Swiss Federal Institute of Technology
Lausanne
Institute of Theoretical Physics

Rutgers
The State University of New Jersey
Departement of Mathematics

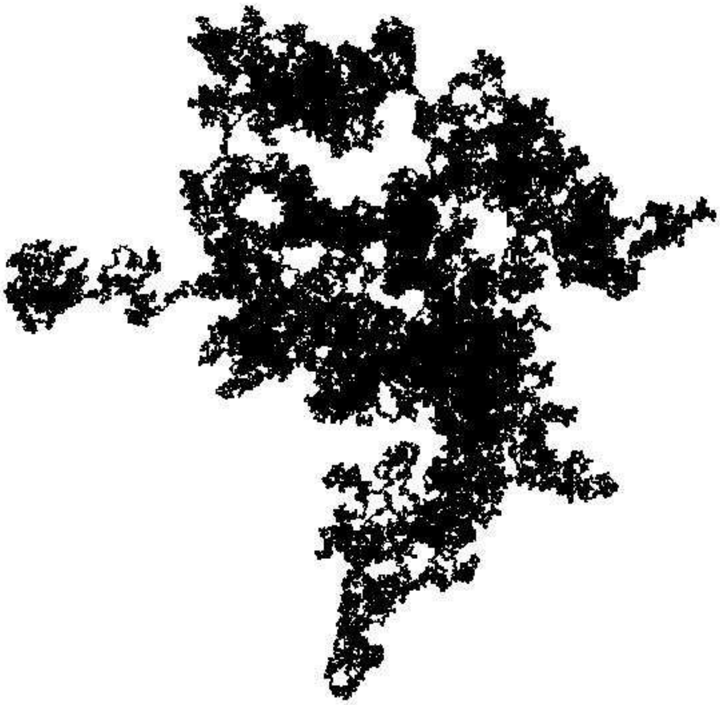
Contact Matrices for Random Walks

Diploma Work of
Patrik Ferrari

Supervised by
Prof. Joel Lebowitz

October 2000 - February 2001

A Christian, Mamma e Papà



Acknowledgments

Innanzitutto ringrazio mio fratello Christian per le particolari opportunità che ho potuto sfruttare, grazie ai suoi consigli, durante questi quattro anni di studio.
Ringrazio inoltre i miei famigliari, in special modo mia mamma, mio papà e mio fratello, per avermi sempre sostenuto durante questo periodo e per essermi stati sempre vicini, particolarmente durante questo soggiorno negli Stati Uniti. Nemmeno l'Atlantico è abbastanza grande per farti sentire solo quando c'è chi ti sta vicino.

In regards to the academic environment, I would like to thank Prof. Joel Lebowitz for giving me the opportunity of coming to Rutgers for my diploma work, for his availability and also for his kindness.
My thanks also go to Prof. Christian Gruber for his help. I would also like to thank Prof. Charles Pfister for helping me and initiating me to the subject of random walks during the TP4. My thanks also go to Prof. Harry Kesten for his useful proof for self-avoiding walks. I am also grateful to the people who read the manuscript and gave me advice.
Finally, I would like to thank Daniel Ueltschi for his hospitality during my stay in the United States.

Contents

Introduction

Self-avoiding random walks on a lattice with nearest neighbor steps are used to model polymers. Variations of such walks, such as the Domb-Joyce model where each walk receives a penalty $e^{-\beta}$ for every self-intersection are also used. But many questions are not known rigorously in the most interesting dimensions (two and three). Known results are in dimension one or dimension five or higher (see [3] for a discussion of one dimensional random polymers).
In [1] the authors proposed a model of protein folding in which the interaction between carbon atoms is replaced by linear springs with Gaussian fluctuations. The equilibrium correlation between fluctuations of two carbon atoms can be expressed in terms of a connectivity matrix (similar to contact maps used in [4]). Their results on the variance of atom positions are in good agreement with experimental data.

In the present work we investigate properties related to self-intersections of random walks on various lattices. We are motivated by [4] where the authors studied contact maps for self-avoiding walks. The contact map of a self-avoiding walk of length N is the matrix $S \in M_{N+1}(\mathbb{R})$ such that $S_{ij} = 1$ if steps i and j $(i \neq j)$ are nearest neighbors and $S_{ij} = 0$ otherwise. They studied some statistical properties of these contact maps analytically and also carried out numerical simulations.
Here we will consider random walks that intersect with or without constraints (like the avoiding of immediate return), the contact matrices of such walks are the matrices $\mathcal{C} \in M_{N+1}(\mathbb{R})$ such that $\mathcal{C}_{ij} = 1$ if steps i and j $(i \neq j)$ are at the same spatial position and $\mathcal{C}_{ij} = 0$ otherwise.
The main quantity we investigate is the exponential growth factor of the number of different contact matrices in the asymptotic limit when the length of the random walk goes to infinity. Our principal result is that, to leading order, the number of contact matrices equals the number of random walks for unweighted random walks. We also present some results on the degeneracy of the different contact matrices (i.e. the number of random walks that cannot be distinguished by looking only at $\mathcal{C}$).

Now we describe the structure of this paper.

- In section 1 we give some definitions and introduce the problem to be studied.
- In section 2 we consider the one dimensional random walk.
- In section 3 we consider random walks on a strip. We give the asymptotic behavior of the number of contact matrices. We also compare the end-to-end distance of two cases.
- In section 4 we consider random walks on two dimensional lattices. We give some results on the growth factor, on the degeneracy and on the end-to-end distance. The recurrence property of these walks will be important.

- In section 5 we consider simple random walks on $\mathbb{Z}^d$. We give a result on the degeneracy (proved for $d = 3$) and on the growth factor. In this case the result will be determined using properties on large deviation of the support of simple random walks.

- In section 6 we look at self-avoiding and bond-self-avoiding (i.e. each bond cannot be occupied more than once). In these case we have a different result on the growth factor. First we adapt a theorem of Kesten on self-avoiding walks for bond-self-avoiding walks. After that we give the result on the exponential growth factor.

- In section 7 we consider a particular case of random walk. Each walk receives a probability weight as function of the number of intersections. We give a result directly related to the growth factor but for a weighted number of matrices.

1 Definitions

We consider the set of random walks on a d-dimensional lattice $\mathcal{L}$ starting from the origin. The random walks are not necessarily simple and while $\mathcal{L}$ is often $\mathbb{Z}^d$, it is not always so (for example it can be isomorphe to a subset of $\mathbb{Z}^d$, like for the honeycomb lattice). First some definitions.

Definition 1. *The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is defined as follows:*

- *Sample space: $\Omega \doteqdot \{\omega = (0, \omega(1), \omega(2), \dots)$ s.t. $\omega(i) \in \mathcal{L}$ for $i \in \mathbb{N}^*\}$,*
- *Set of events: the σ-algebra is the one generated by the set of cylinders $\mathcal{C}$, which is defined by*

  $$\mathcal{C} \doteqdot \{A_n = \{\omega \in \Omega \ s.t. \ \omega(k) = a_k \in \mathcal{L} \ for \ k = 1, \dots, n\}, n \geq 0\}.$$

  Let us consider a cylinder

  $$A_n = \{\omega \in \Omega \ s.t. \ \omega(k) = a_k \in \mathcal{L} \ for \ k = 1, \dots, n\} \equiv [a_1, \dots, a_n].$$

  The complementary set A_n^c is given by

  $$\begin{aligned} A_n^c &= \{\omega \in \Omega \ s.t. \ \omega(k) \neq a_k \in \mathcal{L} \ for \ at \ least \ a \ k \in \{1, \dots, n\}\} \\ &\equiv \bigcup_{\substack{b_1, \dots, b_n \ s.t. \\ \exists i \ s.t. \ a_i \neq b_i}} [b_1, \dots, b_n] \end{aligned}$$

  and the union of two cylinders $A_n = [a_1, \dots, a_n]$ and $B_m = [b_1, \dots, b_m]$ is given by

  $$A_n \cup B_m = \{\omega \in \Omega \ s.t. \ \omega(i) \in \{a_i, b_i\} \ for \ i = 1, \dots, \min\{n, m\}\}.$$

  Let $\widetilde{\mathcal{C}}$ be the algebra generated by $\mathcal{C}$. Then $\mathcal{F} = \tau(\widetilde{\mathcal{C}})$ is the σ-algebra generated by $\widetilde{\mathcal{C}}$.

- *Probability measure: we define the probability measure on the cylinders.*

  $$\begin{aligned} \mathbb{P}(A_n) &\doteqdot \mathbb{P}(\omega(1) = a_1, \dots, \omega(n) = a_n) \\ \mathbb{P}(A_0) &\doteqdot 1. \end{aligned}$$

  The probability measure depends on the constraints on random walks. Since it is bounded ($\mathbb{P}(\Omega) = 1$) there exists an unique *extension of the measure $\mathbb{P}$ on the σ-algebra $\mathcal{F}$.*

We will often consider the random walks with a fixed length N. In this case $\Omega'_N \doteqdot \{\omega = (0, \omega(1), \dots, \omega(N))$ s.t. $\omega(i) \in \mathcal{L}\}$, the σ-algebra considered is the one generated by the set of cylinders of length N at most.

The probability $\mathbb{P}_N$ is defined in the same way[1]. In fact we are interested in those $\omega \in \Omega'_N$ that occur with non-zero probability. Therefore we define

$$\Omega_N \doteqdot \{\omega \in \Omega'_N \text{ s.t. } \mathbb{P}_N(\omega) \neq 0\}.$$

Let the cardinality of Ω_N be denoted by $\mathrm{Card}(\Omega_N)$. In what follows, with the only exception of section 7, we give the same weight to each $\omega \in \Omega_N$:

$$\mathbb{P}_N(\omega) = \frac{1}{\mathrm{Card}(\Omega_N)}.$$

Definition 2 (Random walk). *Let $\mathcal{L}$ be a d-dimensional lattice (e.g. $\mathbb{Z}^d$). A random walk of length N, a N-random walk, is an element of Ω_N.*

Definition 3 (Simple random walk). *The simple random walk on $\mathbb{Z}^d$ is the one such that the vector-valued displacements X_k are independent and identically distributed with $\mathbb{P}(X_k = e_i) = \mathbb{P}(X_k = -e_i) = \frac{1}{2d}$ where $\mathbb{Z}^d \ni (e_i)_k = \delta_{k,i}$, $i = 1, \dots, d$.*

Definition 4 (Concatenation). *The concatenation of two walks ω_1 and ω_2 of length N_1 and N_2 respectively is the walk of length $N_1 + N_2$ defined as follows:*

$$\omega(k) = \begin{cases} \omega_1(k) & \text{for } k = 0, \dots, N_1, \\ \omega_1(N_1) + \omega_2(k - N_1) - \omega_2(0) & \text{for } k = N_1 + 1, \dots, N_1 + N_2. \end{cases}$$

Definition 5 (Support or Range of a random walk).
The support (or range) of a N-random walk $\omega = (0, \omega(1), \dots, \omega(N))$ is the number of different lattice sites visited by the random walk, i.e.

$$R_N \doteqdot \mathrm{Card}(\{0, \omega(1), \dots, \omega(N)\}).$$

Definition 6 (Intersections). *The number of intersections of a random walk is defined to be the number of steps for which the random walk visits a place that was already visited by it. Therefore the number of intersection is $N + 1 - R_N$.*

Definition 7 (Contact matrices). *Let $\omega = (0, \omega(1), \dots, \omega(N)) \in \Omega_N$ be a random walk of length N. We define the application $\mathcal{C}$*

$$\begin{aligned} \mathcal{C} : \Omega_N &\longmapsto M_{N+1}(\mathbb{R}) \\ \omega &\longrightarrow \mathcal{C}_{i,j}(\omega),\ i, j \in \{0, \dots, N\} \end{aligned}$$

where $\mathcal{C}_{i,j}(\omega) = \begin{cases} 1 & \text{if } \omega(i) = \omega(j), i \neq j, \\ 0 & \text{otherwise.} \end{cases}$
The contact matrix of ω is its image by $\mathcal{C}$. The ***degeneracy*** *of a contact matrix is the number of random walks corresponding to that contact matrix*[2].

[1] In what follows the index N on $\mathbb{P}_N$ will not be written explicitly when not necessary.
[2] See figure 1, page 4.

Definition 8 (Growth factor). *Let $W(N)$ be the total number of* ***different*** *contact matrices of the N-random walks. The growth factor of $W(N)$ is given by*

$$\gamma_N \doteq \frac{\ln W(N)}{\ln \mathrm{Card}(\Omega_N)}$$

The limit of the growth factor is denoted by $\bar{\gamma}$:

$$\bar{\gamma} \doteq \lim_{N\to\infty} \gamma_N$$

In the case considered (except for the m-strip with $m > 2$) the cardinality of Ω_N is well defined because for each step we have a fixed number of choices[3], say C, then $\mathrm{Card}(\Omega_N) = C^N$ and $\frac{\ln \mathrm{Card}(\Omega_N)}{N} = \ln C$. For the m-strip with $m > 2$ we expect that exists a C such that $\lim_{N\to\infty} \frac{\ln \mathrm{Card}(\Omega_N)}{N} = \ln C$ (but it is not necessary for the proof). The existence of the limit is assured by a superadditivity property.

Proposition 9.
The total number of different contact matrices $W(N)$ satisfies

$$W(N+M) \geq W(N)W(M) \tag{1}$$

and this implies that

$$\lim_{N\to\infty} \frac{\ln W(N)}{N} \text{ exists in } (-\infty, \infty] \tag{2}$$

and is equal to

$$\lim_{N\to\infty} \frac{\ln W(N)}{N} = \sup_{N\geq 1} \frac{\ln W(N)}{N} \tag{3}$$

Proof.
Let us consider two contact matrices $P_1 \in M_{N+1}(\mathbb{R})$ and $P_2 \in M_{M+1}(\mathbb{R})$. Then for each choice of ω_1 and ω_2 whose contact matrices are P_1 and P_2 respectively, the contact matrix $P \in M_{M+N+1}(\mathbb{R})$ of $\omega = \omega_1 \circ \omega_2$, the concatenation[4] of ω_1 and ω_2, is of the form

$$P = \begin{pmatrix} P_1 & Q \\ Q^T & P_2 \end{pmatrix}$$

where Q is a matrix in $M_{M,N}(\mathbb{R})$ that depends on the intersection of ω_1 and ω_2, $(P_1)_{N+1,N+1}$ and $(P_2)_{1,1}$ overlap. Since generally each contact matrix corresponds to more than one random walk, whose intersections with another walk can be different, $W(N+M)$ is greater than $W(N)W(M)$ and equation (1) is verified.
The sequence $\{\ln W(N)\}_{N\geq 1}$ is superadditive. It also clearly has an upper bound $N \ln D$ (D is the coordinations number of $\mathcal{L}$), so (2) and (3) hold (see lemma 1.2.2 [7]). □

[3] The first step can have a different number of choices, C', in which case $\mathrm{Card}(\Omega_N) = C^{N-1}C'$ but the behavior is exactly the same.

[4] It is always possible to choose, for each P_2, a ω_2 such that the concatenation satisfies an eventual constraint related to the previous step because of the first step freedom.

Definition 10 (End-to-end distance). *Let us consider a random walk on a lattice* $\mathcal{L}$ *and let* $X_1, \ldots, X_N$ *be the vector-valued displacements. Then the total displacement is given by the random variable* $S_N \doteqdot \sum_{k=1}^{N} X_k$. *The end-to-end distance is defined by* $\sqrt{\mathbb{E}(S_N^2)}$.

Formulation of problem

The main question we study is the growth factor of the number of contact matrices. It can be expressed as

$$W(N) = \sum_{\omega \in \Omega_N} \frac{1}{\deg \mathcal{C}(\omega)}$$

where the degeneracy of the matrix $\mathcal{C}(\omega)$, $\deg \mathcal{C}(\omega)$, is the number of different ω which give the same $\mathcal{C}(\omega)$.

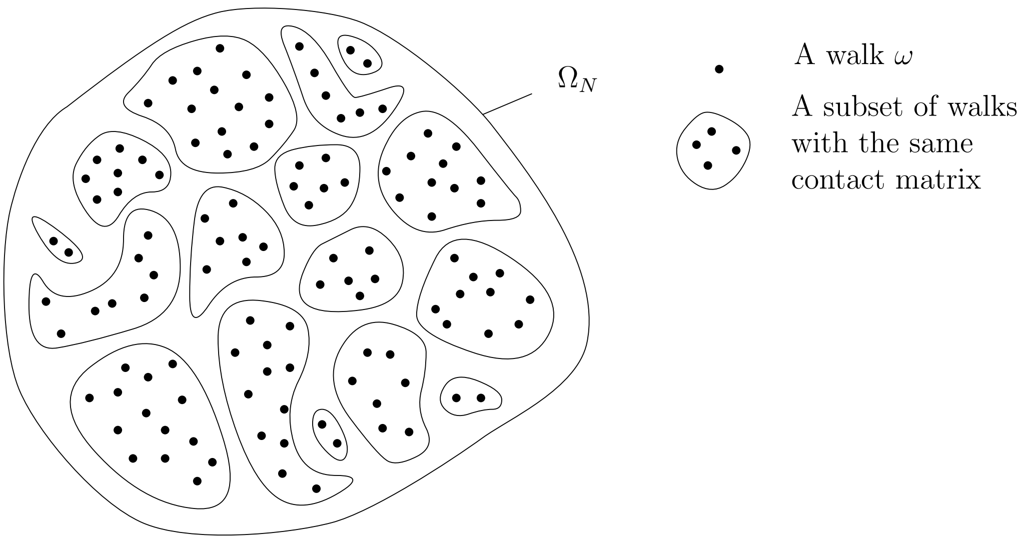
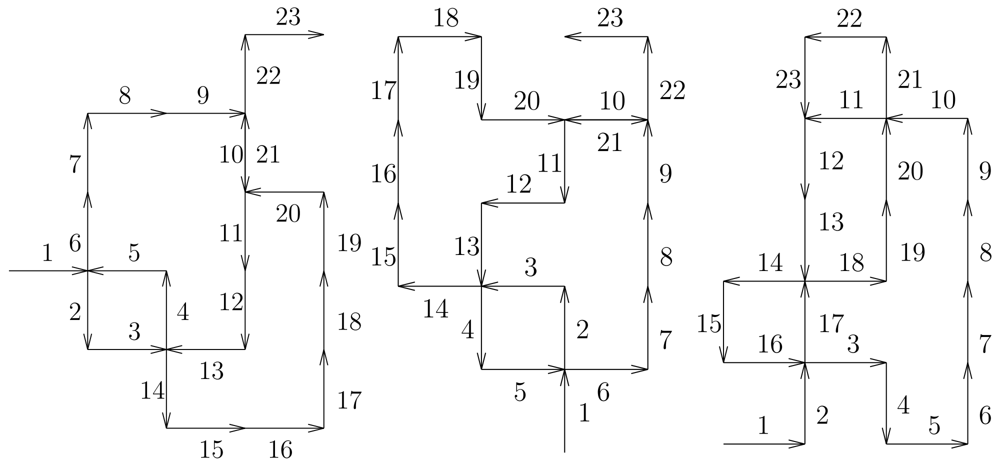


Figure 1: Illustration of contact matrices and random walks.

Example 11. *In the next figure, the first two random walks have the same contact matrix and the third one has a different contact matrix.*



walk ω_1 walk ω_2 walk ω_3

Figure 2: Three random walks of same length (23 steps). ω_1 and ω_2 have the same contact matrix because they have the same intersections ($1 = 5$, $3 = 13$, $9 = 21$, $10 = 20$). The contact matrix of ω_3 is different because its intersections are $2 = 16$, $10 = 20$, $11 = 23$, $13 = 17$.

