}}\mathrm{d}w\oint_{\Gamma_0}\mathrm{d}z\frac{w^{n_1}(1+pw)^{t_1-n_1+1}}{(1+w)^{x_1+n_1+1}} \times\frac{(1+z)^{x_2+n_2}(1+2z+pz^2)}{z^{n_2}(1+pz)^{t_2-n_2+2}}\frac{1}{(w-z)(w+\frac{1+z}{1+pz})}. \tag{4.20}$$

At this point both simple poles $w=z$ and $w=-(1+z)/(1+pz)$ are inside the integration path $\Gamma_{0,-1}$, but the integrand does not have any pole anymore at $w=0$. Thus we will drop the 0 in $\Gamma_{0,-1}$. Separating the contribution from the pole at $w=z$ we get

$$\begin{aligned}\sum_{k=1}^{n_2}\Psi_{n_1-k}^{n_1,t_1}(x_1)\Phi_{n_2-k}^{n_2,t_2}(x_2) &= \widetilde{K}((n_1,t_1),x_1;(n_2,t_2),x_2) \qquad (4.21)\\ &+\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_0}\left(\frac{1+pz}{z}\right)^{n_2-n_1}(1+z)^{n_2+x_2-n_1-x_1-1}.\end{aligned}$$

Moreover, we also have

$$\begin{aligned}\phi^{*((n_1,t_1),(n_2,t_2))}(x_1,x_2) &= \phi^{((n_1,t_1),(n_2,t_2))}(x_1,x_2) \qquad (4.22)\\ &+\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma_0}\left(\frac{1+pz}{z}\right)^{n_2-n_1}(1+z)^{n_2+x_2-n_1-x_1-1}.\end{aligned}$$

Thus the last two terms of (4.21) and (4.22) cancel out, leading to (4.19). ☐

With Proposition 14 we almost obtained Theorem 1. What remains to do is to focus far enough into the negative axis, where the influence of the finiteness of the number of particles is not present anymore. There the kernel is equal to the kernel for the initial conditions $y_i=-2i$, $i\in\mathbb{Z}$.

Proof of Theorem 1. The kernel for the flat case is obtained by considering the region satisfying $x_1+n_1+1\leq 0$ where the effect of the boundary in the TASEP is absent. Here the pole at $w=-1$ vanishes. Computing the residue at $w=-(1+z)/(1+pz)$ in Proposition 14 gives the kernel (2.5) up to a factor $(-1)^{n_1-n_2}$ which we cancel by a conjugation of the kernel. ☐

5 Proof of Theorem 3

From Theorem 1 we have that $\mathbb{P}(x_n(t) \geq x) = \det(\mathbb{1} - \mathbb{1}_{(-\infty,x)} K \mathbb{1}_{(-\infty,x)})$. We have such a situation but with $x = y - sT^{1/3}$. With this change of variable, we get $\mathbb{P}(x_n(t) \geq y - sT^{1/3}) = \det(\mathbb{1} - \mathbb{1}_{(s,\infty)} K_T^{\rm resc} \mathbb{1}_{(s,\infty)})$ where $K_T^{\rm resc}(\xi_1, \xi_2) = T^{1/3} K(x_1 - \xi_1 T^{1/3}, x_2 - \xi_2 T^{1/3})$ (here we did not write explicitly the (n,t) entries). Taking into account the scaling (2.10), we thus have to analyze the rescaled kernel

$$\begin{aligned} & K_T^{\rm resc}(u_1, \xi_1; u_2, \xi_2) \\ = \; & T^{1/3} K\big((n(u_1), t(u_1)), x(u_1) - \xi_1 T^{1/3}; (n(u_2), t(u_2)), x(u_2) - \xi_2 T^{1/3}\big), \end{aligned} \tag{5.1}$$

with $x(u) = -2n(u) + \mathbf{v} t(u)$, $\mathbf{v} = 1 - \sqrt{q}$. In particular, we have to prove that, for u_1, u_2 fixed, $K_T^{\rm resc}$ (or a conjugate kernel of it) converges to the kernel $\kappa_{\rm v}^{-1} K_{\mathcal{A}_1}(\kappa_{\rm h}^{-1} u_1, \kappa_{\rm v}^{-1} \xi_1, \kappa_{\rm h}^{-1} u_2, \kappa_{\rm v}^{-1} \xi_2)$ uniformly on bounded sets and have enough control (bounds) on the decay of $K^{\rm resc}$ in the variables ξ_1, ξ_2 such that also the Fredholm determinant converges.

In order to have a proper limit of the kernel as $T \to \infty$, we have to consider the conjugate kernel $K_T^{\rm conj}$ given by

$$K_T^{\rm conj}(u_1, \xi_1; u_2, \xi_2) = K_T^{\rm resc}(u_1, \xi_1; u_2, \xi_2) \left(\frac{\sqrt{q}}{1 + \sqrt{q}} \right)^{x_1 - x_2} q^{n_1 - n_2} q^{-(t_1 - t_2)/2}. \tag{5.2}$$

The new kernel does not change the determinantal measure, being just a conjugation of the old one. So, in the following we will determine the limit of (5.2) as $T \to \infty$.

Proposition 15 (Uniform convergence on compact sets)**.** *For u_1, u_2 fixed, according to (2.10), set*

$$x_i = \left[-2n(u_i) + \mathbf{v} t(u_i) - \xi_i T^{1/3} \right], \tag{5.3}$$

$$n_i = n(u_i), \quad t_i = t(u_i). \tag{5.4}$$

Then, for any fixed $L > 0$, we have

$$\lim_{T \to \infty} K_T^{\rm conj}(n_1, x_1; n_2, x_2) T^{1/3} = \kappa_{\rm v}^{-1} K_{\mathcal{A}_1}(\kappa_{\rm h}^{-1} u_1, \kappa_{\rm v}^{-1} \xi_1; \kappa_{\rm h}^{-1} u_2, \kappa_{\rm v}^{-1} \xi_2) \tag{5.5}$$

uniformly for $(\xi_1, \xi_2) \in [-L, L]^2$, with the kernel $K_{\mathcal{A}_1}$ given by (2.7) and the constants $\kappa_{\rm v}$ and $\kappa_{\rm h}$ given by (2.13).