2 The one dimensional random walk

The one dimensional case is almost trivial. In fact, let us consider a N-random walk on $\mathbb{Z}$ (with unit steps), $\omega = (0, \omega(1), \ldots, \omega(N))$. The cardinality of Ω_N is 2^N. Because of the symmetry of the positive and negative direction, the random walks $\omega = (0, \omega(1), \ldots, \omega(N))$ and $\omega' \doteqdot -\omega = (0, -\omega(1), \ldots, -\omega(N))$ correspond to the same contact matrix. Let us consider the class of random walks such that $\omega_1 = 1$. Then for each choice of the next $N-1$ steps the corresponding contact matrix will be different. In fact for the step k the contact matrix will have $\mathcal{C}_{k,k-2} = 1$ in the case that the random walk come back immediately or $\mathcal{C}_{k,k-2} = 0$ in the case that the random walk does not turn at step k.
Therefore the total number of different contact matrices $W(N)$ for the set of N-random walk is

$$W(N) = 2^{N-1}.$$

The degeneracy of each contact matrix is exactly two and the growth factor is

$$\gamma_N = \frac{N-1}{N} \longrightarrow \bar{\gamma} = 1 \text{ as } N \to \infty.$$

3 Random walk on a strip

We consider a random walk on a m-strip, i.e. the lattice $\mathbb{Z} \times \{0, \ldots, m-1\}$. We prove that, if the probability that the end-to-end distance of an N-random walk satisfies some weak decreasing conditions, then the growth factor goes to 1 as N goes to infinity. To achieve this goal we find a lower bound of γ_N that converges to 1 as N goes to infinity.
For the case of a 2-strip the cardinality of the total set of random walk is well defined. For the unconstrained random walk we have $\mathrm{Card}(\Omega_N) = 3^N$ and for the case in which the random walk cannot come back immediately $\mathrm{Card}(\Omega_N) = 3 \cdot 2^{N-1}$. In the case of a m-strip the cardinality of Ω_N grows exponentially in N, $\mathrm{Card}(\Omega_N) = e^{\tau_N N}$ (clearly $\tau_N \geq 2$)[5].

First we give the proposition for $m \in \mathbb{N}$ fixed. After that we verify the hypothesis used in the proposition for the x-component of the position (the one whose value can be in all $\mathbb{Z}$) for $m = 2$, both for an unconstrained random walk and for a random walk that cannot come back immediately. We also give an argument for the m-strip.

Let us consider a random walk on a m-strip starting from $(0, k)$ with $0 \leq k \leq m-1$. We are interested in the position of the x-component of the random walk after N steps: S_N. We introduce the random variables of the displacement in the x-direction $X_i, i = 1, \ldots, N$ and the random variable $S_N \doteqdot \sum_{i=1}^{N} X_i$.

[5]For the random walk on a m-strip without constraints, τ_N goes to $4 - \frac{3}{2m-1}$ when $N \to \infty$.

Proposition 12.
Let $X_1,\dots,X_N$ be 1D random variables corresponding to the x-component of a N-random walk (the only non-zero probabilities are $\mathbb{P}(X_i=1)$, $\mathbb{P}(X_i=0)$ and $\mathbb{P}(X_i=-1)$) and $S_N \doteqdot \sum_{i=1}^N X_i$. If S_N satisfies the following property[6]*:*

$$1. \quad B \doteqdot \limsup_{N\to\infty} \max_{0\le i\le N} N\mathbb{P}\left(|S_i| \ge N^\alpha\right) < 1 \text{ for some } \alpha < 1,$$

then

$$\bar{\gamma} = \lim_{N\to\infty} \frac{\ln(W(N))}{\ln(\mathrm{Card}(\Omega_N))} = 1.$$

Proof.
Let Ω_N be the set of all random walks on an m-strip starting from $(0,y_0)$. In the unconstrained case (case a) $3^N \le \mathrm{Card}(\Omega_N) \le 4^N$ and in the second case (case b) $2^N \le \mathrm{Card}(\Omega_N) \le 3^N$.
We subdivide Ω_N in two disjoint subsets Ω^0 and $(\Omega^0)^c$:

1. $\Omega_N \supset \Omega^0 \doteqdot \{\omega \in \Omega_N \text{ s.t. } |S_i| < N^\alpha, \forall\, i \in \{0,\dots,N\}\}$,
2. $\Omega_N \supset (\Omega^0)^c = \{\omega \in \Omega_N \text{ s.t. } \exists i \in \{0,\dots,N\} \text{ s.t. } |S_i| \ge N^\alpha\}$.

First we find an upper bound for $\mathrm{Card}((\Omega^0)^c) = \mathrm{Card}(\Omega_N)\mathbb{P}(\omega \in (\Omega^0)^c)$.

$$\begin{aligned} \mathbb{P}(\omega \in (\Omega^0)^c) &= \mathbb{P}\left(\{\exists i \in \{1,\dots,N\} \text{ s.t. } |S_i| \ge N^\alpha\}\right) \\ &= \mathbb{P}\left(\textstyle\bigcup_{i=1}^N \{|S_i| \ge N^\alpha\}\right) \\ &\le \textstyle\sum_{i=1}^N \mathbb{P}(|S_i| \ge N^\alpha) \\ &\le NT_N, \end{aligned}$$

where $T_N \doteqdot \max_{0\le i\le N} \mathbb{P}(|S_i| \ge N^\alpha)$.
Then

$$\mathrm{Card}((\Omega^0)^c) \le \mathrm{Card}(\Omega_N) NT_N.$$

But as $\Omega_N = \Omega^0 \cup (\Omega^0)^c$ and $\Omega^0 \cap (\Omega^0)^c = \varnothing$,

$$\mathrm{Card}(\Omega^0) = \mathrm{Card}(\Omega_N) - \mathrm{Card}((\Omega^0)^c) \ge \mathrm{Card}(\Omega_N)\left(1 - NT_N\right). \tag{4}$$

We consider a subset of Ω^0:

$$\Omega^0 \supset \widetilde{\Omega} \doteqdot \{\omega \in \Omega^0 \text{ s.t. } \omega(i) = \widetilde{\omega}(i), \forall\, i \in \{0,\dots,m(N,\alpha)\}\}$$

where $m(N,\alpha) \le [N^\alpha - 1](2m+2) + 3(m-1)$ and $\widetilde{\omega}$ is the following (fixed) $m(N,\alpha)$-random walk[7]. $\widetilde{\omega}$ starts from $(0,y_0)$, finishes in $(0,y_0)$ and covers all points of the strip such that $|x| \le [N^\alpha] - 1$. The construction of such a walk is simple. First $\widetilde{\omega}$ goes from $(0,y_0)$ to $(-[N^\alpha]+1, y_0)$ directly, then to $(-[N^\alpha]+1, 0)$ directly. After

[6] In fact this property can be replaced by a more general one: $\frac{1}{N}\ln\left(1 - N\mathbb{P}(|S_i| \ge N^\alpha)\right) \to 0$ as $N \to \infty$, for all $0 \le i \le N$.
[7] $[z]$ is the integer part of z.

that we fill all the desired points (remaining always in the rectangle to be filled). In fact we can choose $\widetilde{\omega}$ such that it goes straight from left to right, then increase the y-component of by one unit and goes from the right to the left directly. This procedure is repeated until the rectangle is filled. After that $\widetilde{\omega}$ goes from corner to the starting point in less than $m + [N^\alpha - 1]$ steps.

Now we estimate Card($\widetilde{\Omega}$):

$$\begin{aligned}
\mathbb{P}(\omega \in \widetilde{\Omega}) &= \mathbb{P}(\omega \in \Omega^0 \cap \omega(0) = \widetilde{\omega}(0) \cap \ldots \cap \omega(m(N,\alpha)) = \widetilde{\omega}(m(N,\alpha))) \\
&= \mathbb{P}(\omega \in \Omega^0)\mathbb{P}\left(\bigcap_{i=0}^{m(N,\alpha)} \omega(i) = \widetilde{\omega}(i) \,\middle|\, \omega \in \Omega^0 \right) \\
&= \mathbb{P}(\omega \in \Omega^0) \prod_{i=1}^{m(N,\alpha)} \mathbb{P}\left(\omega(i) = \widetilde{\omega}(i) \,\middle|\, \bigcap_{k=0}^{i-1} \omega(k) = \widetilde{\omega}(k), \omega \in \Omega^0 \right) \\
&\geq \mathbb{P}(\omega \in \Omega^0) \begin{cases} 4^{-m(N,\alpha)} & \text{in the case a,} \\ 3^{-m(N,\alpha)} & \text{in the case b.} \end{cases}
\end{aligned}$$

We have obtained:

$$\mathrm{Card}(\widetilde{\Omega}) \geq \mathrm{Card}(\Omega^0) \begin{cases} 4^{-m(N,\alpha)} & \text{in the case a,} \\ 3^{-m(N,\alpha)} & \text{in the case b.} \end{cases} \tag{5}$$

Putting together equations (4) and (5) we obtain,
for the case a:

$$\mathrm{Card}(\widetilde{\Omega}) \geq 4^{-m(N,\alpha)}(1 - N\mathbb{P}(|S_N| \geq N^\alpha))\,\mathrm{Card}(\Omega_N)$$

and then

$$\begin{aligned}
\lim_{N\to\infty} \frac{\ln \mathrm{Card}(\widetilde{\Omega})}{\ln \mathrm{Card}(\Omega_N)} &= \lim_{N\to\infty} \left(1 - \frac{m(N,\alpha)\ln 4}{N \ln \tau_N} + \frac{\ln \overbrace{(1 - NT_N)}^{\to 1-B>0, N\to\infty}}{N \ln \tau_N} \right) \\
&= 1 \text{ because } \alpha < 1 \text{ and hypothesis 1.}
\end{aligned} \tag{6}$$

For the case b we proceed exactly in the same way. Now we can conclude the proof because $W(N) \geq \mathrm{Card}(\widetilde{\Omega})$. In fact, let $\omega \in \widetilde{\Omega}$, then for each step from $m(N,\alpha)+1$ to N the path intersects one of the first $m(N,\alpha)$ steps, therefore for each choice of these $N - m(N,\alpha) + 1$ steps we obtain a different contact matrix. Moreover, since $W(N) \leq \mathrm{Card}(\Omega_N)$, we have

$$1 \geq \lim_{N\to\infty} \frac{\ln W(N)}{\ln \mathrm{Card}(\Omega_N)} \geq \lim_{N\to\infty} \frac{\ln \mathrm{Card}(\widetilde{\Omega})}{\ln \mathrm{Card}(\Omega_N)} = 1.$$

$\square$

3.1 The unconstrained random walk on a 2-strip

In this case the random variables of the x-displacements $X_i, i = 1, \ldots, N$ are iid and such that

$$\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \mathbb{P}(X_i = 0) = \tfrac{1}{3}.$$

When $X_i = 0$ the random walk moves in the vertical direction.
Now we compute the distribution of S_N when $N \to \infty$. Let $\Phi_{S_N}(\xi)$ be the characteristic function of the random variable S_N and $\Phi_{X_1}(\xi)$ the one of X_1. Then

$$\Phi_{S_N}(\xi) = \mathbb{E}\left(e^{i\xi S_N}\right) \overset{\text{iid}}{=} \left(\mathbb{E}\left(e^{i\xi X_1}\right)\right)^N = \left(\Phi_{X_1}(\xi)\right)^N.$$

For the random variable X_1 we have

$$\Phi_{X_1}(\xi) = \tfrac{1}{3}\left(e^{i\xi} + 1 + e^{-i\xi}\right).$$

Therefore

$$\begin{aligned} \mu &\doteqdot \mathbb{E}(X_1) = 0, \\ \sigma^2 &\doteqdot \mathbb{E}\left(X_1 - \mathbb{E}(X_1)\right)^2 \overset{\mu=0}{=} (-i)^2 \tfrac{d^2}{d\xi^2} \Phi_{X_1}(\xi)\,\big|_{\xi=0} = \tfrac{2}{3}. \end{aligned}$$

For a $1D$ random walk with transition function $p(x,y) = \mathbb{P}(X_i = y - x)$ that satisfies $\mu = 0$ and $\sigma^2 < \infty$, $\frac{S_N}{\sqrt{N}} = \frac{1}{\sqrt{N}} \sum_{i=1}^N X_i$ converges weakly[8] to the Gaussian distribution $\mathcal{N}(0, \sigma^2)$ (with mean 0 and variance σ^2). Therefore

$$\mathbb{P}\left(\frac{S_N}{\sqrt{N}} \le x\right) = \frac{1}{\sqrt{4\pi/3}} \int_{-\infty}^{x} e^{-\frac{y^2}{4/3}}\, \mathrm{d}y \text{ as } N \to \infty.$$

Now we derive the property used in the proposition 12. By Markov-Chebyshev's inequality

$$\mathbb{P}\left(|S_N| \ge \delta\right) \le \frac{\mathbb{E}\left(|S_N|^4\right)}{\delta^4}.$$

In our case for N large enough $\mathbb{E}\left(|S_N|^4\right) \cong 3N^2\sigma^2$. Therefore taking $\delta = N^\alpha$ with $\alpha = 7/8$ we obtain (for large N)

$$\mathbb{P}\left(|S_N| \ge N^{7/8}\right) \le 3\sigma^2 N^{-3/2}.$$

Consequently

$$N\mathbb{P}\left(|S_N| \ge N^{7/8}\right) \le \frac{3\sigma^2}{\sqrt{N}} \overset{N\to\infty}{\longrightarrow} 0.$$

When $N >> 1$, $N^\alpha >> 1$ too and therefore

1. $\mathbb{P}(|S_i| \ge N^\alpha) = 0$ for $0 \le i < N^\alpha$ and
2. $\mathbb{P}(|S_i| \ge N^\alpha) \le \frac{3\sigma^2 i^2}{N^{7/2}} \le 3\sigma^2 N^{-3/2}$ for $N^\alpha \le i \le N$.

Property 1. is verified.

[8] See [11] proposition 8 chapter 2, paragraph 6, page 64. The method is the same that we use in the other cases.

3.2 The random walk on a 2-strip without immediate return

The end-to-end distribution is more difficult to compute in this case because the displacement in the x-direction depends on the previous one. We prove that the probability distribution of S_N converges weakly to a Gaussian with variance $2N$. Once we have this result, the property verified in the previous case are true also in this case (with a different value of σ^2).
Let us define the following random variables[9]:

$$\begin{aligned} a(N,x) &\doteqdot \mathbb{P}(S_N = x | S_{N-1} = x-1) \\ b(N,x) &\doteqdot \mathbb{P}(S_N = x | S_{N-1} = x+1) \\ c(N,x) &\doteqdot \mathbb{P}(S_N = x | S_{N-1} = x) \end{aligned}$$

Since the random walk cannot be in the same position after two steps, the relations of these random variables for two consecutive steps are:

$$\begin{aligned} a(N,x) &= \frac{1}{2}\left(a(N-1,x-1) + c(N-1,x-1)\right), \\ b(N,x) &= \frac{1}{2}\left(b(N-1,x+1) + c(N-1,x+1)\right), \\ c(N,x) &= \frac{1}{2}\left(a(N-1,x) + b(N-1,x)\right). \end{aligned}$$

We define the characteristic function of these random variables

$$\Phi_\rho(N,\xi) \doteqdot \sum_{x\in\mathbb{Z}} \rho(N,x) e^{i\xi x} \text{ for } \rho = a, b, c$$

and the characteristic function of S_N

$$\Phi_{S_N}(\xi) \doteqdot \sum_{x\in\mathbb{Z}} \mathbb{P}(S_N = x) e^{i\xi x} = \Phi_a(N,\xi) + \Phi_b(N,\xi) + \Phi_c(N,\xi).$$

Finally we define

$$\Phi(N,\xi) \doteqdot \begin{pmatrix} \Phi_a(N,\xi) \\ \Phi_b(N,\xi) \\ \Phi_c(N,\xi) \end{pmatrix}.$$

With these definitions we obtain a matrix relation between $\Phi(N,\xi)$ and $\Phi(N-1,\xi)$:

$$\Phi(N,\xi) = \frac{1}{2}\underbrace{\begin{pmatrix} e^{i\xi} & 0 & e^{i\xi} \\ 0 & e^{-i\xi} & e^{-i\xi} \\ 1 & 1 & 0 \end{pmatrix}}_{\doteqdot T} \Phi(N-1,\xi)$$

We choose the initial condition to be $\Phi(0,\xi) = \frac{1}{3}(1,1,1)$, i.e. the initial position is $x = 0$ and we have equal probability to go right, left or vertically. To obtain the

[9] $\mathbb{P}(A|B)$ is the conditional probability of A knowing B.

asymptotic probability distribution of S_N we need to compute $\Phi(N,\xi)$ and then sum over its three components. In order to compute $\Phi(N,\xi)=\frac{1}{2^N}T^N\Phi(0,\xi)$ we diagonalise T, whose eigenvalues are

$$\begin{aligned}\lambda_1 &= -1,\\ \lambda_2 &= \nu+\sqrt{\nu^2-2},\\ \lambda_3 &= \nu-\sqrt{\nu^2-2},\end{aligned}$$

with $\nu=\frac{e^{i\xi}+1+e^{-i\xi}}{2}$.
A calculation gives:

$$\Phi_{S_N}(\xi)=\frac{\lambda_2^N-\lambda_3^N}{6\cdot 2^N}\cdot\frac{\nu}{\sqrt{\nu^2-2}}+\frac{\lambda_2^N+\lambda_3^N}{2\cdot 2^N}$$

Now we want to study the characteristic function of S_N when N goes to infinity. For ξ near to 0, $|\lambda_2|>|\lambda_3|$, so that $1\pm\left(\frac{\lambda_3}{\lambda_2}\right)^N\stackrel{N\to\infty}{\longrightarrow}1$, $\frac{\nu}{3\sqrt{\nu^2-2}}=1+\frac{8}{3}\xi^2+\mathcal{O}(\xi^4)$, $\lambda_2=2(1-\xi^2)+\mathcal{O}(\xi^4)$. Therefore $\mathbb{E}(S_N)=0$ (by symmetry) and $\mathbb{E}(S_N^2)=(-i)^2\frac{d^2}{d\xi^2}\Phi_{S_N}(\xi)\big|_{\xi=0}=2N$.
Let us consider the rescaled random variable

$$A_N\doteqdot\frac{S_N}{\sqrt{2N}}.$$

As $N\to\infty$, for each fixed ξ, $\frac{\xi}{\sqrt{2N}}$ goes to zero and the terms $(\lambda_2/\lambda_3)^N$ vanish. Therefore

$$\begin{aligned}\Phi_{A_N}(\xi) &= \mathbb{E}\left(e^{i\xi A_N}\right)=\mathbb{E}\left(e^{i\frac{\xi S_N}{\sqrt{2N}}}\right)=\Phi_{S_N}\left(\xi'\doteqdot\frac{\xi}{\sqrt{2N}}\right)\\ &= \frac{\lambda_2\left(\xi'\right)^N}{2^N}\cdot\left(1+\mathcal{O}\left(\frac{\xi^2}{N}\right)\right)\\ &= \frac{1}{2^N}e^{N\ln\lambda_2\left(\frac{\xi}{\sqrt{2N}}\right)}\cdot\left(1+\mathcal{O}\left(\frac{\xi^2}{N}\right)\right)\\ &= \frac{1}{2^N}e^{N\left(\ln 2+\ln\left(1-\frac{\xi^2}{2N}+\mathcal{O}(\xi^4/N^2)\right)\right)}\cdot\left(1+\mathcal{O}\left(\frac{\xi^2}{N}\right)\right)\\ &= e^{-\frac{\xi^2}{2}+\mathcal{O}(\xi^4/N)}\cdot\left(1+\mathcal{O}\left(\frac{\xi^2}{N}\right)\right)\stackrel{N\to\infty}{\longrightarrow}e^{-\frac{\xi^2}{2}}.\end{aligned}$$

$e^{-\xi^2/2}$ is a continuous function at $\xi=0$ and corresponds to the characteristic function of the normal distribution. Therefore the Paul-Lévy theorem implies that the sequence of random variables $\left\{A_N=\frac{S_N}{\sqrt{2N}}\right\}_{N\geq 1}$ converges weakly (in law) to the normal distribution $\mathcal{N}(0,1)$.
The probability distribution of S_N converges to

$$\mathbb{P}\left(\frac{S_N}{\sqrt{N}}\leq x\right)=\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{x}e^{-\frac{y^2}{2\sigma^2}}\,\mathrm{d}y \text{ with } \sigma^2=2$$

as N goes to infinity.
Now that we have the probability distribution of S_N, hypothesis 1. of proposition 12 holds (their verification is the same as in section 3.1). Then for the random walk on a 2-strip with the constraint that cannot come back immediately we have $\bar{\gamma} = 1$.

3.3 Comparison of the end-to-end distance

We can also compare the end-to-end distance of the two random walks considered[10].

Type of random walk	$\sqrt{\mathbb{E}(S_N^2)}$
unconstrained random walk	$\sqrt{\frac{2N}{3}}$
random walk without immediate return	$\sqrt{2N}$

Table 1: The end-to-end distance on a 2-strip.

The effect of the constraint on the end-to-end distance is not negligible. In fact the end-to-end distance increases by a factor $\sqrt{3}$ due to the constraint.

3.4 Complete asymptotic behavior of $W(N)$

In this section we give the expected behavior of $W(N)$ using numerical values of $\frac{W(N)}{\mathrm{Card}(\Omega_N)}$ for $N \in \{15, \ldots, 25\}$, for the 2-strip with the condition of avoiding an immediate return. We do not give an analytical proof but an argument for this behavior.
The asymptotic guessed form of $W(N)$ is $W(N) = A\,2^N$.
The expected number of visited points on the 2-strip behaves like $C\sqrt{N}$. Moreover it returns to the origin a lot of times ($C'\sqrt{N}$). Each of the $C\sqrt{N}$ steps for which the random walk visits a new site, it has at most two possible choices ("degrees of freedom"), which do not affect the contact matrices (until this step). But the contact matrices can be different due to later steps, and since it comes back frequently and it has only two possibilities to do that in the y-direction, a lot of "degrees of freedom" are destroyed for the typical random walks. Clearly there are some remaining degrees of freedom, mostly of them at the "boundary of the walk".
Let us define $A(N) \doteqdot \frac{W(N)}{\mathrm{Card}(\Omega_N)}$. Then 2 cases can occur:

1. there exists a $K > 0$ such that $W(N) \geq K2^N$ for N large enough,
2. there does not exists a $K > 0$ such that $W(N) \geq K2^N$ for N large enough.

[10]The asymptotic end-to-end distance is equal to the one for the x-component because $\mathbb{E}\left((S_N^y)^2\right) \in [0, m^2]$ becomes negligible with respect to the one of the x-component that grows linearly in N. Moreover $\frac{1}{\sqrt{2\pi N\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\pi N\sigma^2}} x^2 \, \mathrm{d}x = N\sigma^2$.

In the second case $\liminf_{N\to\infty} A(N) = 0$. In the 2-strip case without immediate return, the data shows (see figures 3 and 4) that $A(N)$ "increases"[11] with increasing N, therefore we expect that $A(N)$ goes to a limit value $A > 0$ when $N \to \infty$. Therefore we are in the first case. Note that this behavior is the same as in the one dimensional case[12].
If the $A(N) \sim N^{-\sigma}$ for a $\sigma > 0$, i.e. a power correction, then the limit of $A(N)$ would be 0.

Remark 13. *In all the 2D cases*[13] *for which we have numerical results (for small values of N) the situation is different. In fact $A(N)$ "decreases" when N increases*[14].

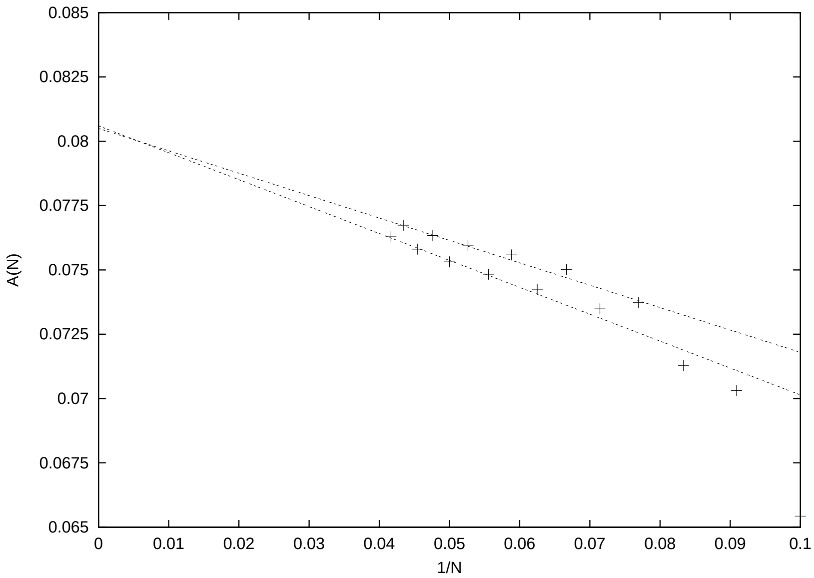


Figure 3: $A(N)$ as a function of $\frac{1}{N}$ for random walk on a 2-strip with immediate return avoided.

3.5 The random walk on a m-strip

For the case of a m-strip we give an argument that $\bar{\gamma} = 1$.
First let us consider the unconstrained random walk on a m-strip. In this case $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}\mathbb{P}(X_i = 0) = \frac{1}{4}$ when the random walk is not on the boundary of the strip, i.e. the y-component is different from 0 and $m-1$, in which cases we have $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \mathbb{P}(X_i = 0) = \frac{1}{3}$. Then if we take the sequence of the x-displacements, $\{X_i\}_{i=1}^{N}$, the probability of having a non-zero x-displacement is

[11] In the sense that the global behavior increases, not that $A(N)$ is increasing strictly speaking.
[12] The fact that a model on a strip is closer to a one dimensional model than a two dimensional is of course not new (see [3], section 2.3).
[13] See table 4 in the appendix for the enumeration.
[14] See plots 10 to 14 in the appendices.

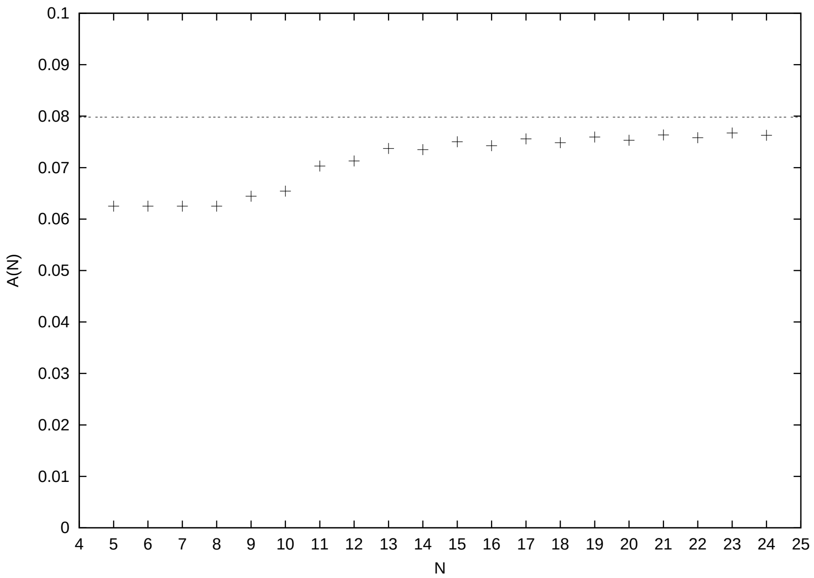


Figure 4: $A(N)$ as a function of N for random walk on a 2-strip with immediate return avoided.

less than in the 2-strip case. Then the probability distribution of S_N cannot be more spread out than in the 2-strip case. Therefore the probability distribution satisfies the hypothesis of proposition 12.
In the case of a random walk on a m-strip for which coming-back immediately is avoided, the same argument applies, and proposition 12 too[15].

[15]In [10] the authors computed that the diffusion coefficient for random walks on strips of finite width m is $K = \frac{m}{2m-1}$ for the unconstrained random walk (except for the boundary constraints), but they did not compute the exact distribution of S_N.

4 Random walks on two dimensional lattices

We consider random walks on a 2-dimensional lattice. We have looked at the following different cases:

1. the unconstrained random walk on $\mathbb{Z}^2$,
2. the random walk on $\mathbb{Z}^2$ that cannot come back immediately,
3. the random walk on $\mathbb{Z}^2$ that turns 90 degrees at each step,
4. the unconstrained random walk on the honeycomb lattice,
5. the random walk on the honeycomb lattice that cannot come back immediately.

For all these cases we have some exact numerical result for small N. If we look at the data[16] for the $2D$ walks we could guess that the total number of contact matrices $W(N)$ grows less rapidly than $\mathrm{Card}(\Omega_N)$ in the sense that $\bar{\gamma} < 1$. Nevertheless we can prove analytically that also in this case $\bar{\gamma} = 1$. This result is related to the asymptotic behavior of the range (the number of different visited points) of the random walk. For recurrent random walks the proposition holds also if the displacements are not nearest neighbors. First we prove that $\bar{\gamma} = 1$, after that we verify that the used hypothesis hold for the considered random walks. We also compute the end-to-end distance of these different random walks and give two result on the degeneracy of the contact matrices.

4.1 Growth factor

In what follows $X_k, 1 \leq k \leq N$ denote the vector-valued displacement at the k^{th} step, and the total displacement is given by $S_N = \sum_{k=1}^{N} X_k$ and $S_0 = 0$ (the starting point is fixed at the origin).
Let G be the expected number of visits to the origin and F the probability of coming back to the origin. A random walk is **recurrent** if $G = \infty$ ($F = 1$), otherwise the random walk is **transient**.
For random walks with iid displacements (not necessarily nearest neighbors) we have the following result.

Proposition 14 (Erdös, Dvoretzky 1951).
If a random walk is transient, then $\frac{R_N}{N}$ converges in measure to $\frac{1}{G} = 1 - F > 0$ and, if it is recurrent $\frac{R_N}{N}$ converges in measure to $0 = 1 - F$.

In other words, for a recurrent random walk we have:

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{R_N}{N} > \varepsilon\right) = 0 \text{ for all } \varepsilon > 0. \tag{7}$$

[16] See plots 8 and 9 in the appendix for two examples.

Following the proof given by Spitzer[17] we can prove the same proposition for non iid random walk for which the probability of a displacement depends on the previous one but not on previous occupancy of lattice points. We prove it for the recurrent random walk for which we use it.

Proposition 15.
Let us consider a random walk as above.
Then $\lim_{N\to\infty} \mathbb{P}\left(\frac{R_N}{N} > \varepsilon\right) = 0$ *for all* $\varepsilon > 0$ *holds.*

Proof.
The range of the N-random walk is given by

$$R_N = \sum_{k=0}^{N} \varphi_k$$

where $\varphi_0(S_0 = 0) = 1, \varphi_k(S_1, \ldots, S_k) = \begin{cases} 1 \text{ if } S_k \neq S_j, \forall\, 0 \leq j \leq k-1, \\ 0 \text{ otherwise.} \end{cases}$
Then by linearity

$$\mathbb{E}(R_N) = \mathbb{E}\left(\sum_{k=0}^{N} \varphi_k\right) = \sum_{k=0}^{N} \mathbb{E}(\varphi_k).$$

For $k = 0$, $\mathbb{E}(\varphi_0) = 1$ and for $k \geq 1$ we have:

$$\begin{aligned} \mathbb{E}(\varphi_k) &= \mathbb{P}(\varphi_k = 1) \cdot 1 + \mathbb{P}(\varphi_k = 0) \cdot 0 \\ &= \mathbb{P}(S_k \neq S_{k-1} \cap \ldots \cap S_k \neq S_1 \cap S_k \neq 0) \\ &= \mathbb{P}(S_k - S_{k-1} \neq 0 \cap \ldots \cap S_k \neq 0) \\ &= \mathbb{P}(X_k \neq 0 \cap X_k + X_{k-1} \neq 0 \cap \ldots \cap X_k + \ldots + X_1 \neq 0). \end{aligned}$$

If the X_k were iid we could replace directly X_{k-j} by $X_{1+j}\, \forall\, j \in \{0, \ldots, k-1\}$. Since the X_k are not iid, we need to proceed in a slightly different way.
Let $\mathbb{P}_N(A)$ denote the probability of the event $A \in \mathcal{F}_N$ and similarly $\mathbb{P}_k(B)$ the probability of the event $B \in \mathcal{F}_k$. Let us define the one-to-one application in Ω_k:

$$\begin{aligned} T:\ \Omega_k &\longmapsto \Omega_k \\ \omega &\longrightarrow \omega' = T(\omega) \end{aligned}$$

where $\omega = (0, S_1(\omega), \ldots, S_k(\omega))$ and $\omega' \doteqdot (0, S_1'(\omega), \ldots, S_k'(\omega))$ with $S_j'(\omega) \doteqdot X_k(\omega) + \ldots + X_{k-j+1}(\omega)$.
Since the event $A = \{X_k \neq 0 \cap X_k + X_{k-1} \neq 0 \cap \ldots \cap X_k + \ldots + X_1 \neq 0\}$ is in $\mathcal{F}_k$ we have:

$$\begin{aligned} \mathbb{E}(\varphi_k) &= \mathbb{P}_N(X_k \neq 0 \cap X_k + X_{k-1} \neq 0 \cap \ldots \cap X_k + \ldots + X_1 \neq 0) \\ &= \mathbb{P}_k(X_k \neq 0 \cap X_k + X_{k-1} \neq 0 \cap \ldots \cap X_k + \ldots + X_1 \neq 0) \\ &= \mathbb{P}_k(S_1', \neq 0 \cap \ldots \cap S_k' \neq 0) \\ &\overset{T:1-1}{=} \mathbb{P}_k(S_1, \neq 0 \cap \ldots \cap S_k \neq 0) \\ &= \mathbb{P}_N(S_1, \neq 0 \cap \ldots \cap S_k \neq 0) \end{aligned}$$

[17]See [11], chapter 1, paragraph 4, theorem 1, page 35.

The probability of an event $A \in \mathcal{F}_k$ is proportional to the cardinality of the subset of random walks that satisfies A, then because of the bijectivity of the application T step four holds.

For the probability of never coming back to the origin we have the relation

$$\mathbb{P}(S_j \neq 0, \forall\, j = 1, \ldots, N) \leq \frac{1}{G_N(0,0)}$$

where $G_N(0,0)$ is the expected number of visits to the origin after N steps[18].
Then for a recurrent random walk

$$\lim_{N\to\infty} \mathbb{E}(\varphi_N) = \lim_{N\to\infty} \mathbb{P}(S_i \neq 0, \forall\, i \in \{1, \ldots, N\}) \leq \lim_{N\to\infty} \frac{1}{G_N(0,0)} = \frac{1}{G} = 0.$$

Therefore

$$\lim_{N\to\infty} \frac{\mathbb{E}(R_N)}{N} = \lim_{N\to\infty} \sum_{k=0}^{N} \frac{\mathbb{E}(\varphi_k)}{N} = 0.$$

We can now prove the proposition.

$$\begin{aligned} \forall\, \varepsilon > 0, \quad \mathbb{P}\left(\frac{R_N}{N} > \varepsilon\right) &= \sum_{k>N\varepsilon} \mathbb{P}(R_N = k) \leq \sum_{k>N\varepsilon} \mathbb{P}(R_N = k)\frac{k}{N\varepsilon} \\ &\leq \frac{1}{N\varepsilon} \sum_{k\geq 0} k\mathbb{P}(R_N = k) = \frac{1}{\varepsilon}\frac{\mathbb{E}(R_N)}{N}, \end{aligned}$$

therefore

$$\forall\, \varepsilon > 0, \quad \lim_{N\to\infty} \mathbb{P}\left(\frac{R_N}{N} > \varepsilon\right) \leq \frac{1}{\varepsilon} \lim_{N\to\infty} \frac{\mathbb{E}(R_N)}{N} = 0.$$

□

What we need to show now is first that the different random walks considered here satisfy $\mathbb{P}(S_j \neq 0, \forall\, j \in \mathbb{N}) = 0$ (i.e. the random walk are recurrent). If this is true, we can apply proposition 17 to these random walk.

Lemma 16.
Let ω be a N-random walk with a support equal to M and $\mathcal{C}(\omega)$ its contact matrix. Then

$$\deg(\mathcal{C}(\omega)) \leq D^M$$

where D is the number of choices for each step such that the position at $k+1$ does not coincide with the position at $k-1$, $\omega(k+1) \neq \omega(k-1)$.

PROOF.
Let us consider a fixed ω and its contact matrix $\mathcal{C}(\omega)$. $\mathcal{C}(\omega)$ has $N+1-M$ columns with an "1" in the upper triangular part, because we have $N+1-M$ intersections. Let us construct all random walks ω' such that $\mathcal{C}(\omega') = \mathcal{C}(\omega)$. Consider its k^{th} step, then there are two possible cases:

[18]This follows from the proof of the proposition 4, chapter 1, paragraph 1, page 7 of [11].

1. there exists a $i < k$ such that $(\mathcal{C}(\omega))_{ik} = 1$,
2. for all $i < k$ $(\mathcal{C}(\omega))_{ik} = 0$.

In the first case, the position of the k^{th} step is fixed by the i^{th} and therefore we have only one choice for it.
In the second case, the k^{th} step will occupy a place which was never occupied before. Therefore we have *at most* D possibles choices[19].
For a ω' with a contact matrix $\mathcal{C}(\omega') = \mathcal{C}(\omega)$ there are $M - 1$ steps[20] for which we are in the second case and $N + 1 - M$ steps for which we are in the first one.
Therefore there is *at most* $D^{M-1} \leq D^M$ different ω' satisfying $\mathcal{C}(\omega') = \mathcal{C}(\omega)$, i.e.

$$\deg(\mathcal{C}(\omega)) \leq D^M.$$

□

In cases 1 and 2, $D = 3$, in cases 3,4,5, $D = 2$ and for a simple random walk on $\mathbb{Z}^d$ is $2d - 1$.

Proposition 17 (Growth factor).
Let us consider recurrent random walks such that each step depends only on the previous displacement. Then

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{R_N}{N} > \varepsilon\right) = 0 \textit{ for all } \varepsilon > 0$$

holds and

$$\bar{\gamma} = \lim_{N\to\infty} \frac{\ln W(N)}{\ln \operatorname{Card}(\Omega_N)} = 1.$$

Proof.
Proposition 15 proves the first part (the convergence in measure of R_N/N to 0). Let us prove the second part.
First we divide the set Ω_N in two disjoint subsets: $\Omega_N^\delta \cup (\Omega_N^\delta)^c$ where

$$\Omega_N^\delta \doteqdot \left\{\omega \in \Omega_N \text{ s.t. } \frac{R_N(\omega)}{N} < \delta\right\}$$

and

$$(\Omega_N^\delta)^c = \left\{\omega \in \Omega_N \text{ s.t. } \frac{R_N(\omega)}{N} \geq \delta\right\}.$$

In what follows we consider random walks $\omega \in \Omega_N^\delta$ because its cardinality goes to $\operatorname{Card}(\Omega_N)$ when N goes to infinity, i.e. $\frac{\operatorname{Card}(\Omega_N^\delta)}{\operatorname{Card}(\Omega_N)} \to 1$ as $N \to \infty$.

[19] In fact there is only one exception if we do not fix the first step.
[20] Because the starting point is fixed at the origin.