Proof of Proposition 15. First we consider the first term in (2.4). We thus consider (2.6) with the above replacements and conjugation. This term has to be considered only for $u_2 > u_1$. The change of variable $w = -1 + \sqrt{q}z$ leads then to

$$\frac{T^{1/3}}{2\pi \mathrm{i}} \oint_{\Gamma_0} \frac{\mathrm{d}z}{z} e^{T^{2/3}(g_0(z)-g_0(z_c))+T^{1/3}(g_1(z)-g_1(z_c))} \tag{5.6}$$

with $z_c = (1+\sqrt{q})^{-1}$ and

$$\begin{aligned} g_0(z) &= (u_2-u_1)(\pi'(\theta)+1)\big(\ln(\sqrt{q}+(1-q)z) - (1-\sqrt{q})\ln(z)\big) \\ &\quad + (u_2-u_1)(1-\pi'(\theta))\ln\left(\frac{\sqrt{q}+(1-q)z}{z(1-\sqrt{q}z)}\right), \\ g_1(z) &= -(u_2^2-u_1^2)\tfrac{1}{2}\pi''(\theta)\big(\sqrt{q}\ln(z)+\ln(1-\sqrt{q}z)\big) - (\xi_2-\xi_1)\ln(z). \end{aligned} \tag{5.7}$$

The function g_0 has a critical point at $z = z_c$. The series expansions around $z = z_c$ are

$$\begin{aligned} g_0(z) &= g_0(z_c) + (u_2-u_1)\kappa_1(z-z_c)^2 + \mathcal{O}((z-z_c)^3), \\ g_1(z) &= g_1(z_c) - (\xi_2-\xi_1)(1+\sqrt{q})(z-z_c) + \mathcal{O}((z-z_c)^2), \end{aligned} \tag{5.8}$$

where

$$\kappa_1 = \sqrt{q}(1+\sqrt{q})^2\left[(\pi'(\theta)+1)\frac{1-\sqrt{q}}{2} + 1 - \pi'(\theta)\right]. \tag{5.9}$$

To prove convergence of (5.6) we have to show that the contribution coming around the critical point dominates in the $T\to\infty$ limit. We do it by finding a steep descent path[1] for g_0 passing by $z = z_c$. Consider the path $\Gamma_0 = \{\rho e^{i\phi}, \phi \in [-\pi,\pi)\}$. Then, on Γ_0, $\frac{\mathrm{d}}{\mathrm{d}\phi}\mathrm{Re}(\ln(z)) = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}\phi}\mathrm{Re}(\ln(\sqrt{q}+(1-q)z)) = -\frac{\sqrt{q}(1-q)\rho\sin(\phi)}{|\sqrt{q}+(1-q)z|^2}, \tag{5.10}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\phi}\mathrm{Re}(-\ln(1-\sqrt{q}z)) = -\frac{\sqrt{q}\rho\sin(\phi)}{|1-\sqrt{q}z|^2}. \tag{5.11}$$

Thus Γ_0 is a steep descent path for g_0. Now we set $\rho = z_c$. Then, the real part of $g_0(z)$ is maximal at $z = z_c$ and strictly less then $g(z_c)$ for all other points on Γ_0. Therefore, we can restrict the integration from Γ_0 to $\Gamma_0^\delta = \{z \in \Gamma_0 \,|\, |z-z_c| \leq \delta\}$. For δ small, the error made is just of order

[1] For an integral $I = \int_\gamma \mathrm{d}z e^{tf(z)}$, we say that γ is a steep descent path if (1) $\mathrm{Re}(f(z))$ is maximal at some $z_0 \in \gamma$: $\mathrm{Re}(f(z)) < \mathrm{Re}(f(z_0))$ for $z \in \gamma \setminus \{z_0\}$ and (2) $\mathrm{Re}(f(z))$ is monotone along $\gamma \setminus \{z_0\}$ except, if γ is closed, at a single point where $\mathrm{Re}(f)$ is minimal.

$\mathcal{O}(e^{-cT^{2/3}})$ with $c>0$ ($c\sim\delta^2$ as $\delta\ll 1$). In the integral over Γ_0^δ we can use (5.8) to get

$$\begin{aligned}
&\frac{(1+\sqrt{q})T^{1/3}}{2\pi\mathrm{i}}\oint_{\Gamma_0^\delta}\mathrm{d}z e^{T^{2/3}(u_2-u_1)\kappa_1(z-z_c)^2-T^{1/3}(\xi_2-\xi_1)(1+\sqrt{q})(z-z_c)}\\
\times\;& e^{\mathcal{O}(T^{2/3}(z-z_c)^3,T^{1/3}(z-z_c)^2,(z-z_c))}. \qquad (5.12)
\end{aligned}$$

We use $|e^x-1|\leq |x|e^{|x|}$ to control the difference between (5.12) and the same expression without the error terms. By taking δ small enough and the change of variable $(z-z_c)T^{1/3}=W$, we obtain that this difference is just of order $\mathcal{O}(T^{-1/3})$, uniformly for ξ_1,ξ_2 in a bounded set. At this point we remain with (5.12) without the error terms. We extend the integration path to $z_c+\mathrm{i}\mathbb{R}$ and this, as above, gives an error of order $\mathcal{O}(e^{-cT^{2/3}})$. Thus we have

$$\begin{aligned}
(5.6) &= \mathcal{O}(e^{-cT^{2/3}},T^{-1/3}) \qquad (5.13)\\
&+ \frac{(1+\sqrt{q})T^{1/3}}{2\pi\mathrm{i}}\oint_{z_c+\mathrm{i}\mathbb{R}}\mathrm{d}z e^{T^{2/3}(u_2-u_1)\kappa_1(z-z_c)^2-T^{1/3}(\xi_2-\xi_1)(1+\sqrt{q})(z-z_c)}.
\end{aligned}$$

Therefore, uniformly for ξ_1,ξ_2 in bounded sets,

$$\lim_{T\to\infty}(5.6)=\frac{1}{\sqrt{4\pi(u_2-u_1)\alpha}}\exp\left(-\frac{(\xi_2-\xi_1)^2}{4(u_2-u_1)\alpha^2}\right) \qquad (5.14)$$

with $\alpha^2=\kappa_1/(1+\sqrt{q})^2=\kappa_{\mathrm{v}}^2/\kappa_{\mathrm{h}}$.

Now we have to consider the second term in (2.4). Notice that this time the restriction $u_2>u_1$ does not apply. Set $z_c=-1/(1+\sqrt{q})$. Then

$$\widetilde{K}_T^{\mathrm{conj}}(u_1,\xi_1;u_2,\xi_2)=\frac{-T^{1/3}}{2\pi\mathrm{i}}\oint_{\Gamma_0}\mathrm{d}z\frac{e^{Tf_0(z)+T^{2/3}f_1(z)+T^{1/3}f_2(z)+f_3(z)}}{e^{T^{2/3}f_1(z_c)+T^{1/3}f_2(z_c)}} \qquad (5.15)$$

with

$$\begin{aligned}
f_0(z) &= (\pi(\theta)+\theta)\Big[(1-\sqrt{q})\ln\left(\frac{1+z}{-z}\right)-(1+\sqrt{q})\ln(1+(1-q)z)+\sqrt{q}\ln(q)\Big],\\
f_1(z) &= (\pi'(\theta)+1)\Big[u_1((1-\sqrt{q})\ln(-z)+\sqrt{q}\ln((1+(1-q)z)/q))\\
&\qquad -u_2((1-\sqrt{q})\ln(1+z)-\ln(1+(1-q)z))\Big]\\
&\quad +(1-\pi'(\theta))(u_1-u_2)\ln\left(\frac{(1+z)(-z)}{1+(1-q)z}\right), \qquad (5.16)
\end{aligned}$$

$$
\begin{aligned}
f_2(z) = \frac{\pi''(\theta)}{2}\Big[&u_1^2(\ln(1+z)+\sqrt{q}\ln(-qz)-(1+\sqrt{q})\ln(1+(1-q)z)) \\
&-u_2^2(\sqrt{q}\ln(1+z)+\ln(-z))\Big] \\
&+\xi_1\ln(-qz/(1+(1-q)z))-\xi_2\ln(1+z), \\
f_3(z) = -\ln(&-z(1+(1-q)z)).
\end{aligned} \tag{5.17}
$$

The function f_0 has a double critical point at $z = z_c$ and the series expansions around $z = z_c$ of the f_i's are given by

$$
\begin{aligned}
f_0(z) &= \tfrac{1}{3}\kappa_2(z-z_c)^3 + \mathcal{O}((z-z_c)^4), \\
f_1(z) &= f_1(z_c) + \frac{\kappa_1}{q}(u_2-u_1)(z-z_c)^2 + \mathcal{O}((z-z_c)^3), \\
f_2(z) &= f_2(z_c) - (\xi_1+\xi_2)\frac{1+\sqrt{q}}{\sqrt{q}}(z-z_c) + \mathcal{O}((z-z_c)^2), \\
f_3(z) &= \ln((1+\sqrt{q})/\sqrt{q}) + \mathcal{O}((z-z_c)),
\end{aligned} \tag{5.18}
$$

with κ_1 given in (5.9) and

$$
\kappa_2 = \frac{(\pi(\theta)+\theta)(1-q)(1+\sqrt{q})^3}{q}. \tag{5.19}
$$

The leading contribution in the $T \to \infty$ limit will come from the region around the double critical point. The first step is to choose for γ_0 a steep descent path for f_0. First we consider $\gamma_0 = \{-\rho e^{\mathrm{i}\phi}, \phi \in [-\pi,\pi)\}$, $\rho \in (0, 1/(1+\sqrt{q})]$. The only part in $\mathrm{Re}(f_0(z))$ which is not constant along γ_0 is the term $(\pi(\theta)+\theta)(1-\sqrt{q})$ multiplied by $A(z) = \ln|1+z| - a\ln|1+(1-q)z|$, $a = (1+\sqrt{q})/(1-\sqrt{q})$. Simple computations lead then to

$$
\frac{\mathrm{d}}{\mathrm{d}\phi}A(z) = -\frac{\sin(\phi)\rho\sqrt{q}\big(2+\sqrt{q}-4\rho(1+\sqrt{q})\cos(\phi)+\rho^2(2-\sqrt{q})(1+\sqrt{q})^2\big)}{|1+z|^2|1+(1-q)z|^2}. \tag{5.20}
$$

This expression is strictly less than zero along γ_0 except at $\phi = 0, -\pi$, provided that the last term is strictly positive for $\phi \neq 0, -\pi$. This is easy to check because the last term reaches his minimum at $\cos(\phi) = -1$. Solving a second degree equations, we get that on $\rho \in (0, 1/(1+\sqrt{q}))$ it is strictly positive and at $\rho = 1/(1+\sqrt{q})$ is zero. Thus, the path γ_0 is steep descent for f_0.

But close to the critical point, the steepest descent path leaves with an angle $\pm\pi/3$. Therefore, consider for a moment $\gamma_1 = \{z = z_c + e^{-\mathrm{i}\pi\,\mathrm{sgn}(x)/3}|x|, x \in \mathbb{R}\}$. By symmetry we can restrict the next computations to $x \geq 0$. We have to see that $B(z)=\ln|1+z|-\ln|z|-a\ln|1+(1-q)z|$ is maximum at $x=0$ and decreasing for $x>0$. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}B(z) &= -\frac{x^2}{2|1+z|^2|z|^2|1+(1-q)z|^2} \qquad (5.21)\\ \times &\ \big(2q+2(1-q)\sqrt{q}x-(1-3\sqrt{q}+q)(1+\sqrt{q})^2x^2+2(1-q)(1+\sqrt{q})^3x^3\big).\end{aligned}$$

The term in the second line is always positive for all $x\geq 0$. To see this, remark that it is a polynomial of third degree which goes to ∞ as $x\to\infty$ and at $x=0$ is already positive and has positive slope. Therefore one just computes its stationary points and, if reals, takes the right-most one. There, the term under consideration turns out to be positive, which concludes the argument. Consequently, γ_1 is also a steep descent path.