For all $\omega \in \Omega_N^\delta$, $R_N(\omega) < N\delta$, then the number of intersection is bigger than $N(1-\delta)$. For each $\omega \in \Omega_N^\delta$, the corresponding matrix $\mathcal{C}(\omega)$ is at most $D^{N\delta}$ times degenerate (by lemma 16). Therefore we have

$$\deg(\mathcal{C}(\omega)) \leq D^{N\delta}.$$

The total number of different matrices $W(N)$ satisfies:

$$\forall\, \delta > 0, \quad W(N) \geq W(N)\Big|_{\Omega_N^\delta} \geq \frac{\mathrm{Card}(\Omega_N^\delta)}{D^{N\delta}} = \frac{\mathrm{Card}(\Omega_N)\,\mathbb{P}(\omega \in \Omega_N^\delta)}{D^{N\delta}}.$$

If we define $\tau \doteqdot \limsup_{N\to\infty} \tau_N$ and $\tau_N \doteqdot \frac{\ln \mathrm{Card}(\Omega_N)}{N}$ then

$$W(N) \geq \mathbb{P}\left(\omega \in \Omega_N \text{ s.t. } \frac{R_N(\omega)}{N} < \delta\right) e^{N(\tau_N - \delta \ln D)}.$$

Then

$$\begin{aligned}
\forall\, \delta > 0, \quad 1 \;&\geq\; \lim_{N\to\infty} \frac{\ln W(N)}{\ln \mathrm{Card}(\Omega_N)} \\
&\geq\; \lim_{N\to\infty} \frac{N(\tau_N - \delta \ln D) + \ln \overbrace{\left(\mathbb{P}\left(\omega \in \Omega_N \text{ s.t. } \frac{R_N(\omega)}{N} < \delta\right)\right)}^{\to 1 \text{ as } N\to\infty}}{N\tau_N} \\
&\geq\; 1 - \delta \frac{\ln D}{\tau}.
\end{aligned}$$

Let us define $\varepsilon = \delta \frac{\ln D}{\tau} > 0$, then

$$\forall\, \varepsilon > 0, \quad 1 \geq \lim_{N\to\infty} \frac{\ln W(N)}{\ln \mathrm{Card}(\Omega_N)} \geq 1 - \varepsilon.$$

this implies that

$$\lim_{N\to\infty} \frac{\ln W(N)}{\ln \mathrm{Card}(\Omega_N)} = 1.$$

□

4.2 Results on the degeneracy

Now we give some results on the degeneracy of contact matrices.

Proposition 18.
Let us consider recurrent random walks such that each step depends only on the previous displacement. If the random walk is recurrent then, for each $\mu > 0$, we have

$$\lim_{N\to\infty} \mathbb{P}\left(\omega \in \Omega_N \ s.t.\ \deg(\mathcal{C}(\omega)) \geq e^{\mu N}\right) = 0.$$

Proof.
Let D be as in propositon 17. Then

$$\mathbb{P}\left(\omega \in \Omega_N \text{ s.t. } \deg(\omega) > D^{\beta N}\right) = 1 - \mathbb{P}\left(\omega \in \Omega_N \text{ s.t. } \deg(\omega) \leq D^{\beta N}\right).$$

Moreover

$$\mathbb{P}\left(\omega \in \Omega_N \text{ s.t. } \deg(\omega) \leq D^{\beta N}\right) \geq \mathbb{P}\left(\omega \in \Omega_N \text{ s.t. } R_N(\omega) < \beta N\right)$$

because all N-random walks with a range less than βN are less degenerate than $D^{\beta N}$. Then for each $\mu > 0$ we can choose $\beta = \frac{\mu}{\ln D}$ so that

$$\begin{aligned}
\mathbb{P}\left(\omega \in \Omega_N \text{ s.t. } \deg(\omega) > e^{\mu N}\right) &\leq 1 - \mathbb{P}\left(\omega \in \Omega_N \text{ s.t. } R_N(\omega) < \beta N\right) \\
&= 1 - \mathbb{P}\left(\omega \in \Omega_N \text{ s.t. } \frac{R_N(\omega)}{N} < \beta\right) \\
&\longrightarrow 0 \text{ as } N \to \infty.
\end{aligned}$$

□

In the next part we consider *simple random walks* in two dimensions and we apply a result of [12]. Let $\omega \in \Omega_N$, then for each lattice site let us define the function

$$m_N(x) = m_N(x,\omega) = \begin{cases} 1 & \text{if } x \text{ has not been visited by } \omega, \\ 0 & \text{otherwise.} \end{cases}$$

We consider the observables[21] that can be written in the following form

$$M_N = M_N(\omega) = \sum_A \mu_A M_{A,N}(\omega) \text{ where } M_{A,N}(\omega) = \sum_{x\in\mathbb{Z}^2}\prod_{a\in A} m_N(x+a,\omega)$$

where A is a finite subset of lattice vectors. $M_N(\omega)$ is the number of times that the pattern M appears in the support of the walk. Van Wijland, Caser and Hilhorst proved in [12] that in two dimensions[22] $\Delta M_N = M_N - \mathbb{E}(M_N)$ satisfies in the asymptotic limit

$$\Delta M_N \simeq (k+1)\frac{\mathcal{A}}{\ln 8N}\mathbb{E}(M_N)\gamma(N) \tag{8}$$

[21] An observable is a pattern M composed by visited and unvisited sites.
[22] We will not write always explicitly that the observables are functions of ω.

where $\gamma(N)$ is a random variable[23] with mean 0 and variance 1. $\mathcal{A}$ is a constant whose value is $1.3034\ldots$ and k is an integer depending on the observable. The expectation of such an observable M_N is given by

$$\mathbb{E}(M_N) = \frac{\pi N}{\ln 8N} m_0 + \frac{\pi N}{(\ln 8N)^2}((C-1)m_0 + \pi m_1) + \mathcal{O}\left(\frac{N}{(\ln 8N)^3}\right),$$

where C is the Euler's constant ($C = 0.577215\ldots$) and m_0, m_1 are coefficients computed in what follows.
Now we apply (8) to the following observable. Let us consider the pattern Q[24] composed of the following sets of visited sites A_v, and unvisited sites A_u:

$$A_v(Q) = \{(0,0) \quad , \quad (-1,0)\}$$

and

$$A_u(Q) = \{(1,0) \quad , \quad (0,-1) \quad , \quad (0,1) \quad , \quad (-1,1)\}.$$

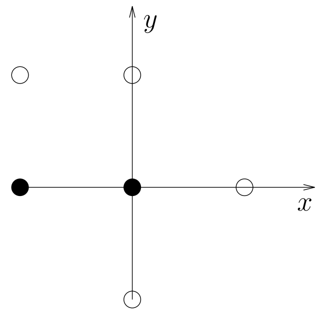


Figure 5: The pattern Q. The visited places of Q are the black ones and the unvisited places the white ones.

Then the number of times that Q appears in the support of a random walk is given by

$$\begin{aligned} Q_N = Q_N(\omega) &= \sum_{x\in\mathbb{Z}^2} m_N(x+e_1)\cdot m_N(x+e_2)\cdot m_N(x-e_2)\cdot \\ &\cdot \quad m_N(x+e_2-e_1)\cdot(1-m_N(x))\cdot(1-m_N(x-e_1)) \\ &= \sum_{i=1}^{4} \mu_{A_i} M_{A_i}(N) \end{aligned}$$

where $e_1 = (1,0)$ and $e_2 = (0,1)$. The sets A_i and the coefficients μ_{A_i} are given in the next table. The coefficients m_l are expressed as

$$m_l = -\sum_{A\neq\varnothing} \mu_A g_A^l,$$

[23] It is not the normal distribution, but is the Varadhan's renormalized local time of self-intersections (see [6]).
[24] The letter Q is used for this particular observable, the letter M for a general one.

where g_A are coefficients to be computed. For observables whose sets of visited and unvisited points are both non empty, we have $\sum_{A\neq\varnothing}\mu_A = \sum_A \mu_A = 0$, therefore in our case we have $m_0 = 0$. From the computations we find that $m_1 \neq 0$. The integer k in equation (8) is the smallest one for which $m_k \neq 0$, therefore in this case $k = 1$. In table 2 we report also the coefficients g_{A_i} that we have computed[25].

i	A_i	μ_{A_i}	g_{A_i}
1	$\{e_1, e_2, -e_2, e_2 - e_1\}$	-1	$\frac{-2\pi(\pi^3-15\pi^2+68\pi-88)}{\pi^4-20\pi^3+152\pi^2-512\pi+576}$
2	$\{0, e_1, e_2, -e_2, e_2 - e_1\}$	1	$\frac{2\pi(\pi^4-15\pi^3+73\pi^2-128\pi+64)}{3\pi^4-48\pi^3+240\pi^2-384\pi+128}$
3	$\{e_1, -e_1, e_2, -e_2, e_2 - e_1\}$	1	$\frac{2\pi(\pi^2-9\pi+16)}{\pi^3-8\pi^2+32}$
4	$\{0, e_1, -e_1, e_2, -e_2, e_2 - e_1\}$	-1	$\frac{2\pi(\pi^2-9\pi+16)}{\pi^3-8\pi^2+32}$

Table 2: The sets A_i with the coefficients μ_{A_i} and g_{A_i}.

From these values we obtain

$$m_1 = \frac{-2(\pi-2)^2(\pi-4)^2(\pi^2-10\pi+20)^2\pi}{(\pi^4-20\pi^3+152\pi^2-512\pi+576)(3\pi^4-48\pi^3+240\pi^2-384\pi+128)} = 2.78\cdot 10^{-3}.$$

Proposition 19.
Let us consider the simple random walks on $\mathbb{Z}^2$. Then there exists a $\nu > 0$ such that

$$\lim_{N\to\infty}\mathbb{P}\left(\omega\in\Omega_N \text{ s.t. } \deg(\mathcal{C}(\omega)) \geq e^{\frac{\nu N}{(\ln N)^2}}\right) = 1.$$

Proof.
Suppose that the pattern Q, up to a translation of $\zeta\in\mathbb{Z}^2$, exists in the support of a random walk ω. Let us consider the following transformation:

$$\begin{array}{rcl} T_\zeta : \Omega_N & \longmapsto & \Omega_N \\ \omega & \longrightarrow & T_\zeta(\omega) \doteqdot \omega' = \begin{cases} \omega'_i = \omega_i & \text{if } \omega_i \neq \zeta, \\ \omega'_i = \zeta + (-1,-1) & \text{if } \omega_i = \zeta. \end{cases} \end{array}$$

In other words we exchange the points ζ and $\zeta + (-1,-1)$. This application does not change the contact matrix of the random walk, i.e. $\mathcal{C}(\omega) = \mathcal{C}(\omega')$ because $\zeta + (-1,-1)$ is connected only with $\zeta + (-1,0)$.
Therefore if we prove that the probability of having at least λN times the pattern Q in the support of a random walk goes to 1 as N goes to infinity, then the degeneracy of the corresponding contact matrix is at least $2^{\lambda N}$. In fact we can apply or not apply T_ζ independently for each ζ such that Q appears in the support of ω (centered in ζ).

[25]For more details, see [12].

Now we want an upper bound of $\mathbb{P}\left(Q_N < \alpha\frac{\mu N}{(\ln N)^2}\right)$ for $\alpha \in (0,1)$ and where $\mu \doteqdot m_1\pi^2$.
For each $k > 0$, there exists a N_0 such that for $N \geq N_0$,

$$\mathbb{P}\left(Q_N < \alpha\frac{\mu N}{(\ln N)^2}\right) \leq \mathbb{P}\left(Q_N - \mathbb{E}(Q_N) \leq -ka_Q\frac{N}{(\ln 8N)^3}\right)$$

with $a_Q = 2\mu\mathcal{A}$.
In fact, for N large enough, $\mathbb{E}(Q_N) = \frac{\mu N}{(\ln 8N)^2} + \mathcal{O}(N/(\ln 8N)^3)$ and therefore for each $\alpha < 1$, $\frac{\mu N}{(\ln 8N)^2} + \mathcal{O}(N/(\ln 8N)^3) - ka_Q\frac{N}{(\ln 8N)^3} \geq \alpha\frac{\mu N}{(\ln N)^2}$. Therefore for each $k > 0$ and N large enough

$$\begin{aligned}
\mathbb{P}\left(Q_N < \frac{\mu N}{(\ln N)^2}\right) &\leq \mathbb{P}\left(Q_N - \mathbb{E}(Q_N) \leq -ka_Q\frac{N}{(\ln 8N)^3}\right)\\
&\leq \mathbb{P}\left(|Q_N - \mathbb{E}(Q_N)| \geq ka_Q\frac{N}{(\ln 8N)^3}\right)\\
&\leq \frac{\mathbb{E}\left(Q_N - \mathbb{E}(Q_N)\right)^2}{k^2a_Q^2\frac{N^2}{(\ln 8N)^6}} \overset{N\to\infty}{\longrightarrow} \frac{1}{k^2}.
\end{aligned}$$

Therefore for each $\alpha \in (0,1)$ we have

$$\forall\, k > 0,\ \lim_{N\to\infty} \mathbb{P}\left(Q_N < \alpha\frac{\mu N}{(\ln N)^2}\right) \leq \frac{1}{k^2}.$$

This implies that for all $\alpha \in (0,1)$,

$$\lim_{N\to\infty} \mathbb{P}\left(Q_N \geq \alpha\frac{\mu N}{(\ln N)^2}\right) = 1.$$

All the random walks with a number of pattern Q on their support more than $\alpha\frac{\mu N}{(\ln N)^2}$ are more degenerate than $e^{\alpha\frac{\mu N\ln 2}{(\ln N)^2}}$. Then for all choice of $\nu < \mu\ln 2$ (because $\alpha \in (0,1)$) we have

$$\lim_{N\to\infty} \mathbb{P}\left(\omega \in \Omega_N \text{ s.t. } \deg(\mathcal{C}(\omega)) > e^{\frac{\nu N}{(\ln N)^2}}\right) = 1.$$

$\square$

Proposition 20.
Let us consider simple random walks on $\mathbb{Z}^2$. Then there exists a $\beta > 0$ such that

$$\lim_{N\to\infty} \mathbb{P}\left(\omega \in \Omega_N \textit{ s.t. } \deg(\mathcal{C}(\omega)) \geq e^{\frac{\beta N}{\ln N}}\right) = 0.$$

Proof.
For this proof we consider the observable to be the support R_N. It has $m_0 = 1$ and $m_l = 0$ for $l \geq 1$. Its expectation and variance are given (for large N) by

$$\mathbb{E}(R_N) = \frac{\pi N}{\ln 8N}\left(1 + \mathcal{O}\left(1/\ln N\right)\right)$$

and
$$\mathbb{E}\left(R_N - \mathbb{E}(R_N)\right)^2 = \frac{\mathcal{A}^2\pi^2 N^2}{(\ln 8N)^2}.$$
Moreover by lemma 16
$$\mathbb{P}\left(\deg(\mathcal{C}(\omega)) \geq e^{\beta\frac{N}{\ln N}}\right) \leq \mathbb{P}\left(R_N \geq \beta \ln 3 \frac{N}{\ln N}\right)$$
and taking $\beta > \frac{\pi}{\ln 3}$ and $\varepsilon \doteqdot \beta \ln 3 - \pi > 0$ we obtain
$$\begin{aligned}
\mathbb{P}\left(R_N \geq \beta \ln 3 \frac{N}{\ln N}\right) &= \mathbb{P}\left(R_N - \mathbb{E}(R_N) \geq \varepsilon \frac{N}{\ln N} + \mathcal{O}\left(\frac{N}{(\ln N)^2}\right)\right) \\
&\leq \mathbb{P}\left(|R_N - \mathbb{E}(R_N)| \geq \frac{\varepsilon N}{\ln N} + \mathcal{O}\left(\frac{N}{(\ln N)^2}\right)\right) \\
&\leq \frac{\mathbb{E}\left(R_N - \mathbb{E}(R_N)\right)^2}{\frac{\varepsilon^2 N^2}{(\ln N)^2}(1 + \mathcal{O}(1/\ln N))} \\
&\overset{N\to\infty}{\longrightarrow} \lim_{N\to\infty} \frac{\mathcal{A}^2\pi^2 N^2/(\ln 8N)^4}{\varepsilon^2 N^2/(\ln N)^2} = \lim_{N\to\infty} \frac{\mathcal{A}^2\pi^2}{\varepsilon^2(\ln N)^2} = 0.
\end{aligned}$$
Therefore for all $\beta > \frac{\pi}{\ln 3}$,
$$\lim_{N\to\infty} \mathbb{P}\left(\omega \in \Omega_N \text{ s.t. } \deg(\mathcal{C}(\omega)) \geq e^{\frac{\beta N}{\ln N}}\right) = 0.$$
□

4.3 End-to-end distance

In the next pages we compute the end-to-end distance for the five different types of random walks mentioned at the beginning of this section.

4.3.1 Unconstrained random walk on $\mathbb{Z}^2$

This is the easiest case for the computation of the end-to-end distance. Let us consider an unconstrained random walk on a square lattice and let $X_i, 1 \leq i \leq N$ be iid displacements whose probabilities are:
$$\mathbb{P}(X_1 = (1,0)) = \mathbb{P}(X_1 = (-1,0)) = \mathbb{P}(X_1 = (0,1)) = \mathbb{P}(X_1 = (0,-1)) = \frac{1}{4}.$$
Then the characteristic function of $S_N = \sum_{i=1}^N X_i$ is
$$\Phi_{S_N}(\xi \doteqdot (\xi_1,\xi_2)) = \mathbb{E}\left(e^{i\xi\cdot S_N}\right) \overset{\text{iid}}{=} \left(\mathbb{E}\left(e^{i\xi\cdot X_1}\right)\right)^N = \left(\Phi_{X_1}(\xi)\right)^N,$$
where
$$\Phi_{X_1}(\xi) = \frac{1}{4}\left(e^{i\xi_1} + e^{-i\xi_1} + e^{i\xi_2} + e^{-i\xi_2}\right).$$

The mean and the variance of X_1 are respectively $\mu = 0$ and $\sigma^2_{X_1} = 1$. Moreover, $\Phi_{X_1}(\xi_1, \xi_2) = 1 - \frac{1}{4}(\xi_1^2 + \xi_2^2) + \mathcal{O}(\xi_i^4)$. Defining $A_N \doteqdot \frac{S_N}{\sqrt{N/2}}$ we obtain

$$\Phi_{A_N}(\xi) = \Phi_{S_N}\left(\frac{\xi}{\sqrt{N/2}}\right) = e^{N \ln\left(1 - \frac{1}{2N}(\xi_1^2+\xi_2^2) + \mathcal{O}(\xi_i^4/N^2)\right)} \xrightarrow{N\to\infty} e^{-\frac{\xi_1^2+\xi_2^2}{2}}.$$

Since the limit of the sequence $\Phi_{A_N}(\xi)$ is continuous at $(0,0)$ and corresponds to the characteristic function of the normal distribution, by the Paul-Lévy theorem, $\left\{A_N = \frac{S_N}{\sqrt{N/2}}\right\}_{N\geq 1}$ converges weakly to the two dimensional normal distribution $\mathcal{N}(0,1)$.
Therefore probability distribution of S_N converges to

$$\mathbb{P}\left(S_N = (x,y)\right) = \frac{1}{2\pi N\sigma^2} e^{-\frac{x^2+y^2}{2N\sigma^2}} \text{ with } \sigma^2 = \frac{1}{2} \tag{9}$$

as N goes to infinity.
The expected number of visits at the origin G is given by:

$$G = \sum_{N=0}^{\infty} \mathbb{P}(S_N = (0,0)) = \infty \text{ because } \mathbb{P}(S_N = (0,0)) \sim \frac{1}{N} \text{ as } N \to \infty,$$

and the probability of never coming back to the origin is $1/G = 1 - F = 0$. Therefore proposition 17 holds that means $\bar{\gamma} = 1$.