We choose a steep descent path Γ_0 as follows. We follow γ_1 starting from the critical point until we intersect it with γ_0, and then we follow γ_0. Since Γ_0 is steep descent for f_0, we can integrate only on $\Gamma_0^\delta=\{z\in\Gamma_0||z-z_c|\leq\delta\}$. The error made by this cut is just of order $\mathcal{O}(e^{-cT})$ for some $c=c(\delta)>0$ (with $c\sim\delta^3$ as $\delta\to 0$). Around the critical point we use the series expansions (5.18). Thus we have

$$\begin{aligned}&\widetilde{K}_T^{\mathrm{conj}}(u_1,\xi_1;u_2,\xi_2)=\mathcal{O}(e^{-cT})\\ &+\frac{-T^{1/3}}{2\pi\mathrm{i}}\int_{\Gamma_0^\delta}\mathrm{d}z\frac{1+\sqrt{q}}{\sqrt{q}}e^{\mathcal{O}((z-z_c)^4T,(z-z_c)^3T^{2/3},(z-z_c)^2T^{1/3},(z-z_c))}\\ &\times e^{\frac{1}{3}\kappa_2(z-z_c)^3T+\frac{\kappa_1}{q}(u_2-u_1)(z-z_c)^2T^{2/3}-(\xi_1+\xi_2)\frac{1+\sqrt{q}}{\sqrt{q}}(z-z_c)T^{1/3}} \qquad (5.22)\end{aligned}$$

We want to cancel the error terms. The difference between (5.22) and the same expression without the error terms is bounded using $|e^x-1|\leq e^{|x|}|x|$, applied to $x=\mathcal{O}(\cdots)$. Then, this error term becomes

$$\begin{aligned}&\Big|\frac{T^{1/3}}{2\pi\mathrm{i}}\int_{\Gamma_0^\delta}\mathrm{d}z\frac{1+\sqrt{q}}{\sqrt{q}}\mathcal{O}((z-z_c)^4T,(z-z_c)^3T^{2/3},(z-z_c)^2T^{1/3},(z-z_c))\\ &\times e^{c_1\frac{1}{3}\kappa_2(z-z_c)^3T+c_2\frac{\kappa_1}{q}(u_2-u_1)(z-z_c)^2T^{2/3}-c_3(\xi_1+\xi_2)\frac{1+\sqrt{q}}{\sqrt{q}}(z-z_c)T^{1/3}}\Big| \qquad (5.23)\end{aligned}$$

for some c_1,c_2,c_3 depending on δ. As $\delta\to 0$, the $c_i\to 1$. Thus, for δ small enough, we have $c_1>0$. By the change of variable $(z-z_c)T^{1/3}=W$ we obtain that (5.23) is just of order $\mathcal{O}(T^{-1/3})$. Thus we have

$$\begin{aligned}&\widetilde{K}_T^{\mathrm{conj}}(u_1,\xi_1;u_2,\xi_2)=\mathcal{O}(e^{-cT},T^{-1/3}) \qquad (5.24)\\ &+\frac{-T^{1/3}}{2\pi\mathrm{i}}\int_{\Gamma_0^\delta}\mathrm{d}z\frac{1+\sqrt{q}}{\sqrt{q}}e^{\frac{1}{3}\kappa_2(z-z_c)^3T+\frac{\kappa_1}{q}(u_2-u_1)(z-z_c)^2T^{2/3}-(\xi_1+\xi_2)\frac{1+\sqrt{q}}{\sqrt{q}}(z-z_c)T^{1/3}}.\end{aligned}$$

The extension of the path Γ_0^δ to a path going from $e^{\mathrm{i}\pi/3}\infty$ to $e^{-\mathrm{i}\pi/3}\infty$ accounts into an error $\mathcal{O}(e^{-cT})$ only. We do the change of variable $Z = \kappa_2^{1/3} T^{1/3}(z-z_c)$ and we define

$$\kappa_{\mathrm{v}} = \frac{\kappa_2^{1/3}\sqrt{q}}{1+\sqrt{q}}, \quad \kappa_{\mathrm{h}} = \frac{\kappa_2^{2/3} q}{\kappa_1}. \tag{5.25}$$

Then,

$$\lim_{T\to\infty} \widetilde{K}_T^{\mathrm{conj}}(u_1,\xi_1;u_2,\xi_2) = \kappa_{\mathrm{v}}^{-1} \frac{-1}{2\pi\mathrm{i}} \int_{\gamma_\infty} \mathrm{d}Z e^{\frac{1}{3}Z^3 + (u_2-u_1)Z^2\kappa_{\mathrm{h}}^{-1} - (\xi_1+\xi_2)Z\kappa_{\mathrm{v}}^{-1}} \tag{5.26}$$

where γ_∞ is any path going from $e^{\mathrm{i}\pi/3}\infty$ to $e^{-\mathrm{i}\pi/3}\infty$. The proof ends by using the Airy function representation (A.5). □

Proposition 16 (Bound for the diffusion term of the kernel)**.** *Let n_i, t_i, and x_i be defined as in Proposition 15. Then, for $u_2 - u_1 > 0$ fixed and for any $\xi_1, \xi_2 \in \mathbb{R}$, the bound*

$$\left| \phi^{((n_1,t_1),(n_2,t_2))}(x_1,x_2) T^{1/3} \left(\frac{\sqrt{q}}{1+\sqrt{q}}\right)^{x_1-x_2} q^{n_1-n_2} q^{-(t_1-t_2)/2} \right| \leq \mathrm{const}\, e^{-|\xi_1-\xi_2|} \tag{5.27}$$

holds for T large enough and const *independent of T.*

Proof of Proposition 16. We start with (5.6). The difference now is that the contribution coming from large $|\xi_1 - \xi_2|$ can be of the same order as the one from $g_0(z)$. We consider as path $\Gamma_0 = \{\rho e^{\mathrm{i}\phi}, \phi \in [-\pi,\pi)\}$.

The difference is that now we choose ρ as follows. For an ε with $0 < \varepsilon \ll 1$ and $z_c = 1/(1+\sqrt{q})$,

$$\rho = \begin{cases} z_c + \frac{(\xi_2-\xi_1)T^{-1/3}}{(u_2-u_1)\kappa_1}, & \text{if } |\xi_2-\xi_1| \leq \varepsilon T^{1/3}, \\ z_c + \frac{\varepsilon}{(u_2-u_1)\kappa_1}, & \text{if } \xi_2-\xi_1 \geq \varepsilon T^{1/3}, \\ z_c - \frac{\varepsilon}{(u_2-u_1)\kappa_1}, & \text{if } \xi_2-\xi_1 \leq -\varepsilon T^{1/3}. \end{cases} \tag{5.28}$$

By (5.10) and (5.11), Γ_0 is a steep descent path for $g_0(z)$ plus the term proportional to $\xi_1 - \xi_2$ in $g_1(z)$. So, integrating on $\Gamma_0^\delta = \{z = \rho e^{\mathrm{i}\phi}, \phi \in (-\delta,\delta)\}$ instead of Γ_0 we do only an error of order $\mathcal{O}(e^{-cT^{2/3}})$ times the value at $\phi = 0$, for some $c > 0$. Thus

$$\begin{aligned} \text{LHS of (5.27)} &= e^{T^{2/3}(g_0(\rho)-g_0(z_c)) + T^{1/3}(g_1(\rho)-g_1(z_c))} \qquad (5.29) \\ &\times \left(\mathcal{O}(e^{-cT^{2/3}}) + \frac{T^{1/3}}{2\pi\mathrm{i}} \int_{\Gamma_0^\delta} \frac{\mathrm{d}z}{z} e^{T^{2/3}(g_0(z)-g_0(\rho)) + T^{1/3}(g_1(z)-g_1(\rho))} \right). \end{aligned}$$

On Γ_0^δ, the ξ_i-dependent term in $\mathrm{Re}(g_1(z)-g_1(\rho))$ is equal to zero. With the same procedure as in Proposition 15 one shows that the integral is bounded by a constant, uniformly in T.

It remains to estimate the first factor in (5.29). With our choice (5.28), we need just series expansions of g_0 and g_1 around ρ. Namely, by (5.8)

$$\begin{aligned} T^{2/3}(g_0(\rho)-g_0(z_c)) &= (u_2-u_1)\kappa_1(\rho-z_c)^2T^{2/3}(1+\mathcal{O}(\rho-z_c)), \\ T^{1/3}(g_1(\rho)-g_1(z_c)) &= (\xi_1-\xi_2)(1+\sqrt{q})(\rho-z_c)T^{1/3}(1+\mathcal{O}(\rho-z_c)) \\ &+ \mathcal{O}((\rho-z_c)^2)T^{1/3}. \end{aligned} \tag{5.30}$$

First consider the case $|\xi_2-\xi_1|\leq \varepsilon T^{1/3}$. We replace ρ given in (5.28) into (5.30) and get that the sum of the two contributions in (5.30) writes

$$-\frac{\sqrt{q}(\xi_2-\xi_1)^2}{(u_2-u_1)\kappa_1}\big(1+\mathcal{O}(\varepsilon)+\mathcal{O}(T^{-1/3})\big). \tag{5.31}$$

$\mathcal{O}(\varepsilon)$ comes from $\mathcal{O}(\rho-z_c)$, while the $\mathcal{O}(T^{-1/3})$ from $\mathcal{O}((\rho-z_c)^2)$. Then, by taking ε small enough and T large enough, we get

$$(5.31)\leq -|\xi_2-\xi_1|+\text{const}. \tag{5.32}$$

In the case, $\xi_2-\xi_1>\varepsilon T^{1/3}$, we also replace the appropriate ρ given in (5.28) into (5.30). We explicitly use the bound $\varepsilon T^{1/3}<\xi_2-\xi_1$ to bound $\mathcal{O}((\rho-z_c)^2)\leq(\xi_2-\xi_1)T^{-1/3}\varepsilon$. Then, we obtain the following bound for the sum of the two contributions in (5.30),

$$|\xi_2-\xi_1|\varepsilon T^{1/3}\left(\mathcal{O}(T^{-1/3})-\frac{\sqrt{q}}{(u_2-u_1)\kappa_1}(1+\mathcal{O}(\varepsilon))\right)\leq -|\xi_2-\xi_1| \tag{5.33}$$

by taking a fixed ε small enough and then T large enough. Finally, for $\xi_2-\xi_1<\varepsilon T^{1/3}$, the same result holds in a similar way. ☐

Proposition 17 (Bound for the main term of the kernel)**.** *Let n_i, t_i, and x_i be defined as in Proposition 15. Let $L>0$ fixed. Then, for given u_1,u_2 and $\xi_1,\xi_2\geq -L$, the bound*

$$\big|\widetilde{K}_T^{conj}(u_1,\xi_1;u_2,\xi_2)\big|\leq \text{const}\, e^{-(\xi_1+\xi_2)} \tag{5.34}$$

holds for T large enough and const *independent of T.*

Proof of Proposition 17. For $\xi_1,\xi_2\in[-L,L]$ it is the content of Proposition 15. Thus we consider $\xi_1,\xi_2\in[-L,\infty)^2\setminus[-L,L]^2$. Define $\tilde{\xi}_i=(\xi_i+2L)T^{-2/3}>0$. Then we consider a slight modification of (5.15), namely

$$\widetilde{K}_T^{\rm conj}(u_1,\xi_1;u_2,\xi_2)=\frac{-T^{1/3}}{2\pi\mathrm{i}}\oint_{\Gamma_0}\mathrm{d}z\frac{e^{T\tilde{f}_0(z)+T^{2/3}f_1(z)+T^{1/3}\tilde{f}_2(z)+f_3(z)}}{e^{T\tilde{f}_0(z_c)+T^{2/3}f_1(z_c)+T^{1/3}\tilde{f}_2(z_c)}} \tag{5.35}$$

with $f_1(z)$ and $f_3(z)$ as in (5.16)-(5.17), $\tilde f_2(z)$ as $f_2(z)$ in (5.17) but with ξ_1 and ξ_2 replaced by $-2L$, and finally $\tilde f_0(z)$ is set to be equal to $f_0(z)$ in (5.16) plus the term

$$-\tilde\xi_1 \ln((1+(1-q)z)/(-qz)) - \tilde\xi_2 \ln(1+z). \tag{5.36}$$

We also chose $\Gamma_0 = \{-\rho e^{\mathrm{i}\phi}, \phi\in[-\pi,\pi)\}$. In the proof of Proposition 15 we already proved that Γ_0 is a steep descent path for f_0 for the values $\rho\in(0,z_c]$. Also, since $\tilde\xi_i>0$, Re((5.36)) is also decreasing while $|\phi|$ is increasing. The precise choice of ρ is

$$\rho = \begin{cases} -z_c - ((\tilde\xi_1+\tilde\xi_2)/\kappa_2)^{1/2}, & \text{if } |\tilde\xi_1+\tilde\xi_2|\leq\varepsilon,\\ -z_c - (\varepsilon/\kappa_2)^{1/2}, & \text{if } |\tilde\xi_1+\tilde\xi_2|\geq\varepsilon,\end{cases} \tag{5.37}$$

for some small $\varepsilon>0$ which can be chosen later. Let us define

$$Q(\rho) = e^{\mathrm{Re}\left(T(\tilde f_0(-\rho)-\tilde f_0(z_c))+T^{2/3}(f_1(-\rho)-f_1(z_c))+T^{1/3}(\tilde f_2(-\rho)-\tilde f_2(z_c))\right)}. \tag{5.38}$$