4.3.2 Random walk on $\mathbb{Z}^2$ that cannot come back immediately

In this case the end-to-end distribution is more difficult to compute because each step depend on the previous one. For this case we prove that the probability distribution converges weakly to Gaussian with variance $2N$. Once we have this result, the property of recurrence computed in the previous case is also true in this case (with a different value of σ^2).
Let us define the following random variables:

$$\begin{aligned}
a(N,x,y) &\doteqdot \mathbb{P}(S_N = (x,y)|S_{N-1} = (x-1,y)) \\
b(N,x,y) &\doteqdot \mathbb{P}(S_N = (x,y)|S_{N-1} = (x+1,y)) \\
c(N,x,y) &\doteqdot \mathbb{P}(S_N = (x,y)|S_{N-1} = (x,y-1)) \\
d(N,x,y) &\doteqdot \mathbb{P}(S_N = (x,y)|S_{N-1} = (x,y+1))
\end{aligned}$$

Since the random walks cannot be in the same position after two steps, the relations of these random variables for two consecutive steps are:

$$\begin{aligned}
a(N,x,y) &= \frac{1}{3}\left(a(N-1,x-1,y)+c(N-1,x-1,y)+d(N-1,x-1,y)\right)\\
b(N,x,y) &= \frac{1}{3}\left(b(N-1,x+1,y)+c(N-1,x+1,y)+d(N-1,x+1,y)\right)\\
c(N,x,y) &= \frac{1}{3}\left(a(N-1,x,y-1)+b(N-1,x,y-1)+c(N-1,x,y-1)\right)\\
d(N,x,y) &= \frac{1}{3}\left(a(N-1,x,y+1)+b(N-1,x,y+1)+d(N-1,x,y+1)\right)
\end{aligned}$$

We define the characteristic function of these random variables

$$\Phi_\rho(N,\xi_1,\xi_2) \doteqdot \sum_{(x,y)\in\mathbb{Z}^2} \rho(N,x,y)e^{i(\xi_1 x+\xi_2 y)} \text{ for } \rho=a,b,c,d$$

and the characteristic function of S_N

$$\begin{aligned}
\Phi_{S_N}(\xi_1,\xi_2) &\doteqdot \sum_{(x,y)\in\mathbb{Z}^2} \mathbb{P}(S_N=x)e^{i(\xi_1 x+\xi_2 y)}\\
&= \Phi_a(N,\xi_1,\xi_2)+\Phi_b(N,\xi_1,\xi_2)+\Phi_c(N,\xi_1,\xi_2)+\Phi_d(N,\xi_1,\xi_2).
\end{aligned}$$

Finally we define:

$$\Phi(N,\xi_1,\xi_2) \doteqdot \begin{pmatrix} \Phi_a(N,\xi_1,\xi_2)\\ \Phi_b(N,\xi_1,\xi_2)\\ \Phi_c(N,\xi_1,\xi_2)\\ \Phi_d(N,\xi_1,\xi_2)\end{pmatrix}.$$

With these definitions we obtain a matrix relation between $\Phi(N,\xi_1,\xi_2)$ and $\Phi(N-1,\xi_1,\xi_2)$:

$$\Phi(N,\xi) = \frac{1}{3}\underbrace{\begin{pmatrix} e^{i\xi_1} & 0 & e^{i\xi_1} & e^{i\xi_1}\\ 0 & e^{-i\xi_1} & e^{-i\xi_1} & e^{-i\xi_1}\\ e^{i\xi_2} & e^{i\xi_2} & e^{i\xi_2} & 0\\ e^{-i\xi_2} & e^{-i\xi_2} & 0 & e^{-i\xi_2}\end{pmatrix}}_{\doteqdot T}\Phi(N-1,\xi_1,\xi_2).$$

We takes as initial condition $\Phi(0,\xi_1,\xi_2)=\frac{1}{4}(1,1,1,1)$, i.e. the initial position is $x=0$ and we have equal probability of going in each directions. To obtain the asymptotic probability distribution of S_N we need to compute $\Phi(N,\xi_1,\xi_2)$ and then sum over its four components.
In order to compute $\Phi(N,\xi_1,\xi_2)=\frac{1}{3^N}T^N\Phi(0,\xi_1,\xi_2)$ we diagonalise T. The eigenvalues of T are

$$\begin{aligned}
\lambda_1 &= 1,\\
\lambda_2 &= -1,\\
\lambda_3 &= \nu+\sqrt{\nu^2-3},\\
\lambda_4 &= \nu-\sqrt{\nu^2-3},
\end{aligned}$$

where $\nu = \frac{e^{i\xi_1}+e^{-i\xi_1}+e^{i\xi_2}+e^{-i\xi_2}}{2}$.
A calculation gives:

$$\Phi_{S_N}(\xi_1,\xi_2) = \frac{\lambda_3^N - \lambda_4^N}{4 \cdot 3^N} \cdot \frac{\nu}{\sqrt{\nu^2-3}} + \frac{\lambda_3^N + \lambda_4^N}{2 \cdot 3^N}.$$

Now we want to study the characteristic function of S_N when N goes to infinity.
For (ξ_1,ξ_2) near to $(0,0)$, $|\lambda_3| > |\lambda_4|$, so that $1 \pm \left(\frac{\lambda_4}{\lambda_3}\right)^N \stackrel{N\to\infty}{\longrightarrow} 1$, $\frac{\nu}{2\sqrt{\nu^2-3}} = 1 + \frac{3}{4}(\xi_1^2 + \xi_2^2) + \mathcal{O}(\xi^4)$[26], $\lambda_3 = 3(1 - \frac{1}{2}(\xi_1^2+\xi_2^2)) + \mathcal{O}(\xi^4)$. Then $\mathbb{E}(S_N) = 0$ (by symmetry) and $\mathbb{E}(S_N^2) = (-i)^2 \left(\frac{d^2}{d\xi_1^2} + \frac{d^2}{d\xi_2^2}\right) \Phi_{S_N}(\xi)\big|_{(\xi_1,\xi_2)=(0,0)} = 2N$.
Let us consider the rescaled random variable

$$A_N \doteqdot \frac{S_N}{\sqrt{N}}.$$

As $N \to \infty$, for each fixed $\xi = (\xi_1,\xi_2)$, $\frac{(\xi_1,\xi_2)}{\sqrt{N}}$ goes to $(0,0)$ and the terms $(\lambda_3/\lambda_4)^N$ vanish. Therefore

$$\begin{aligned}
\Phi_{A_N}(\xi) &= \mathbb{E}\left(e^{i\xi\cdot A_N}\right) = \mathbb{E}\left(e^{i\frac{\xi\cdot S_N}{\sqrt{N}}}\right) = \Phi_{S_N}\left(\xi' \doteqdot \frac{\xi}{\sqrt{N}}\right) \\
&= \frac{\lambda_3\left(\xi'\right)^N}{3^N} \cdot \left(1 + \mathcal{O}\left(\frac{\xi_1^2}{N}, \frac{\xi_2^2}{N}\right)\right) \\
&= \frac{1}{3^N} e^{N \ln \lambda_3\left(\frac{\xi}{\sqrt{N}}\right)} \cdot \left(1 + \mathcal{O}\left(\frac{\xi_1^2}{N}, \frac{\xi_2^2}{N}\right)\right) \\
&= \frac{1}{3^N} e^{N\left(\ln 3 + \ln\left(1 - \frac{\xi_1^2+\xi_2^2}{2N} + \mathcal{O}(\xi^4/N^2)\right)\right)} \cdot \left(1 + \mathcal{O}\left(\frac{\xi_1^2}{N}, \frac{\xi_2^2}{N}\right)\right) \\
&= e^{-\frac{\xi_1^2+\xi_2^2}{2} + \mathcal{O}(\xi^4/N)} \cdot \left(1 + \mathcal{O}\left(\frac{\xi_1^2}{N}, \frac{\xi_2^2}{N}\right)\right) \stackrel{N\to\infty}{\longrightarrow} e^{-\frac{\xi_1^2+\xi_2^2}{2}}
\end{aligned}$$

$e^{-\frac{\xi_1^2+\xi_2^2}{2}}$ is a continuous function at $(\xi_1,\xi_2) = (0,0)$, therefore the Paul-Lévy theorem implies that the sequence of random variables $\left\{A_N = \frac{S_N}{\sqrt{N}}\right\}_{N\geq 1}$ converges weakly to the $2D$ normalized normal distribution $\mathcal{N}(0,1)$.
Consequently the probability distribution of S_N converges to

$$\mathbb{P}(S_N = (x,y)) = \frac{1}{2\pi N\sigma^2} e^{-\frac{x^2+y^2}{2N\sigma^2}} \text{ with } \sigma^2 = 1$$

as N goes to infinity.
By the same argument used for the unconstrained random walk (section 4.3.1) we conclude that proposition 17 holds, i.e. $\bar{\gamma} = 1$.

[26] $\mathcal{O}(\xi^4)$ is used for $\mathcal{O}(\xi_1^p \xi_2^q)$ for $q + p = 4$.

4.3.3 Random walk on $\mathbb{Z}^2$ that turns 90 degrees at each step

Let us consider the random walk on $\mathbb{Z}^2$ which turns 90 degrees at each step. In this case the displacements divide in the iid $X_{2i}, 1 \leq i \leq [N/2]$, for which $\mathbb{P}(X_{2i} = (0,1)) = \mathbb{P}(X_{2i} = (0,-1)) = \frac{1}{2}$, and $X_{2i-1}, 1 \leq i \leq [(N+1)/2]$, for which $\mathbb{P}(X_{2i-1} = (1,0)) = \mathbb{P}(X_{2i-1} = (-1,0)) = \frac{1}{2}$. All these displacements are independent. The probability distribution is the convolution of the one of the odd steps with the one of the even steps. If we define $T_N^e \doteqdot \sum_{i=1}^{[N/2]} X_{2i}$ and $T_N^o \doteqdot \sum_{i=1}^{[(N+1)/2]} X_{2i-1}$ then $\Phi_{S_N}(\xi_1, \xi_2) = \Phi_{T_N^e}(\xi_1, \xi_2) \cdot \Phi_{T_N^o}(\xi_1, \xi_2)$.
When $N \to \infty$, $\Phi_{T_N^o}\left(\frac{\xi_1}{\sqrt{N/2}}\right) = e^{-\frac{\xi_1^2}{2}}$ and $\Phi_{T_N^e}\left(\frac{\xi_2}{\sqrt{N/2}}\right) = e^{-\frac{\xi_2^2}{2}}$, then the probability distribution of $\frac{S_N}{\sqrt{N/2}}$ converges to $\mathcal{N}(0,1)$.
Consequently the probability distribution of S_N converges to

$$\mathbb{P}(S_N = (x,y)) = \frac{1}{2\pi N\sigma^2} e^{-\frac{x^2+y^2}{2N\sigma^2}} \text{ with } \sigma^2 = \frac{1}{2} \tag{10}$$

as N goes to infinity and proposition 17 holds and $\bar{\gamma} = 1$.

4.3.4 Unconstrained random walk on honeycomb lattice

Let us consider the unconstrained random walk on honeycomb lattice such that the distance between two nearest neighbors is equal to one.

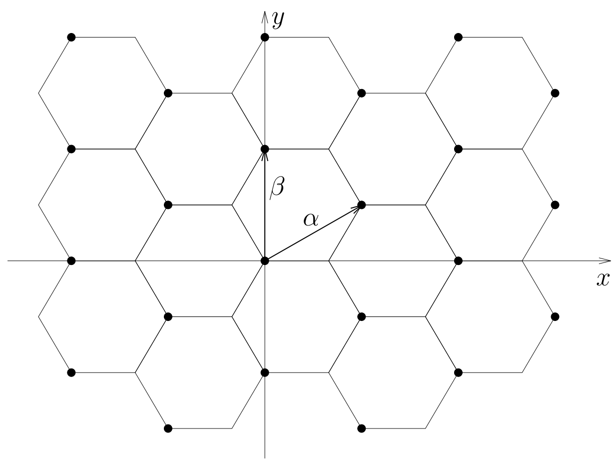


Figure 6: The honeycomb lattice. The sublattice considered is the emphasized one.

The honeycomb lattice is an union of two triangular sublattices. At each step the random walk passes from a sublattice to the other one, so that after two steps the random walk returns on the initial sublattice. We consider the sublattice containing the origin and we compute the end-to-end distribution of a $2N$-random walk.

The sublattice considered is

$$\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \alpha M_1 + \beta M_2 \text{ where } M_1, M_2 \in \mathbb{Z} \text{ and } \alpha = \begin{pmatrix} 3/2 \\ \sqrt{3}/2 \end{pmatrix}, \beta = \begin{pmatrix} 0 \\ \sqrt{3} \end{pmatrix} \right\}$$

Let $\{Y_k\}_{k=1}^N$ be iid the random variable of the double steps. Then the position after $2N$ steps is given by the random variable $S_{2N} \doteqdot \sum_{k=1}^N Y_k$.
The probability distribution of Y_k is given by:

$$\begin{aligned} \mathbb{P}(Y_k = \alpha) &= \mathbb{P}(Y_k = -\alpha) = \mathbb{P}(Y_k = \beta) = \mathbb{P}(Y_k = -\beta) \\ &= \mathbb{P}(Y_k = \alpha - \beta) = \mathbb{P}(Y_k = \beta - \alpha) = \frac{1}{9}, \\ \mathbb{P}(Y_k = 0) &= \frac{1}{3}, \end{aligned}$$

and the characteristic function of S_{2N} is given by

$$\begin{aligned} \Phi_{S_{2N}}(\xi = (\xi_1, \xi_2)) &= \mathbb{E}\left(e^{iS_{2N}\cdot\xi}\right) \\ &\overset{\text{iid}}{=} \left(\mathbb{E}\left(e^{iY_1\cdot\xi}\right)\right)^N \\ &= \left(\Phi_{Y_1}(\xi)\right)^N \end{aligned}$$

where $\Phi_{Y_1}(\xi) = \frac{1}{9}\left(3 + e^{i\alpha\cdot\xi} + e^{-i\alpha\cdot\xi} + e^{i\beta\cdot\xi} + e^{-i\beta\cdot\xi} + e^{i(\alpha-\beta)\cdot\xi} + e^{-i(\alpha-\beta)\cdot\xi}\right)$

By symmetry $\mathbb{E}(Y_1) = 0$ and $\sqrt{\mathbb{E}(Y_1^2)} = \sqrt{2}$. Let us define $A_{2N} \doteqdot \frac{S_{2N}}{\sqrt{2N}}$. Then $\Phi_{A_{2N}}(\xi) = \left(\Phi_{Y_1}\left(\frac{\xi}{\sqrt{2N}} \doteqdot \xi'\right)\right)^N$ and by the same method used for the previous cases:

$$\lim_{N\to\infty} \Phi_{A_{2N}}(\xi) = e^{-\frac{\xi_1^2+\xi_2^2}{2}}$$

and the probability distribution of S_N converges to

$$\mathbb{P}(S_N = (x,y)) = \frac{1}{2\pi N\sigma^2} e^{-\frac{x^2+y^2}{2N\sigma^2}} \text{ with } \sigma^2 = \frac{1}{2}.$$

4.3.5 Random walk on a honeycomb lattice that cannot come back immediately

To compute the end-to-end distance we use the same method as above, but this time there is a little complication due to the lattice geometry.
Like for the random walk on honeycomb lattice without constraints, we consider the sublattice of the even steps. Each of its points are given by the couple of integer (M_1, M_2) in the basis of $\{\alpha, \beta\}$. Let us consider the position after N double displace-

ments of the random walk, noted Z_N. We define the following random variables:

$$\begin{aligned}
a_1(N,M_1,M_2) &\doteqdot \mathbb{P}(Z_N=(M_1,M_2)|Z_{N-1}=(M_1-1,M_2))\\
a_2(N,M_1,M_2) &\doteqdot \mathbb{P}(Z_N=(M_1,M_2)|Z_{N-1}=(M_1,M_2-1))\\
b_1(N,M_1,M_2) &\doteqdot \mathbb{P}(Z_N=(M_1,M_2)|Z_{N-1}=(M_1+1,M_2-1))\\
b_2(N,M_1,M_2) &\doteqdot \mathbb{P}(Z_N=(M_1,M_2)|Z_{N-1}=(M_1+1,M_2))\\
c_1(N,M_1,M_2) &\doteqdot \mathbb{P}(Z_N=(M_1,M_2)|Z_{N-1}=(M_1,M_2+1))\\
c_2(N,M_1,M_2) &\doteqdot \mathbb{P}(Z_N=(M_1,M_2)|Z_{N-1}=(M_1-1,M_2+1))
\end{aligned}$$

Since the random walks cannot be in the same position after two steps, the relations of these random variables for two consecutive steps are (the dots replace the argument of the previous term):

$$\begin{aligned}
a_1(N,M_1,M_2) &= \frac{1}{4}\left(a_1(N-1,M_1-1,M_2)+a_2(\dots)+c_1(\dots)+c_2(\dots)\right)\\
a_2(N,M_1,M_2) &= \frac{1}{4}\left(a_1(N-1,M_1,M_2-1)+a_2(\dots)+b_1(\dots)+b_2(\dots)\right)\\
b_1(N,M_1,M_2) &= \frac{1}{4}\left(a_1(N-1,M_1+1,M_2-1)+a_2(\dots)+b_1(\dots)+b_2(\dots)\right)\\
b_2(N,M_1,M_2) &= \frac{1}{4}\left(b_1(N-1,M_1+1,M_2)+b_2(\dots)+c_1(\dots)+c_2(\dots)\right)\\
c_1(N,M_1,M_2) &= \frac{1}{4}\left(b_1(N-1,M_1,M_2+1)+b_2(\dots)+c_1(\dots)+c_2(\dots)\right)\\
c_2(N,M_1,M_2) &= \frac{1}{4}\left(a_1(N-1,M_1-1,M_2+1)+a_2(\dots)+c_1(\dots)+c_2(\dots)\right)
\end{aligned}$$

We define the characteristic function of these random variables

$$\Phi_\rho(N,\zeta_1,\zeta_2)\doteqdot\sum_{(M_1,M_2)\in\mathbb{Z}^2}\rho(N,M_1,M_2)e^{i(\zeta_1M_1+\zeta_2M_2)}\text{ for }\rho=a_1,a_2,b_1,b_2,c_1,c_2$$

and the characteristic function of Z_N

$$\begin{aligned}
\Phi_{Z_N}(\zeta_1,\zeta_2) &\doteqdot \sum_{(M_1,M_2)\in\mathbb{Z}^2}\mathbb{P}(Z_N=(M_1,M_2))e^{i(\zeta_1M_1+\zeta_2M_2)}\\
&= \sum_{\rho=a_1,\dots,c_2}\Phi_\rho(N,\zeta_1,\zeta_2).
\end{aligned}$$

Defining a vector with the characteristic function of $a_1,\dots,c_2$ we can find a matrix relation between the steps $N-1$ and N. Then we diagonalise the matrix, we take its N^{th} power and we sum over its six component. One more time we take as initial condition that the initial position is $x=0$ and we have equal probability of going in each directions.

But what we want is the distribution of S_N, the actual position after N steps. First remark that $\mathbb{P}(S_{2N}) = (x,y)) = \mathbb{P}(Z_N = (M_1 = \frac{2}{3}x, M_2 = \frac{y}{\sqrt{3}} - \frac{x}{3}))$ and therefore we have

$$\begin{aligned} \Phi_{S_{2N}}(\xi_1,\xi_2) &\doteqdot \sum_{(x,y)\in \text{ lattice}} \mathbb{P}(S_{2N} = (x,y))e^{i(x\xi_1+y\xi_2)} \\ &= \Phi_{Z_N}(\zeta_1,\zeta_2) \text{ with } \begin{cases} \zeta_1 = \frac{3}{2}\xi_1 + \frac{\sqrt{3}}{2}\xi_2 \\ \zeta_2 = \sqrt{3}\xi_2 \end{cases} \end{aligned}$$

Finally from the computations we obtain that the probability distribution of S_N converges to

$$\mathbb{P}(S_N = (x,y)) = \frac{1}{2\pi N\sigma^2} e^{-\frac{x^2+y^2}{2N\sigma^2}} \text{ with } \sigma^2 = \frac{3}{4}$$

as N goes to infinity.
By the same argument used for the unconstrained random walk (section 4.3.1) we conclude that proposition 17 holds, i.e. $\bar{\gamma} = 1$.

4.3.6 Comparison of the end-to-end distance

All the end-to-end distributions we have computed converge to a two dimensional normal distribution with mean 0, $\mathcal{N}(0, N\sigma^2)$, with a different value of σ^2. Therefore the end-to-end distance[27] of these random walks are $\sqrt{2N\sigma^2}$. In the following table we sum up them.

Type of random walk	$\sqrt{\mathbb{E}(S_N^2)}$
Simple random walk on $\mathbb{Z}^2$	$\sqrt{N}$
Random walk without immediate return	$\sqrt{2N}$
Random walk that turns 90° at each step	$\sqrt{N}$
Random walk on honeycomb lattice without immediate return	$\sqrt{\frac{3N}{2}}$
Unconstrained random walk on honeycomb lattice	$\sqrt{N}$

Table 3: The different values of the end-to-end distance

We see that the effect of the constraint of avoiding the immediate return is important. In fact the end-to-end distance increases of a factor $\sqrt{2}$ for the square lattice and $\sqrt{3/2}$ for the honeycomb lattice. It is interesting to notice that for unconstrained random walk on a honeycomb and on a square lattice the end-to-end distance is asymptotically the same. This is no more true if we consider the random walks that cannot come back immediately. In this case the end-to-end distance for the square lattice is bigger than the one of the honeycomb lattice.

[27] $\frac{1}{2\pi N\sigma^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2\pi N\sigma^2}}(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y = 2N\sigma^2$.

5 Simple random walks in $d \geq 3$

5.1 Growth factor for random walk on $\mathbb{Z}^d$

In this section we prove that, due to the left heavy tail of the probability distribution of the support, we have $\bar{\gamma} = 1$ in all dimension (for i.i.d. random walks). Nevertheless the number of different contact matrices can be very far to the total number of walks. In fact $\bar{\gamma} = 1$ implies only that the correction to the total number of walks is less important than e^{-qN} for each $q > 0$.

Proposition 21 (Growth factor). *Let us consider simple random walks on $\mathbb{Z}^d$. Then*

$$\bar{\gamma} = \lim_{N\to\infty} \frac{\ln W(N)}{N \ln 2d} = 1.$$

PROOF.
Let $\Omega_N^\delta \doteqdot \{\omega \in \Omega_N \text{ s.t. } R_N(\omega) \leq N\delta\}$. Then using lemma 16, page 17,

$$W(N) \geq W(N)|_{\Omega_N^\delta} \geq \frac{\mathrm{Card}(\Omega_N)\mathbb{P}(R_N \leq N\delta)}{(2d-1)^{N\delta}}.$$

Therefore

$$\begin{aligned} 1 \geq \lim_{N\to\infty} \frac{\ln W(N)}{\ln \mathrm{Card}(\Omega_N)} &\geq \lim_{N\to\infty} \frac{\ln \mathbb{P}(R_N \leq N\delta) + N \ln 2d - N\delta \ln (2d-1)}{N \ln 2d} \\ &= 1 - \delta \frac{\ln (2d-1)}{\ln 2d} \end{aligned}$$

because[28]

$$\lim_{N\to\infty} \frac{1}{N} \ln \mathbb{P}(R_N \leq N\delta) = 0 \text{ for all } \delta > 0.$$

Let us define $\delta' \doteqdot \delta \frac{\ln (2d-1)}{\ln 2d}$. Then

$$\forall\, \delta' > 0, \quad 1 \geq \lim_{N\to\infty} \frac{\ln W(N)}{\ln \mathrm{Card}(\Omega_N)} \geq 1 - \delta'$$

that implies

$$\lim_{N\to\infty} \frac{\ln W(N)}{\ln \mathrm{Card}(\Omega_N)} = 1.$$

□

[28] See introduction of [2].

In the next proposition we give a lower bound for the first correction on γ_N.

Proposition 22. *Let* $\gamma_N \doteqdot \frac{\ln W(N)}{\ln \mathrm{Card}(\Omega_N)}$, *then there exists a* $C > 0$ *such that*

$$\gamma_N \geq 1 - \frac{C}{N^{2/(d+2)}}.$$

Proof.
When N goes to infinity, $\mathbb{P}(R_N = X) \sim e^{-a\frac{N}{X^{2/d}}}$ for $d \geq 2$ and in the scaling limit $X \to \infty$ and $\frac{X}{N} \to 0$ (see [9]).
Therefore for $\alpha \in (0, 1)$,

$$\mathbb{P}(R_N = [N^\alpha]) \sim e^{-aN^{1-2\alpha/d}}.$$

But for an ω with $R_N = [N^\alpha]$, $\deg(\mathcal{C}(\omega)) \leq (2d-1)^{N^\alpha}$ (see lemma 16). Therefore

$$W(N) \geq \frac{(2d)^N \mathbb{P}(R_N = [N^\alpha])}{(2d-1)^{N^\alpha}}$$

that implies, for large N,

$$1 \geq \gamma_N \geq 1 - \sigma(N, \alpha)$$

where

$$\sigma(N, \alpha) = \frac{N^\alpha \ln(2d-1)}{N \ln 2d} + \frac{aN^{-2\alpha/d}}{\ln 2d}.$$

The correction will be smaller if $\sigma(N, \alpha)$ is smaller, then let us search the $\alpha \in (0, 1)$ such that $\sigma(N, \alpha)$ is the smallest when $N \to \infty$. This is the case for α such that the two terms of $\sigma(N, \alpha)$ are of the same order, i.e. $\alpha - 1 = -2\alpha/d$. Then $\alpha = \frac{d}{d+2}$ (and $\alpha - 1 = -\frac{2}{d+2}$). Taking $C = \frac{\ln(2d-1)+a}{\ln 2d}$ we end the proof. $\square$

This proposition implies that for N large we have (up to smaller corrections),

$$\frac{W(N)}{\mathrm{Card}(\Omega_N)} \geq e^{-CN^{d/(d+2)} \ln 2d}.$$

We believe that this lower bound is closer to the actual value for high dimensions than for low dimension, because the upper bound for the degnerancy of a contact matrix will be closer to the actual behaviour in these dimension. In fact for low dimension the constraints given by the matrix will eliminate some “degrees of freedom”, more in low than in high dimensions.

5.2 Degeneracy for random walk on $\mathbb{Z}^3$

The random walks start from the origin 0. Consider a random walk $\omega \in \Omega_N$, then for each lattice site let us define the function

$$m_N(x) = m_N(x,\omega) = \begin{cases} 1 & \text{if } x \text{ has not been visited by } \omega, \\ 0 & \text{otherwise.} \end{cases}$$

Let us consider the observables that can be written in the following form

$$M_N = M_N(\omega) = \sum_A \mu_A M_{A,N}(\omega) \text{ where } M_{A,N}(\omega) = \sum_{x\in\mathbb{Z}^3} \prod_{a\in A} m_N(x+a,\omega)$$

where A is a finite subset of lattice vectors. Van Wijland and Hilhorst proved in [13] that in three dimensions $\Delta M_N = M_N - \mathbb{E}(M_N)$ satifies, in the asymptotic limit[29],

$$\Delta M_N \simeq a_M \sqrt{N \ln N}\zeta(N) \tag{11}$$

where $\zeta(N)$ is a random variable with normal distribution $\mathcal{N}(0,1)$ and a_M is a constant.
More precisely they proved that the expectation of M_N behaves as:

$$\mathbb{E}(M_N) = \begin{cases} m_1 N + \sqrt{\frac{54}{\pi^3}} m_2 \sqrt{N} + \mathcal{O}(1) & (d=3) \\ m_1 N + \frac{4}{\pi^2} m_2 \ln N + \mathcal{O}(1) & (d=4) \\ m_1 N + \mathcal{O}(1) & (d\geq 5) \end{cases} \tag{12}$$

and the variance has the form:

$$\mathbb{E}((\Delta M_N)^2) = \begin{cases} \frac{27}{2\pi^2} m_2^2 N \ln N + c_M N + o(N) & (d=3) \\ C_M N + o(N) & (d \geq 4) \end{cases} \tag{13}$$

In the article they also explain how to compute the coefficients m_1, m_2, c_M, C_M. We apply this results to a particular observable in order to prove that the probability of having a random walk whose contact matrix is exponentially degenerate goes to 1 as N goes to infinity. Nevertheless we prove that $\bar{\gamma} = 1$ in dimensions $d \geq 3$. This is due to the left tail of the distribution of the support of the random walk not decreasing rapidly enough (the right tail instead decreases exponentially in N, see [2]).

The considered observable is a particular pattern P in which there is a local transformation that changes a random walk into another but does not change the corresponding contact matrix. Therefore if the probability of finding this pattern at least a λN times, $\lambda > 0$, goes to 1 (asymptotically), then using the transformation for each pattern independently we can prove that most of the different contact matrices are exponentially degenerate in N (see proposition 23).
Now let us define the pattern P. It consists of a set of visited points A_v and a set of unvisited points A_u as follows:

[29] This does not imply that the tails of all observables decreases in the same way, in fact this is not true, see for example [14].

- Visited points:
  $A_v(P) = \{(0,0,0) \quad , \quad (-1,0,0)\}$

- Unvisited points:
  $A_u(P) = \left\{ \begin{array}{ccccc} (1,0,0) & , & (0,1,0) & , & (0,-1,0) \\ (0,0,1) & , & (0,0,-1) & , & (-1,1,0) \end{array} \right\}$

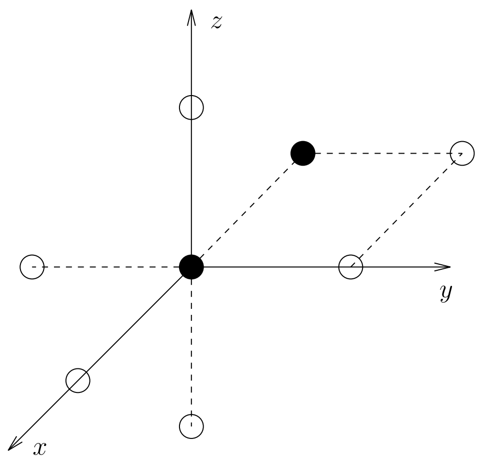


Figure 7: The pattern P.

The number P_N of times that P appears in the support of the N-random walk is given by:

$$P_N = P_N(\omega) = \sum_{x\in\mathbb{Z}^3} \prod_{a\in A_u} m_N(x+a) \prod_{b\in A_v} (1 - m_N(x+b)).$$

Explicitly we have

$$\begin{aligned} P_N \quad = \quad & \sum_{x\in\mathbb{Z}^3} m_N(x+e_1)\cdot m_N(x+e_2)\cdot m_N(x-e_2)\cdot m_N(x+e_3)\cdot \\ \cdot \quad & m_N(x-e_3)\cdot m_N(x+e_2-e_1)\cdot(1-m_N(x))\cdot(1-m_N(x-e_1)) \end{aligned}$$

where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$.

We computed[30] the coefficients m_1, m_2 and a_P that are:

$$m_1 = 2.5\cdot 10^{-3} \quad , \quad m_2 = 9.9\cdot 10^{-3} \quad \text{and} \quad a_P = 1.2\cdot 10^{-2}.$$

Proposition 23.
Let us consider a simple random walk on $\mathbb{Z}^3$. Then there exists a $\nu > 0$ such that

$$\lim_{N\to\infty} \mathbb{P}\left(\omega \in \Omega_N \ s.t.\ \deg(\mathcal{C}(\omega)) \geq e^{\nu N}\right) = 1.$$

[30]We used numerical values from [8].

Proof.

Suppose that a pattern P, up to a translation of $\zeta \in \mathbb{Z}^3$, exists in the support of a random walk ω. Let us consider the following transformation:

$$\begin{array}{rcl} S_\zeta : \Omega_N & \longmapsto & \Omega_N \\ \omega & \longrightarrow & S_\zeta(\omega) \doteqdot \omega' = \left\{ \begin{array}{ll} \omega'_i = \omega_i & \text{if } \omega_i \neq \zeta, \\ \omega'_i = \zeta + (-1,-1,0) & \text{if } \omega_i = \zeta. \end{array} \right. \end{array}$$

In other words we exchange the points ζ and $\zeta + (-1,-1,0)$. This transformation does not change the contact matrix of the random walk, i.e. $\mathcal{C}(\omega) = \mathcal{C}(\omega')$ because $\zeta + (-1,-1,0)$ is connected only with $\zeta + (-1,0,0)$.

Therefore if we prove that the probability of having at least λN times the pattern P in the support of a random walk goes to 1 as N goes to infinity, then the degeneracy of the corresponding contact matrices is at least $2^{\lambda N}$. In fact we can apply or not apply T_ζ independently for each ζ such that P appears in the support of ω (centered in ζ).

We want an upper bound of $\mathbb{P}(P_N < \alpha m_1 N)$ for $\alpha \in (0,1)$.

For each $k > 0$, there exists a N_0 such that for $N \geq N_0$,

$$\mathbb{P}(P_N < \alpha m_1 N) \leq \mathbb{P}(P_N - \mathbb{E}(P_N) \leq -k a_P \sqrt{N \ln N})).$$

In fact, for N large enough, $\mathbb{E}(P_N) = m_1 N + \mathcal{O}(\sqrt{N})$ and therefore for each $\alpha < 1$, $m_1 N + \mathcal{O}(\sqrt{N}) - k a_P \sqrt{N \ln N} \geq \alpha m_1 N$. Therefore for each $k > 0$ and N large enough

$$\begin{array}{rcl} \mathbb{P}(P_N < \alpha m_1 N) & \leq & \mathbb{P}(P_N - \mathbb{E}(P_N) \leq -k a_P \sqrt{N \ln N}) \\ & \leq & \mathbb{P}(|P_N - \mathbb{E}(P_N)| \geq k a_P \sqrt{N \ln N}) \\ & \leq & \dfrac{\mathbb{E}\left(P_N - \mathbb{E}(P_N)\right)^2}{k^2 a_P^2 N \ln N} \overset{N\to\infty}{\longrightarrow} \dfrac{1}{k^2}. \end{array}$$

Consequently for each $\alpha \in (0,1)$ we have

$$\forall\, k > 0,\ \lim_{N\to\infty} \mathbb{P}(P_N < \alpha m_1 N) \leq \frac{1}{k^2}.$$

This implies that for all $\alpha \in (0,1)$,

$$\lim_{N\to\infty} \mathbb{P}(P_N \geq \alpha m_1 N) = 1.$$

All random walks with a number of pattern P in their support more than $\alpha m_1 N$ are more degenerate than $e^{\alpha m_1 N \ln 2}$. Then for all choice of $\nu < m_1 \ln 2$ (because $\alpha \in (0,1)$) we have

$$\lim_{N\to\infty} \mathbb{P}\left(\omega \in \Omega_N \text{ s.t. } \deg(\mathcal{C}(\omega)) > e^{\nu N}\right) = 1.$$

$\square$

Remark 24. *Since in dimension $d \geq 4$ the variance goes like N and the expectation of each observables goes like N too, we expect that this result is true also in dimension greater than three. In order to prove that one should for example prove that m_1 is not zero also for higher dimension for a pattern similar to P.*

6 The self-avoiding and bond-self-avoiding walks on $\mathbb{Z}^d$

In this section we prove that for random walks with the constraint to occupy at most one time each bond, $\bar{\gamma} < 1$.

Definition 25.
*A **bond-self-avoiding** random walk is a random walk that passes at most one time through each bond.*
*A **self-avoiding** random walk is a random walk that occupies at most one time each lattice point.*

We consider self-avoiding walks (SAW) and bond-self-avoiding walks (BAW) on the d-dimensional lattice $\mathbb{Z}^d$.
Let us make some considerations on the BAW. Each lattice point is connected to $2d$ nearest neighbors and therefore each lattice point can never be occupied more than d times, with only one exception, the case for which the first and the last step of the random walk are at the same position and all the $2d$ bonds are occupied. But in this case the random walk cannot continue (it is trapped). Therefore the properties of BAW will be closer to the ones of SAW than the ones of simple random walks.

Definition 26 (Contact matrices for SAW).
Let $\omega = (0, \omega(1), \ldots, \omega(N)) \in \Omega_N$ be a SAW of length N. We define the application $\mathcal{C}$

$$
\begin{array}{rcl}
\mathcal{C} : \Omega_N & \longmapsto & M(N, \mathbb{R}) \\
\omega & \longrightarrow & \mathcal{C}_{i,j}(\omega)
\end{array}
$$

where $\mathcal{C}_{i,j}(\omega) = \begin{cases} 1 & \textit{if } \|\omega(i) - \omega(j)\|_\infty = 1 \textit{ and } |i-j| > 1, \\ 0 & \textit{otherwise.} \end{cases}$
The contact matrix of ω is its image by $\mathcal{C}$. The degeneracy of a contact matrix is the number of random walks corresponding to that contact matrix.

In order to prove that $\bar{\gamma} < 1$, we prove a theorem for BAW, the corresponding of Kesten Pattern Theorem for SAW (see [5] and [7]). Let χ_N be the number of BAW (or SAW) of length N and $W(N)$ the number of contact matrices of BAW (or SAW). A consequence of the theorem is that

$$
\limsup_{N \to \infty} \frac{\ln W(N)}{N} < \ln \beta \text{ where } \beta \doteqdot \lim_{N \to \infty} (\chi_N)^{1/N}.
$$

The proof of the theorem is split in some lemmas. First we introduce some notations and definitions, then we prove the lemmas which conduce to the theorem. After that we apply it in order to prove[31] that the growth factor is less than 1.

[31] In [4] this property was consistent to their numerical result for small length of the walks but there was no proof. It was proven by Kesten in a personal mail with Lebowitz.

6.1 Definitions and notations

Definition 27. *The* concatenation *of two BAW ω_1 and ω_2 of length N_1 and N_2 respectively is the walk of length $N_1 + N_2$, not necessarily BAW, defined as follows:*

$$\omega(k) = \begin{cases} \omega_1(k) & \text{for } k = 0, \dots, N_1 \\ \omega_1(N_1) + \omega_2(k - N_1) - \omega_2(0) & \text{for } k = N_1 + 1, \dots, N_1 + N_2 \end{cases}$$

Since the walk issue from the concatenation of ω_1 and ω_2 is not necessarily a BAW walk, we have

$$\chi_{N_1+N_2} \leq \chi_{N_1}\chi_{N_2}$$

and therefore

$$\{\ln \chi_N\}_{N\geq 1}$$

is a subadditive sequence. Consequently the following limit exists,

$$\lim_{N\to\infty} \frac{\ln \chi_N}{N} = \inf_{N\geq 1} \frac{\ln \chi_N}{N} \doteqdot \ln \beta. \tag{14}$$

Let $b > 1$ be fixed for all what follows.

Definition 28.

1. *Cube D: $D \doteqdot \{x \in \mathbb{Z}^d$ s.t. $c_i \leq X^{(i)} \leq c_i + b, 1 \leq i \leq d\}$ for some $c = (c_1, \dots, c_d) \in \mathbb{Z}^d$.*

2. *Cube D^1: $D^1 \doteqdot \{x \in \mathbb{Z}^d$ s.t. $c_i - 1 \leq X^{(i)} \leq c_i + b + 1, 1 \leq i \leq d\}$ for the same $c \in \mathbb{Z}^d$ of the cube D.*

3. *Cube D^2: $D^2 \doteqdot \{x \in \mathbb{Z}^d$ s.t. $c_i - 2 \leq X^{(i)} \leq c_i + b + 2, 1 \leq i \leq d\}$ for the same $c \in \mathbb{Z}^d$ of the cube D.*

4. *Let ω be a BAW and $c = (c_1, \dots, c_d) \in \mathbb{Z}^d$. Then we define the cube $D(r, c)$: $D(r, c) \doteqdot \{x \in \mathbb{Z}^d$ s.t. $c_i \leq X^{(i)} - X_r^{(i)}(\omega) \leq c_i + b, 1 \leq i \leq d\}$.*

5. *Let $\overline{D}$ be the boundary of the cube D (similarly for all other cubes).*

6. *Pattern P: $P \doteqdot \{X_i(P), 1 \leq i \leq k\}$ is a BAW with $X_0(P) = 0$.*

Definition 29. *Let ω^N be a BAW of length N.*

1. *P occurs at the r^{th} step of ω^N if $X_{r+j}(\omega^N) - X_r(\omega^N) = X_j(P)$ for all $j = 0, \dots, k$.*

2. *If there exists a cube D with $X_0(P) = 0$ and $X_k(P)$ two of its vertices and such that $X_i(P) \in D^1$ for all $i = 0, \dots, k$, then we say that (P, D) occurs at the r^{th} step ω^N if*

   (a) P occurs at the r^{th} step of ω^N,

(b) ω^N *does not occupies any other points of* D (*except for* $\omega^N(k), k = r, \ldots, r+k$).