Then, since Γ_0 is a steep descent path for $\tilde f_0$,

$$(5.35) = Q(\rho)\mathcal{O}(e^{-cT}) + Q(\rho)\frac{-T^{1/3}}{2\pi\mathrm{i}}\int_{\Gamma_0^\delta}\mathrm{d}z\frac{e^{T\tilde f_0(z)+T^{2/3}f_1(z)+T^{1/3}\tilde f_2(z)+f_3(z)}}{e^{T\tilde f_0(\rho)+T^{2/3}f_1(\rho)+T^{1/3}\tilde f_2(\rho)}}, \tag{5.39}$$

where $\Gamma_0^\delta = \{-\rho e^{\mathrm{i}\phi}, \phi\in(-\delta,\delta)\}$, for a small $\delta>0$. The expansion around $\phi=0$ leads to

$$\mathrm{Re}(\tilde f_0(-\rho e^{\mathrm{i}\phi}) - \tilde f_0(-\rho)) = -\gamma_1\phi^2(1+\mathcal{O}(\phi)) \tag{5.40}$$

with

$$\begin{aligned}\gamma_1 &= \frac{(1-\rho(1+\sqrt{q}))\rho\sqrt{q}(1-\sqrt{q})(\rho q+(1-\rho)(2+\sqrt{q}))}{2(1-\rho)^2(1-\rho(1-q))^2}\\ &+ \frac{\rho}{2}\left(\frac{\tilde\xi_2}{(1-\rho)^2}+\frac{\tilde\xi_1(1-q)}{(1-\rho(1-\rho))^2}\right)\end{aligned} \tag{5.41}$$

which is strictly positive for ρ chosen as in (5.37). Also, $\mathrm{Re}(f_1(-\rho e^{\mathrm{i}\phi}) - f_1(-\rho)) = \gamma_2\phi^2(1+\mathcal{O}(\phi))$ for some bounded γ_2 (we do not write it down explicitly since the precise formula is not relevant). Therefore, the last term in (5.39) is bounded by

$$\mathrm{const}\, Q(\rho)T^{1/3}\int_{-\delta}^{\delta}\mathrm{d}\phi e^{-\gamma\phi^2 T(1+\mathcal{O}(\phi))(1+\mathcal{O}(T^{-1/3}))} \tag{5.42}$$

with $\gamma = \gamma_1 + \gamma_2 T^{-1/3}$. By choosing δ small enough and independent of T, and then T large enough, the error terms can be replaced by $1/2$, and the integral is then bounded by the one on $\mathbb{R}$. Thus

$$(5.42) \leq \text{const}\, Q(\rho) \frac{1}{\sqrt{\gamma T^{1/3}}}. \tag{5.43}$$

In the worse case, when $\gamma \to 0$, which happens when $\rho \to z_c$, we have $\gamma_1 T^{1/3} \simeq (\xi_1 + \xi_2 + 4L)^{1/2} \geq (2L)^{1/2}$, which dominates γ_2 for L large enough.

Therefore we have shown that

$$\left|\widetilde{K}_T^{\text{conj}}(u_1, \xi_1; u_2, \xi_2)\right| \leq Q(\rho)\mathcal{O}(1). \tag{5.44}$$

It thus remains to find an bound on $Q(\rho)$. We have, by (5.18),

$$Q(\rho) = e^{\left[\frac{1}{3}\kappa_2(-\rho-z_c)^3 - (\tilde{\xi}_1+\tilde{\xi}_2)\frac{1+\sqrt{q}}{\sqrt{q}}(-\rho-z_c)T + \frac{\kappa_1}{q}(u_2-u_1)(-\rho-z_c)^2 T^{2/3}\right](1+\mathcal{O}(-\rho-z_c))}. \tag{5.45}$$

In the case $|\tilde{\xi}_1 + \tilde{\xi}_2| \leq \varepsilon$, we then obtain

$$Q(\rho) \leq e^{(\xi_1+\xi_2+4L)^{3/2}\left(\frac{1}{3} - \frac{1+\sqrt{q}}{\sqrt{q}}\right)\kappa_2^{-1/2}(1+\mathcal{O}(\varepsilon)) + (\xi_1+\xi_2+4L)\mathcal{O}(1)} \leq \text{const} e^{-(\xi_1+\xi_2)} \tag{5.46}$$

for $L \gg 1$, $\varepsilon \ll 1$. Finally, when $|\tilde{\xi}_1 + \tilde{\xi}_2| \geq \varepsilon$, we have

$$Q(\rho) \leq e^{-(\xi_1+\xi_2+4L)\left(\left(\frac{1}{3} - \frac{1+\sqrt{q}}{\sqrt{q}}\right)\varepsilon^{1/2} T^{1/3}\kappa_2^{-1/2} + \mathcal{O}(1)\right)} \leq e^{-(\xi_1+\xi_2)} \tag{5.47}$$

by first choosing $\varepsilon > 0$ small and then T large enough. □

Proof of Theorem 3. The proof of Theorem 3 is the complete analogue of Theorem 2.5 in [5]. The results in Propositions 5.1,5.3,5.4, and 5.5 in [5] are replaced by the ones in Proposition 15, 16, and 17. The strategy is to write the Fredholm series of the expression for finite T and, by using the bounds in Propositions 16 and 17, see that it is bounded by a T-independent and integrable function. Once this is proven, one can exchange the sums/integrals and the $T \to \infty$ limit by the theorem of dominated convergence. For details, see Theorem 2.5 in [5]. □

6 Proof of Theorem 5

In this section we prove Theorem 5. By Theorem 1, the right hand side of (2.16) with $n_i = \lfloor (t_i - H_i - x_i)/2 \rfloor$ can be written as Fredholm determinant of the kernel

$$\mathbb{1}(X_i < x_i) K((n_i, t_i), X_i; (n_j, t_j), X_j) \mathbb{1}(X_j < x_j) \tag{6.1}$$

with K given in (2.4). By the change of variable $X_i = -h_i + H_i + x_i$, one obtains the Fredholm determinant of the kernel

$$\mathbb{1}(h_i > H_i)K((n_i, t_i), H_i + x_i - h_i; (n_j, t_j), H_j + x_j - h_j)\mathbb{1}(h_j > H_j). \quad (6.2)$$

With this preparation, we now go to the proof of Theorem 5.