3. E *occurs at the* r^{th} *step of* ω^N *is for some* $c \in \mathbb{Z}^d$ *with* $-b \leq c_i \leq 0$, $i = 1, \ldots, d$, all bonds *of the cube* $D(r,c)$ *are filled by* ω^N (*similarly for* E^1 *with the cube* $D^1(r,c)$).

4. E_k *occurs at the* r^{th} *step of* ω^N *if for some* $c \in \mathbb{Z}^d$, ω^N *passes* at least k *steps in* $D^2(r,c)$.

5. *Let* F *be an event like* E *or* E_k. *Then we say that* F *occurs at the* r^{th} *step of* ω^N *from its* $(r-n_1)^{th}$ *step through its* $(r+n_1)^{th}$ *step if* F *occurs at the* n_1^{th} *step of* $\widetilde{\omega}^N$, *which is the restriction of* ω^N *to the steps in* $[r-n_1, r+n_1]$ (*clearly* $r - n_1 < 0$ *or* $r + n_1 > N$ *has to be replaced by* 0 *and* N *respectively*). *In this case we say that* $F(n_1)$ *occurs at the* r^{th} *step of* ω^N.

Definition 30. *The following definitions are used in lemma 34 and in the theorem 35.*

1. $S_{N,k} \doteqdot \{\omega^N$ *s.t.* E *and* E_k *do not occur in* $\omega^N\}$.

2. k_0 *is the largest integer such that* $\liminf_{N\to\infty} \left(\frac{\mathrm{Card}(S_{N,k})}{\chi_N}\right)^{1/N} < 1$.

3. $\widetilde{E}$ *occurs if* E or E_{k_0} *occur.*

4. T_N *is the set of all* ω^N *such that* E *and* E_{k_0+1} *do not occur and* $E_{k_0}(n_1)$ *occurs* at least $a_3 N$ *times.*

5. $D^\star(r,c) \doteqdot D^2(r,c) \cup \{X_i(\omega^N), r - n_1 \leq i \leq r + n_1\}$.

6. W_N *is the set of all* ω^N *for which* (P,D) *occurs* at most $a_6 N$ *times and* $E^1(n_3)$ *occurs* at least $a_7 N$ *times.*

Definition 31.

1. $\chi_N(j,P)$ *is the number of* ω^N *such that* P *occurs* at most *at* j *steps.*

2. $\chi_N(j,(P,D))$ *is the number of* ω^N *such that* (P,D) *occurs* at most *at* j *steps.*

3. $\chi_N(j,F(n_1))$ *is the number of* ω^N *such that* $F(n_1)$ *occurs* at most *at* j *steps.*

6.2 Lemmas and theorem

Lemma 32.

1. *There exists a walk* ω *of length* $\widetilde{k} = \widetilde{k}(b,d)$ *such that* $0 = X_0(\omega)$ *and* $X_{\widetilde{k}}(\omega)$ *are vertices of* D *and* ω *fills exactly* D *in the following sense:*

   (a) *each bond of* D *are occupied by* ω, *(b)* $X_i(\omega) \in D^1$ *for all* $1 \leq i \leq \widetilde{k}$.

2. *For all* $Y, Z \in \overline{D}^2$ ($Y \neq Z$ *in dimension* 1), *there exists a* ω_1 *of length* $k_1 \leq d(b+5)^{d-1}(b+4)$ *such that* $X_0(\omega_1) = Y$, $X_{k_1}(\omega_1) = Z$, $X_i(\omega_1) \in D^2$ *for all* $0 \leq i \leq k_1$ *and such that* ω_1 *contains* ω.

Proof.

1. In dimension $d = 1$ it is evident because there is only a possibility for the walk from 0 to b. Then $\widetilde{k}(b, 1) = b$.
   Assume the first part of the lemma to be proven for dimension $d-1$. Then, let $D^{(0)} \doteqdot \{X^{(1)} = 0, 0 \leq X^{(i)} \leq b, 2 \leq i \leq d\}$ be a $(d-1)$-cube filled from 0 to $P_0 = (0, P^{(2)}, \ldots, P^{(d)})$, $P^{(i)} = 0$ or b. Connect it by 3-step walk (outside of D) to $P_1 = (1, P^{(2)}, \ldots, P^{(d)})$. Then fill the $(d-1)$-cube $D^{(1)} \doteqdot \{X^{(1)} = 1, 0 \leq X^{(i)} \leq b, 2 \leq i \leq d\}$ similarly to what done for $D^{(0)}$. We can repeat this procedure on $(b-1)$ more $(d-1)$-cubes connecting them always with a 3-step. At this point it remains to occupy the bonds in D in the direction 1. Let $Q = (Q^{(1)}, \ldots, Q^{(d)}), Q^{(i)} = 0$ or b the last visited point until now. We can fill the lasts $(d+1)^{d-1}b$ bounds of D moving along the direction 1 and each time we have finished a line, we pass to another one with a 3-walk. In this way we fill all the bonds of D with a walk ω which is contained in D^1 of length $\widetilde{k}(b, d)$. $\widetilde{k}(b, d)$ is given by: $\widetilde{k}(b, d) = (b+1)\widetilde{k}(b, d-1) + (b+1)^{d-1}(b+3) + 3(b-1)$ and $\widetilde{k}(b, 1) = b$.

2. Let us take $Y^{(1)} = -2$ w.l.o.g.. From Y to $(-2, 0, \ldots, 0)$ there exists a path on $\overline{D}^2$ of length $k_2 - 2$. Then with a 2-walk we connect it to the origin $(0, \ldots, 0)$. After that we fill D as shown in the first part of the lemma (without occupying the bond $(-1, 0, \ldots, 0)$ to $(0, \ldots, 0)$). Then from the last point of ω (of length $\widetilde{k}(b, d)$) we can connect it to $\widetilde{D}^2$ with a 2-walk and then to Z with a $(k_3 - 2)$-walk on $\overline{D}^2$ that does not intersect the first one (it is always possible to do that). In this way we have constructed the desired ω_1 of length $k_1 = k_2 + k_3 + \widetilde{k} \leq d(b+5)^{d-1}(b+4)$.

□

Lemma 33. *If*

$$\liminf_{N\to\infty} \left(\frac{\chi_N(0, F)}{\chi_N} \right)^{1/N} < 1$$

then there exists a $a_1 > 0$ *and a* $n_1 \in \mathbb{N}^\star$ *such that*

$$\limsup_{N\to\infty} \left(\frac{\chi_N(a_1 N, F(n_1))}{\chi_N} \right)^{1/N} < 1.$$

PROOF.
First, remark that since no path of length greater than N can occurs before the N^{th} step, we have

$$\chi_N(0, F(N)) = \chi_N(0, F).$$

By hypothesis $\liminf_{N\to\infty} \chi_N(0,F)^{1/N} < \beta$, therefore there exists a $\varepsilon > 0$ and a $n_1 \in \mathbb{N}^\star$ such that

$$\chi_{n_1}(0,F) = \chi_{n_1}(0, F(n_1)) \leq \beta^{n_1}(1-\varepsilon)^{n_1}.$$

But since $\chi_N^{1/N} \searrow \beta$ when $N \to \infty$, then for n_1 large enough we have $(\chi_{n_1})^{1/n_1} \leq \beta(1+\epsilon)$ and therefore

$$\chi_{n_1} \leq \beta^{n_1}(1+\varepsilon)^{n_1}.$$

Let us consider a BAW ω^{sn_1}. It consists on a concatenation of s n_1-walks. For $F(n_1)$ that occurs *at most* a_2N times is ω^{sn_1}, $F(n_1)$ occurs in *at most* a_2N of its pieces and does not occurs in the others ones (*at least* $s(1-a_2)$). There exists at most $\chi_{n_1}(0,F(n_1))$ choices for a ω^{n_1} where $F(n_1)$ does not occurs, and at most χ_{n_1} choices for the others ones. Moreover, if $F(n_1)$ occurs in j (out of s) different pieces of length n_1, there are $\binom{s}{j}$ ways to choose the j pieces. Consequently we obtain:

$$\begin{aligned}
\chi_{sn_1}(a_2 s, F(n_1)) &\leq \sum_{j=0}^{a_2 s} \binom{s}{j} (\chi_{n_1})^{jn_1} (\chi_{n_1}(0, F(n_1)))^{(s-j)n_1} \\
&\leq \beta^{sn_1} \sum_{j=0}^{a_2 s} \binom{s}{j} (1+\varepsilon)^{jn_1}(1-\varepsilon)^{(s-j)n_1} \\
&\doteqdot \beta^{sn_1}\Lambda(a_2, s)
\end{aligned}$$

Let us study $\lim_{s\to\infty} (\Lambda(a_2,s))^{1/sn_1}$. For small $a_2 > 0$, the term with $j = a_2 s$ is the bigger, therefore

$$\begin{aligned}
\Lambda(a_2, s) &= \sum_{j=0}^{a_2 s} \binom{s}{j} \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{jn_1} (1-\varepsilon)^{sn_1} \\
&\leq (a_2 s + 1)\frac{s!}{(a_2 s)!((1-a_2)s)!} \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{a_2 sn_1} (1-\varepsilon)^{sn_1} \\
&= Power(s) \left(a_2^{a_2}(1-a_2)^{1-a_2}\right)^s \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{a_2 sn_1} (1-\varepsilon)^{sn_1}.
\end{aligned}$$

From this follows

$$\lim_{s\to\infty} (\Lambda(a_2,s))^{1/sn_1} \leq (1-\varepsilon)\Phi(a_2)$$

where $\Phi(a_2) = e^{\frac{1}{n_1}(a_2 \ln a_2 + (1-a_2)\ln(a-a_2))} \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^{a_2}$. Since $\lim_{a_2\to 0}\Phi(a_2) = 1$ and $(1-\varepsilon) < 1$ we obtain that for $a_2 > 0$ but small enough,

$$\lim_{s\to\infty} (\Lambda(a_2,s))^{1/sn_1} < 1.$$

And from this follows immediately

$$\limsup_{s\to\infty} \left(\chi_{sn_1}(a_2 s, F(n_1))\right)^{1/sn_1} < \beta.$$

For $sn_1 \le N \le (s+1)n_1$, $\chi_N(a_2 s, F(n_1)) \le (2d)^{n_1}\chi_{sn_1}(a_2 s, F(n_1))$. Let us take $a_1 = \frac{a_2}{2n_1}$. Then

$$\begin{aligned}\limsup_{N\to\infty} \left(\chi_N(a_1 N, F(n_1))\right)^{1/N} &\le \limsup_{N\to\infty} \left(\chi_N(a_2[N/n_1], F(n_1))\right)^{1/N} \\ &\le \limsup_{s\to\infty} (2d)^{n_1/n_1 s} \left(\chi_{sn_1}(a_2 s, F(n_1))\right)^{1/n_1 s} < \beta.\end{aligned}$$

Finally we have the desired result:

$$\limsup_{N\to\infty} \left(\frac{\chi_N(a_1 N, F(n_1))}{\chi_N}\right)^{1/N} < 1$$

for $a_1 > 0$ small enough. □

Lemma 34.

$$\liminf_{N\to\infty} \left(\frac{\chi_N(0,E)}{\chi_N}\right)^{1/N} < 1.$$

PROOF.
We prove it by contradiction. Assume for instance that the proposition is false, i.e. $\liminf_{N\to\infty}\left(\frac{\chi_N(0,E)}{\chi_N}\right)^{1/N} = 1$. First we prove that k_0 exists. Since $X_r(\omega) \in D(r,c)$ then $X_{r-2}(\omega)$, $X_{r-1}(\omega)$, $X_{r+1}(\omega)$ and $X_{r+2}(\omega)$ are in $D^2(r,c)$ if they exists. Therefore at least 3 points are in $D^2(r,c)$ (the random walk considered is longer than 3 steps). Therefore E_3 occurs always and $\text{Card}(S_{N,3}) = 0$. Consequently $k_0 \ge 3$. Let $\hat{k} \doteqdot d(b+5)^d$. On the other hand, if E does not occurs, then $E_{\hat{k}}$ neither and then the initial assumption would give $\liminf_{N\to\infty}\left(\frac{\text{Card}(S_{N,\hat{k}})}{\chi_N}\right)^{1/N} = 1$. Therefore $k_0 \le \hat{k} - 1$. k_0 cannot be greater because a BAW cannot be into $D^2(r,c)$ more than $\hat{k}$. Therefore there exists a k_0 such that

$$\liminf_{N\to\infty}\left(\frac{\text{Card}(S_{N,k_0})}{\chi_N}\right)^{1/N} < 1 \text{ and } \liminf_{N\to\infty}\left(\frac{\text{Card}(S_{N,k})}{\chi_N}\right)^{1/N} = 1, k > k_0.$$

Since $\widetilde{E}$ occurs if E or E_{k_0} occurs, we have $\chi_N(0,\widetilde{E}) = \text{Card}(S_{N,k_0})$ and therefore $\liminf_{N\to\infty}\left(\frac{\chi_N(0,\widetilde{E})}{\chi_N}\right)^{1/N} < 1$. Using lemma 33, there exists a $a_3 > 0$ and a $n_1 \in \mathbb{N}^\star$ such that $\limsup_{N\to\infty}\left(\frac{\chi_N(a_3 N, \widetilde{E}(n_1))}{\chi_N}\right)^{1/N} < 1$. Let us consider the set T_N. T_N is equal to the difference of S_{N,k_0+1} and the set $\{\omega^N$ s.t. E and E_{k_0+1} do not occur and $E_{k_0}(n_1)$ occurs at most $a_3 N$ times$\}$,

which is included in the set $\{\omega^N$ s.t. $E_{k_0}(n_1)$ or E occur at most a_3N times$\}$.
Therefore

$$\mathrm{Card}(T_N) \geq \mathrm{Card}(S_{N,k_0+1}) - \chi_N(a_3N, \widetilde{E}(n_1)).$$

The previous considerations imply that

$$\lim_{N\to\infty} \left(\frac{\mathrm{Card}(T_N)}{\chi_N}\right)^{1/N} = 1.$$

We will prove that this last result conducts to a contradiction and therefore that the initial assumption is false and consequently the lemma is true.
Let us take a $\omega^N \in T_N$ and let us now consider the $D^\star(r,c)$. Each $D^\star(r,c)$ is contained in a cube, of edgelength $2q$ with $q = b+2+n_1$, noted $\widehat{D}(r)$ that is centered in $X_r(\omega^N)$. $\widehat{D}(r)$ intersect a similar cube $\widehat{D}(s)$ for at most $d(4b+4n_1+0)^d$ values of s (because we need $\|X_r(\omega^N) - X_s(\omega^N)\|_\infty \leq 2q+1$). Moreover $X_0(\omega^N)$ or $X_N(\omega^N)$ belong to a $D^\star(r,c)$ for at most $2(2q+1)^d d$ values of r. Then for N large enough, there are *at least* $\frac{a_3N}{d(8b+8n_1)^d} \doteqdot a_4N$ values of r such that the $D^\star(r,c)$ are pairwise disjoints, because $\frac{a_3N}{d(8b+8n_1)^d} \leq \frac{a_3N}{d(4b+4n_1+9)^d} - 2d(2b+2n_1+5)^d$.
Assume to have chosen a_4N of such values of r and consider one of them r_0 with the cube $D_0 \doteqdot D(r_0,c)$. Since $\omega^N \in T_N$, ω^N occupoes k_0 times a place in D_0^2 from its $(r_0-n_1)^{th}$ to its $(r_0+n_1)^{th}$ steps, but does not occupy all bonds of D_0 and does not occupy a site in D_0^2 before its $(r_0-n_1)^{th}$ step or after its $(r_0+n_1)^{th}$ step.
Let $X_{r'}(\omega^N)$ be the first point of ω^N in $\overline{D}_0^2$ and $X_{r''}(\omega^N)$ the last one. Then by lemma 32 there exists a BAW ω_1 of length $k_1 \leq n_2 \doteqdot b(b+5)^{d-1}(b+4)$ that goes from $X_{r'}(\omega^N)$ to $X_{r''}(\omega^N)$ with $X_i(\omega_1) \in D_0^2$ for $0 \leq i \leq k_1$ and such that ω_1 occupies all bonds of D_0.
Now we make some changes in ω^N. Let us replace the piece of ω^N from $X_{r'}(\omega^N)$ to $X_{r''}(\omega^N)$ with ω_1. With this operation we obtain a BAW of length no more than $N+n_2$. If we make a similar replacement for a_5N values of r out of the a_4N choiced, we obtain *at least* $\binom{a_4N}{a_5N}$ $\mathrm{Card}(T_N)$ BAW of length at most $N(1+a_5n_2)$. The next step is to find an upper bound on the number of times that a BAW $\bar{\omega}$ constructed by the above procedure can be obtained.
In $\bar{\omega}$ E occurs at least a_5N times. In fact it can occurs more than a_5N times because the changes in a cube can create E in another cube that intersect the first one. But the number of s such that $D^2(s,c_s)$ can intersect a $D^2(r,c)$ in which we have changed something are at most $\widetilde{M} = b(4b+9)^d a_5N$. Given a $\bar{\omega}$ and a s such that E occurs at the s^{th} step of $\bar{\omega}$, we have no more than $(b+1)^d(b+5)^{2d}$ choices for the corresponding $\bar{c}$, the first and the last intersection of $\bar{\omega}$ with $D^2(s,\bar{c})$ noted $\bar{X}'$ and $\bar{X}''$ respectively. For the replaced piece from $\bar{X}'$ to $\bar{X}''$ we have at most $H \doteqdot \sum_{k=0}^{n_1} \chi_k$ possibilities (because we know only that the length is at most $2n_1$). Therefore each $\bar{\omega}$ can be obtained at most in τ^{a_5N} different ways, where $\tau \doteqdot \left((b+1)^d(b+5)^{2d}H\right)^{d(4b+9)^d}$. Consequently, since the obtained $\bar{\omega}$ are of length less or equal to $N(1+a_5n_2)$, we obtain the following

relation

$$\binom{a_4N}{a_5N}\mathrm{Card}(T_N)\tau^{-a_5N} \leq \sum_{k=0}^{N(1+a_5n_2)} \chi_k.$$

Taking the N^{th} root and the limit $N \to \infty$ we have:

$$\lim_{N\to\infty}\binom{a_4N}{a_5N}^{1/N}\tau^{-a_5} \leq \beta^{a_5n_2}.$$

$\binom{a_4N}{a_5N}^{1/N} = (Power(N))^{1/N}\frac{a_4^{a_4}}{a_5^{a_5}(a_4-a_5)^{a_4-a_5}}$ and for $a_5 > 0$ small enough, $\lim_{N\to\infty}\binom{a_4N}{a_5N}^{1/N} = e^{a_5(1+\ln a_4 - ln a_5)+\mathcal{O}(a_5^2)}$. Let $\delta \doteqdot 1+\ln a_4 - \ln\tau - n_2\ln\beta$. This implies

$$1 \geq \frac{\lim_{N\to\infty}\binom{a_4N}{a_5N}^{1/N}}{\tau^{a_5}\beta^{a_5n_2}} = e^{a_5(\delta-\ln a_5)+\mathcal{O}(a_5^2)}. \tag{15}$$

But $lim_{a_5\to 0}e^{a_5(\delta-\ln a_5)+\mathcal{O}(a_5^2)} = 1$ and $\lim_{a_5\to 0}\frac{d}{da_5}e^{a_5(\delta-\ln a_5)+\mathcal{O}(a_5^2)} = +\infty$, therefore for $a_5 > 0$ but small enough, this contradicts equation (15). □