Proof of Theorem 5. We have to analyze the kernel (2.4) with entries

$$n_i = \frac{\mathbf{t}_i + \mathbf{x}_i}{\sqrt{q}} - \frac{H_i}{2}, \quad t_i = \frac{2\mathbf{t}_i}{\sqrt{q}}, \quad x_i = -\frac{2\mathbf{x}_i}{\sqrt{q}} + H_i - h_i \quad (6.3)$$

and take the limit $q \to 0$ with h_i, H_i fixed. The scaling of x_i might look different from the one in (2.17) but, as we can see below, (6.3) with the last one replaced by $x_i = -\frac{2\mathbf{x}_i}{\sqrt{q}}$ gives the same limiting kernel. As $q \to 0$, the kernel does not have a well defined limit and, as usual, we first have to consider a conjugate kernel. More precisely, we define

$$K_q((\mathbf{x}_1, \mathbf{t}_1), h_1; (\mathbf{x}_2, \mathbf{t}_2), h_2) = K((n_1, t_1), x_1; (n_2, t_2), x_2)q^{(x_1 - x_2)/2}\frac{q^{n_1 - n_2}}{q^{(t_1 - t_2)/2}}. \quad (6.4)$$

What we have to prove is

$$\lim_{q\to 0} \det(\mathbb{1} - \chi_H K_q \chi_H) = \det(\mathbb{1} - \chi_H K^{\rm PNG} \chi_H). \quad (6.5)$$

First we prove the pointwise convergence and then we obtain bounds allowing us to take the limit inside the Fredholm determinant.

Consider the term coming from (2.6). By the change of variable $w = -1 + \sqrt{q}z$, we get

$$\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \frac{\mathrm{d}z}{z} \frac{(\sqrt{q} + (1-q)z)^{t_1 - t_2}}{z^{x_1 - x_2}} \left(\frac{1 - \sqrt{q}z}{z(\sqrt{q} + (1-q)z)}\right)^{n_1 - n_2} \quad (6.6)$$

and, by inserting (6.3), one obtains

$$\begin{aligned} &\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \frac{\mathrm{d}z}{z} \left(\frac{(\sqrt{q} + (1-q)z)(1 - \sqrt{q}z)}{z}\right)^{(\mathbf{t}_1 - \mathbf{t}_2)/\sqrt{q}} \\ \times &\left(\frac{\sqrt{q} + (1-q)z}{(1 - \sqrt{q}z)z}\right)^{(\mathbf{x}_1 - \mathbf{x}_2)/\sqrt{q}} \left(\frac{\sqrt{q} + (1-q)z}{(1 - \sqrt{q}z)z}\right)^{(H_1 - H_2)/2} \frac{1}{z^{h_2 - h_1}}. \end{aligned} \quad (6.7)$$

Consider $q \leq q_0$ for some $q_0 < 1$ fixed. Then, we can fix the path Γ_0 independent of q, and the $q \to 0$ limit is easily obtained. It results in

$$\begin{aligned} \lim_{q\to 0} (6.7) &= \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \frac{\mathrm{d}z}{z} \frac{1}{z^{h_2 - h_1}} e^{-(\mathbf{t}_1 - \mathbf{t}_2)(z - z^{-1})} e^{(\mathbf{x}_2 - \mathbf{x}_1)(z + z^{-1})} \quad (6.8) \\ &= \left(\frac{\mathbf{x}_2 - \mathbf{x}_1 + \mathbf{t}_1 - \mathbf{t}_2}{\mathbf{x}_2 - \mathbf{x}_1 - \mathbf{t}_1 + \mathbf{t}_2}\right)^{|h_1 - h_2|/2} I_{|h_1 - h_2|}\left(2\sqrt{(\mathbf{x}_2 - \mathbf{x}_1)^2 - (\mathbf{t}_1 - \mathbf{t}_2)^2}\right), \end{aligned}$$

where we applied (A.4).

It is the turn of the term coming from (2.5). We do the change of variable $z = -w/(w+\sqrt{q})$ and then we insert (6.3). The result is

$$\begin{aligned} &\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \frac{\mathrm{d}w}{w(1+\sqrt{q}w)} \left(\frac{\sqrt{q}+w}{w(1+\sqrt{q}w)}\right)^{(\mathbf{t}_1+\mathbf{t}_2)/\sqrt{q}} \left(\frac{w}{1+\sqrt{q}w}\right)^{h_1} (w+\sqrt{q})^{h_2} \\ &\times \left(\frac{w}{(w+\sqrt{q})(1+\sqrt{q}w)}\right)^{(\mathbf{x}_1-\mathbf{x}_2)/\sqrt{q}+(H_2-H_1)/2}. \end{aligned} \tag{6.9}$$

If $\mathbf{x}_2 - \mathbf{x}_1 > \mathbf{t}_1 + \mathbf{t}_2$, then for q small enough, the result is identically equal to zero, because the pole at $w = 0$ vanishes. If $\mathbf{x}_1 - \mathbf{x}_2 > \mathbf{t}_1 + \mathbf{t}_2$, then the result is also zero, because the residues at all other poles, $\sqrt{q}$, $1/\sqrt{q}$, and ∞ vanishes. In the other case, when $|\mathbf{x}_2 - \mathbf{x}_1| < \mathbf{t}_1 + \mathbf{t}_2$, the apparent pole at $w = -\sqrt{q}$ is actually not there. So, we can choose a Γ_0 independent of $q \leq q_0$ for some $q_0 < 1$. Then, we can simply take the limit $q \to 0$ of the integrand, which leads to

$$\begin{aligned} \lim_{q\to 0} (6.9) &= \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \frac{\mathrm{d}w}{w} w^{h_1+h_2} e^{(\mathbf{t}_1+\mathbf{t}_2)(w^{-1}-w)} e^{(\mathbf{x}_2-\mathbf{x}_1)(w+w^{-1})} \qquad (6.10) \\ &= \left(\frac{\mathbf{t}_1+\mathbf{t}_2+\mathbf{x}_2-\mathbf{x}_1}{\mathbf{t}_1+\mathbf{t}_2-\mathbf{x}_2+\mathbf{x}_1}\right)^{(h_1+h_2)/2} J_{h_1+h_2}\left(2\sqrt{(\mathbf{t}_1+\mathbf{t}_2)^2-(\mathbf{x}_2-\mathbf{x}_1)^2}\right), \end{aligned}$$

where in the last step we made the change of variable $w \to 1/w$ and applied (A.3).

To have convergence of the Fredholm determinants we still need some bounds for large values of h_1, h_2. For q small enough, say $q \in [0, q_0]$ for some $q_0 < 1$, we can set in (6.7) $\Gamma_0 = \{z, |z| = \mathrm{e}\}$ in the case $h_2 \geq h_1$, and $\Gamma_0 = \{z, |z| = \mathrm{e}^{-1}\}$ in the case $h_2 < h_1$. Then, we get the bound

$$|(6.7)| \leq C_1 e^{-|h_2-h_1|} \tag{6.11}$$

for some finite constant C_1 independent of q. Moreover, in (6.9) we can choose $\Gamma_0 = \{z, |z| = \mathrm{e}^{-2}\}$, which leads to the bound

$$|(6.9)| \leq C_2 e^{-(h_2+h_1)} \tag{6.12}$$

with $C_2 < \infty$ independent of $q \leq q_0$. These two bounds are enough to have convergence of the Fredholm determinants. The strategy is exactly the same as in the proof of Theorem 3. □

7 Proof of Theorem 6

We analyze the kernel (2.23) with the scalings

$$\mathbf{x}_i = u_i T^{2/3}, \tag{7.1}$$

$$\mathbf{t}_i = \gamma(0)T + \gamma'(0)u_i T^{2/3} + \frac{\gamma''(0)}{2}u_i^2 T^{1/3}, \tag{7.2}$$

$$h_i = \lfloor 2\mathbf{t}_i + \xi_i T^{1/3} \rfloor \tag{7.3}$$

(See (2.24) and (2.25)). The strategy of he proof is the same as that for Theorem 3 and hence we only give the main differences.

First we consider the first term in (2.23). From (6.8) it is rewritten in the from (5.6) with $g_0(z), g_1(z)$ replaced by

$$g_0(z) = (u_2 - u_1)\left(\gamma'(0)(z - 1/z - 2\ln z) + (z + 1/z)\right), \tag{7.4}$$

$$g_1(z) = \frac{\gamma''(0)}{2}(u_2^2 - u_1^2)(z - 1/z - 2\ln z) - (\xi_2 - \xi_1)\ln z. \tag{7.5}$$

The critical point of $g_0(z)$ is now $z_c = 1$. The series expansions around $z = z_c$ are

$$g_0(z) = g_0(z_c) + (u_2 - u_1)(z - z_c)^2 + \mathcal{O}((z - z_c)^3), \tag{7.6}$$

$$g_1(z) = g_1(z_c) - (\xi_2 - \xi_1)(z - z_c) + \mathcal{O}((z - z_c)^2). \tag{7.7}$$

The steep descent path can be taken to be $\Gamma_0 = \{e^{i\phi}, \phi \in [-\pi, \pi)\}$. Then the same arguments as in the proof of Theorem 3 give the first term in (2.7).

Next we consider the second term in (2.23). From (6.10) it is rewritten in the form (5.15) with $f_0(z), f_1(z), f_2(z)$ replaced by

$$f_0(w) = 2\gamma(0)(1/w - w + 2\ln w), \tag{7.8}$$

$$f_1(w) = \gamma'(0)(u_1 + u_2)(1/w - w + 2\ln w) + (u_2 - u_1)(w + 1/w), \tag{7.9}$$

$$f_2(w) = \frac{\gamma''(0)}{2}(u_1^2 + u_2^2)(1/w - w + 2\ln w) + (\xi_1 + \xi_2)\ln w. \tag{7.10}$$

Their series expansions around z_c are

$$f_0(w) = -\frac{2\gamma(0)}{3}(w - 1)^3 + \mathcal{O}((w - 1)^4), \tag{7.11}$$

$$f_1(w) = f_1(z_c) + (u_2 - u_1)(w - 1)^2 + \mathcal{O}((w - 1)^3), \tag{7.12}$$

$$f_2(w) = f_2(z_c) + (\xi_1 + \xi_2)(w - 1) + \mathcal{O}((w - 1)^2). \tag{7.13}$$

The steepest descent path is taken to be $\Gamma_0 = \{\rho e^{i\phi}, \phi \in [-\pi, \pi)\}$ with $0 < \rho < 1$. Using these one obtains the second term in (2.7).

The bounds for the diffusion terms and the main term of the kernel are also proved in the same way as those of Propositions 16 and 17.

A Some integral representations

In this appendix we list some integral representations of the Bessel functions and the modified Bessel functions (we use the conventions of [1]). For $n \in \mathbb{Z}$,

$$J_n(2t) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \frac{\mathrm{d}z}{z} \frac{e^{t(z-z^{-1})}}{z^n}, \tag{A.1}$$

$$I_{|n|}(2t) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \frac{\mathrm{d}z}{z} \frac{e^{t(z+z^{-1})}}{z^n}, \tag{A.2}$$

$$\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \frac{\mathrm{d}z}{z} \frac{e^{b(z-z^{-1})}e^{a(z+z^{-1})}}{z^n} = \left(\frac{b+a}{b-a}\right)^{n/2} J_n\left(2\sqrt{b^2-a^2}\right), \tag{A.3}$$

$$\frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \frac{\mathrm{d}z}{z} \frac{e^{b(z-z^{-1})}e^{a(z+z^{-1})}}{z^n} = \left(\frac{a+b}{a-b}\right)^{|n|/2} I_{|n|}\left(2\sqrt{a^2-b^2}\right), \tag{A.4}$$

$$\frac{-1}{2\pi \mathrm{i}} \int_{\gamma_\infty} \mathrm{d}v e^{v^3/3+av^2+bv} = \mathrm{Ai}(a^2-b)\exp(2a^3/3-ab), \tag{A.5}$$

where γ_∞ is any path from $e^{\mathrm{i}\pi/3}\infty$ to $e^{-\mathrm{i}\pi/3}\infty$.

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