Theorem 35.
Let P be a BAW (or SAW) such that there exists a cube D which has $X_0(P) = 0$ and $X_k(P)$ two of its vertices and contains P. Then

$$\limsup_{N\to\infty}\left(\frac{\chi_N(a_6N,(P,D))}{\chi_N}\right)^{1/N} < 1 \textit{ for some } a_6 > 0.$$

Proof.
The strategy of the proof is similar to the one used for proving lemma 34. We assume the theorem to be false and we obtain a contradiction. Assume therefore the theorem to be false. Then

$$\forall\, a_6 > 0, \limsup_{N\to\infty}\left(\frac{\chi_N(a_6N,(P,D))}{\chi_N}\right)^{1/N} < 1.$$

Let us consider the set W_N. It is equal to the difference the sets

$$\{\omega^N \text{ such that } (P,D) \text{ occurs at most } a_6N \text{ times}\}$$

and

$$\{\omega^N \text{ s.t.} \quad \begin{array}{l}(P,D) \text{ occurs at most } a_6N \text{ times and} \\ E^1(n_3) \text{ occurs less than } a_7N \text{ times}\}.\end{array}$$

The last one is included in $\{\omega^N$ s.t. $E^1(n_3)$ occurs less than a_7N times$\}$. Therefore

$$\mathrm{Card}(W_N) \geq \chi_N(a_6N,(P,D)) - \chi_N(a_7N,E^1(n_3)).$$

Using lemma 34 (with $b+1$ instead of b), there exists a $a_7 > 0$ and a $n_3 \in \mathbb{N}^\star$ such that

$$\limsup_{N\to\infty}\left(\frac{\chi_N(a_7 N, E^1(n_3))}{\chi_N}\right)^{1/N} < 1,$$

and with the initial assumption we obtain

$$\limsup_{N\to\infty}\left(\frac{\mathrm{Card}(W_N)}{\chi_N}\right)^{1/N} = 1.$$

For each $\omega^N \in W_N$, there exists at least $a_8 N \doteqdot \frac{a_7 N}{d(8b+8n_1)^d}$ values of r such that ω^N fills the bonds of $D^1(r,c)$ from its $(r-n_3)^{th}$ to its $(r+n_3)^{th}$ step and such that the corresponding $D^\star(r,c)$ are pairwise disjoints and such that the end points of ω^N are not in these cubes.
Assume to have chosen $a_8 N$ such values of r and consider one of them r_0 (with the corresponding cube $D^1(r_0,c)$ denoted by D_0^1). Let $X_{r'}(\omega^N)$ be the first point of ω^N in D_0^1 and $X_{r''}(\omega^N)$ the last one. Then there exists a ω_1 from $X_{r'}(\omega^N)$ to $X_{r''}(\omega^N)$ such that $X_i(\omega_1)$ are contained in D_0^1 and such that (P, D_0) occurs. The construction of such a ω_1 is simple. Let us take[32] $X_{r'}^{(1)}(\omega^N) = -2$ w.l.o.g.. Connect this point to $(-1, 0, \ldots, 0)$ remaining on $\overline{D}_0^1$. Then connect is to $0 = X_0(P)$ in one step. After that we construct the walk P in D_0. Finally we connect it to $\overline{D}_0^1$ with one step and remaining on it we arrive to $X_{r''}(\omega^N)$. Doing this transformation in $a_9 N$ places out of the $a_8 N$, we obtain $\binom{a_8 N}{a_9 N}\mathrm{Card}(W_N)$ walks of length no more than N.
To finish the proof we need to find an upper bound on the number of ways that a walk $\bar\omega$ can be obtained by this procedure. In $\bar\omega$ there are *at most* $a_6 N + a_8 N d(4b+p)^d 2n_3$ occurrences of (P, D). In fact we make no more than $a_8 N$ changes, each change can create new occurrences of (P, D) only on the cubes that intersect $D^1(r,c)$ (no more than $d(4b+9)^d$) and each changed step can create an occurrence of (P, D) in two ways, adding a point or leaving a point in a cube D. But given a $\bar\omega$ and a s such that $E^1(n_3)$ occurs at the s^{th} step of $\bar\omega$, we have no more than $(b+1)^d(b+3)^{2d}$ choices for $\bar c$, the first and the last step of ω^N that is in $D^1(s,\bar c)$. The length of the piece of ω^N that was replaced is at most $2n_3$ steps long. Therefore if we define $H' \doteqdot \sum_{k=0}^{2n_3}\chi_k$, $\sigma \doteqdot \left((b+1)^d(b+3)^{2d}H'\right)^{2n_3 d(4b+9)^d}$ and $a_6' \doteqdot \frac{a_6}{2n_3 d(4b+9)^d}$, we obtain

$$\binom{a_8 N}{a_9 N}\mathrm{Card}(W_N)\sigma^{-(a_8+a_6')N} \le \sum_{k=0}^{N}\chi_k.$$

Taking the N^{th} root and the limit $N \to \infty$ we have

$$\lim_{N\to\infty}\binom{a_8 N}{a_9 N}^{1/N} \le \sigma^{a_6+a_9}\beta^{a_9}.$$

But this for $a_6' > 0$ and $a_9 > 0$ small enough is false, therefore we have obtained the searched contraddiction. □

[32]For this construction all points are noted up to a translation of $X_{r_0}(\omega^N) + c$.

6.3 Growth factor for SAW and BAW

Proposition 36 (Growth factor).
For the BAW and SAW the growth factor of the number of contact matrices is strictly less than 1*, i.e.*

$$\limsup_{N\to\infty} \frac{\ln W(N)}{\ln \chi_N} < 1.$$

PROOF.
Let us take $b > 2$ and consider a BAW (or SAW) of length $k + 2d$ constructed as follows. The firsts d steps of P connect the points $(0, \dots, 0)$ and $(1, \dots, 1)$. The following k steps connect the points $(1, \dots, 1)$ and $(b-1, \dots, b-1)$ with a BAS (or SAW) remaining always in $D \setminus \overline{D}$. The last d steps of P connect the points $(b-1, \dots, b-1)$ and $(b, \dots, b)$.
Let us divide the set Ω_N into a sum of two disjoint parts: $\Omega_N = \Omega_N^a \cup (\Omega_N^a)^c$ where $\Omega_N^a \doteqdot \{\omega \in \Omega_N$ s.t. (P, D) occurs at most aN times $\}$ and $(\Omega_N^a)^c$ its complementaire set. From theorem 35 follows that

$$\exists\, \zeta > 0 \text{ s.t. } \mathbb{P}(\omega \in \Omega_N^a) \le e^{-\zeta N}.$$

Let us consider a $\omega \in (\Omega_N^a)^c$. Then (P, D) occurs at least aN times. Let D' be the bigger cube enclosed by D that does not intersect $\overline{D}$. Consider an occurrence of (P, D) in the piece of P between its t^{th} and its $(t + k + 2d)^{th}$ steps. We apply an axis rotation of $\frac{2\pi}{d}$ degrees to the cube D', where the axis is its diagonal of direction $(1, \dots, 1)$. This transformation does not change the contact matrix, and we can apply it l times obtaining each time a different BAW (or SAW). For the chosen ω we can make this transformation in at least aN different places independently, therefore the corresponding contact matrix is at least l^{aN} times degenerate.
Now we have the desired upper bound for the total number of contact matrices:

$$\begin{aligned} W(N) &\le \mathbb{P}(\omega \in \Omega_N^a)\chi_N + \mathbb{P}(\omega \in (\Omega_N^a)^c)\frac{\chi_N}{l^{aN}} \\ &\le \left(e^{-\zeta N} + e^{-Na\ln l}\right)\chi_N. \end{aligned}$$

If we define $\alpha_M \doteqdot \max\{\zeta, a\ln l\} > 0$ and $\alpha_m \doteqdot \min\{\zeta, a\ln l\} > 0$, then we have

$$\begin{aligned} \limsup_{N\to\infty} \frac{\ln W(N)}{\ln \chi_N} &\le \lim_{N\to\infty} \frac{\ln\left(e^{-\alpha_m N}\left(1 + e^{-\frac{\alpha_M}{\alpha_m}N}\right)\right) + \ln \chi_N}{\ln \chi_N} \\ &= 1 - \frac{\alpha_m}{\ln \beta} < 1. \end{aligned}$$

□

7 Random walks with intersection penalty

In this section we consider random walks to which we assign a non uniform probability, for example to give a penalty to intersections.

One way to generalize the number of contact matrices is to count it giving a weight to the different walks. For unweighted walks the probability is given by

$$\mathbb{P}(\omega) = \frac{1}{\mathrm{Card}(\Omega_N)} \text{ for all } \omega \in \Omega_N,$$

and

$$A(N) \doteqdot \frac{W(N)}{\mathrm{Card}(\Omega_N)} = \sum_{\omega\in\Omega_N} \frac{\mathbb{P}(\omega)}{\deg \mathcal{C}(\omega)} = \mathbb{E}\left(\frac{1}{\deg \mathcal{C}}\right).$$

Therefore a generalized number of contact matrices can be given by the above expression but with a probability[33] $\widetilde{\mathbb{P}}(\omega)$ non necessarily uniformly distributed on Ω_N.

In what follows we consider random walks on $\mathbb{Z}^d$ and we define $\eta \doteqdot \lim_{N\to\infty} \eta(N) \doteqdot \lim_{N\to\infty} \frac{\ln A(N)}{N \ln 2d}$. Note that

$$\eta = 0 \iff \bar{\gamma} = 1.$$

Let us look at two extreme cases:

1. Simple random walk on $\mathbb{Z}^d$. In this case we proved that $\eta = 0$ because we showed that $\bar{\gamma} = 1$.

2. Self-avoiding walks on $\mathbb{Z}^d$. This is the limit case when the penalty we give to each intersection goes to infinity. In this case there is only one contact matrix according to definition given in section 1. Its degeneracy is $\deg \mathcal{C}(\omega) = \chi_N$, where χ_N is the number of SAW of length N. Therefore we have[34] $\eta = -\frac{\ln \mu}{\ln 2d} < 0$.

The obvious question that arises is, how about other cases? We could for example consider random walks with a drift, i.e. with an asymmetric probability distribution of displacements, or we can give a weight to each walk as a function of the intersections (and consequently a function of the contact matrices), as in the following examples:

1. $\widetilde{\mathbb{P}}(\omega) = \frac{e^{-\beta \,\mathrm{Tr}(\mathcal{C}^2(\omega))}}{Q(\beta)}$, Domb-Joyce model (see [3]),

2. $\widetilde{\mathbb{P}}(\omega) = \frac{e^{-\beta I_N(\omega)}}{Q(\beta)}$ where $I_N \doteqdot N + 1 - R_N$ is the number of intersections.

The question we answer in this section is the following: for $\beta > 0$ but small enough does $\lim_{N\to\infty} \eta_\beta(N) = 0$ or $\lim_{N\to\infty} \eta_\beta(N) < 0$?

In what follows we consider random walks with a probability weight given by

$$\widetilde{\mathbb{P}}(\omega) = \frac{e^{-\beta I_N(\omega)}}{Q(\beta)},$$

[33] In this section $\widetilde{\mathbb{P}}(\omega)$ is the weighted probability of ω, $\widetilde{\mathbb{E}}(\ldots)$ the expectation under $\widetilde{\mathbb{P}}$ and $\mathbb{P}(\omega)$ is used for the unweighted probability of ω.

[34] The asymptotic number of SAW is given by $\chi_N = AN^{\gamma-1}\mu^N$.

where $Q(\beta)$ is the normalization.
First note that

$$\sum_{\omega\in\Omega_N}\frac{\widetilde{\mathbb{P}}(\omega)}{\deg\mathcal{C}(\omega)}=\sum_{I=0}^{N}\sum_{\substack{\omega\in\Omega_N\\ I_N(\omega)=I}}\frac{\widetilde{\mathbb{P}}(\omega)}{\deg\mathcal{C}(\omega)}=\sum_{I=0}^{N}\sum_{\substack{\mathcal{C}(\omega)\text{ s.t.}\\ I_N(\omega)=I}}\widetilde{\mathbb{P}}(\omega).$$

For the mean degeneracy we have

$$\begin{aligned}\langle\deg\mathcal{C}(\omega)\rangle\Big|_{I_N(\omega)=I} &\doteqdot \sum_{\substack{\mathcal{C}(\omega)\text{ s.t.}\\ I_N(\omega)=I}}\frac{\deg\mathcal{C}(\omega)}{\mathrm{Card}(\{\mathcal{C}(\omega)\text{ s.t. } I_N(\omega)=I\})}\\ &= \frac{\mathrm{Card}(\{\omega\text{ s.t. } I_N(\omega)=I\})}{\mathrm{Card}(\{\mathcal{C}(\omega)\text{ s.t. } I_N(\omega)=I\})}.\end{aligned}$$

Therefore

$$A(N)=\sum_{I=0}^{N}\frac{e^{-\beta I}}{Q_\beta(N)}\,\mathrm{Card}(\{\mathcal{C}(\omega)\text{ s.t. } I_N(\omega)=I\})$$

where

$$Q_\beta(N)=\sum_{I=0}^{N}e^{-\beta I}\mathbb{P}(I_N=I)\,\mathrm{Card}(\Omega_N).$$

Consequently we obtain

$$\begin{aligned}A(N) &= \frac{\sum_{I=0}^{N}e^{-\beta I}\mathbb{P}(I_N=I)\left(\langle\deg\mathcal{C}(\omega)\rangle\Big|_{I_N(\omega)=I}\right)^{-1}}{\sum_{I=0}^{N}e^{-\beta I}\mathbb{P}(I_N=I)}\\ &\overset{N>>1}{=} \frac{\int_0^1 e^{-\beta Nk}\mathbb{P}\left(\frac{R_N}{N}=1-k\right)\left(\langle\deg\mathcal{C}(\omega)\rangle\Big|_{\frac{I_N(\omega)}{N}=k}\right)^{-1}\mathrm{d}k}{\int_0^1 e^{-\beta Nk}\mathbb{P}\left(\frac{R_N}{N}=1-k\right)\,\mathrm{d}k}\\ &\doteqdot \frac{P_\beta(N)}{Q_\beta(N)}.\end{aligned}$$

Hypothesis 37. *We suppose that*

$$d(k)\doteqdot\lim_{N\to\infty}\frac{1}{N}\ln\langle\deg\mathcal{C}(\omega)\rangle\Big|_{\frac{I_N(\omega)}{N}=k}$$

exists and $\exists\,\gamma>0$ *s.t.* $d(1-k)>0\quad\forall\,k\in[\pi-\gamma,\pi+\gamma]$.

Note that for all $k\in[0,1]$ we have $0\le d(k)\le\ln 2d$.
In particular $d(0)=\ln\mu^{SAW}>0$ and $d(1)=0$. Let us consider a simple random walk on $\mathbb{Z}^d$, $d\ge 3$. Then we have the following properties:

1. If $\pi \doteqdot \lim_{N\to\infty} \mathbb{E}\left(\frac{R_N}{N}\right)$ then $\lim_{N\to\infty} \mathbb{P}\left(\left|\frac{R_N}{N} - \pi\right| > \alpha\right) = 0, \forall\, \alpha > 0$. For simple random walks in dimensions $d \geq 3$, $\pi > 0$.

2. $\lim_{N\to\infty} -\frac{1}{N} \ln \mathbb{P}\left(\frac{R_N}{N} \geq x\right) = \zeta(x)$ with $\zeta(x) > 0$ if $x > \pi$ (see [5]),

3. $\lim_{N\to\infty} -\frac{1}{N} \ln \mathbb{P}\left(\frac{R_N}{N} \leq x\right) = 0$ for all x[35].

We want an upper bound for η_β, therefore we need an upper bound for $A_\beta(N)$. This implies that we need to compute an upper bound for $P_\beta(N)$ and a lower bound for $Q_\beta(N)$.
In what follows all corrections to leadings terms in $P_\beta(N)$ and $Q_\beta(N)$ will be immediately eliminated (because at the end we have to take the logarithm of these quantities, divide by N and take the limit $N \to \infty$).

Lemma 38 (Upper bound for $P_\beta(N)$).
For $\beta > 0$ but small enough,

$$P_\beta(N) \leq e^{-\beta(1-\pi+\gamma)N}.$$

Proof.
For computing this bound we split the integrals on $[0, 1]$ into the integral on $[0, \pi - \gamma)$, $[\pi - \gamma, \pi + \gamma]$ and $(\pi + \gamma, 1]$ where $\gamma > 0$ is the one of hypothesis 37.
Let $\tilde{d} \doteqdot \min_{[\pi-\gamma\leq k\leq\pi+\gamma]} d(k)$.

$$\begin{aligned}
P_\beta(N) &= \int_0^1 e^{-\beta N k} e^{-N d(k)} \mathbb{P}\left(\frac{R_N}{N} = 1 - k\right) \mathrm{d}k \\
&= e^{-\beta N} \int_0^1 e^{\beta N k} e^{-N d(1-k)} \mathbb{P}\left(\frac{R_N}{N} = k\right) \mathrm{d}k \\
&= e^{-\beta N} \left\{ \int_0^{\pi-\gamma} (\cdots)\ \mathrm{d}k + \int_{\pi-\gamma}^{\pi+\gamma} (\cdots)\ \mathrm{d}k + \int_{\pi+\gamma}^1 (\cdots)\ \mathrm{d}k \right\} \\
&\leq e^{-\beta N} \left\{ e^{\beta N(\pi-\gamma)} \mathbb{P}\left(\frac{R_N}{N} < \pi - \gamma\right) \right. \\
&+ e^{\beta N(\pi+\gamma)} e^{-N\tilde{d}} \mathbb{P}\left(\frac{R_N}{N} \in [\pi - \gamma, \pi + \gamma]\right) \\
&+ \left. e^{\beta N} \mathbb{P}\left(\frac{R_N}{N} > \pi + \gamma\right) \right\} \\
&= e^{-\beta N} \left\{ e^{\beta N(\pi-\gamma)} + e^{\beta N(\pi+\gamma)} e^{-N\tilde{d}} \right. \\
&+ \left. e^{\beta N} e^{-\zeta(\pi+\gamma)N} \right\}
\end{aligned}$$

Let us consider β small, more precisely $\beta \leq \beta_c^P \doteqdot \min\left\{\frac{\tilde{d}}{2\gamma}, \frac{\zeta(\pi+\gamma)}{1-\pi+\gamma}\right\}$. By hypothesis 37 we have $\tilde{d} > 0$. Then $\max\{\beta(\pi-\gamma), \beta(\pi+\gamma) - \tilde{d}, \beta - \zeta(\pi+\gamma)\} = \beta(\pi - \gamma)$. Therefore

[35] It seems that $\lim_{N\to\infty} -\frac{1}{N^{(d-2)/d}} \ln \mathbb{P}\left(\frac{R_N}{N} \leq x\right) = \xi(x)$, see [2].

for $\beta > 0$ but small enough,

$$P_\beta(N) \leq e^{-\beta(1-\pi+\gamma)N}.$$

$\square$

Lemma 39 (Lower bound for $Q_\beta(N)$).
For $\beta > 0$ but small enough,

$$Q_\beta(N) \geq e^{-\beta(1-\pi-\alpha)N}.$$

Proof.
In order to obtain this bound we split the integral on $[0,1]$ into the integral on $[0,\pi-\alpha)$, $[\pi-\alpha,\pi+\alpha]$ and $(\pi+\alpha,1]$ (where $\alpha > 0$).

$$\begin{aligned}
Q_\beta(N) &= \int_0^1 e^{-\beta N k}\mathbb{P}\left(\frac{R_N}{N} = 1-k\right) \mathrm{d}k \\
&= e^{-\beta N}\int_0^1 e^{\beta N k}\mathbb{P}\left(\frac{R_N}{N} = k\right) \mathrm{d}k \\
&= e^{-\beta N}\left\{\int_0^{\pi-\alpha}(\cdots)\ \mathrm{d}k + \int_{\pi-\alpha}^{\pi+\alpha}(\cdots)\ \mathrm{d}k + \int_{\pi+\alpha}^1(\cdots)\ \mathrm{d}k\right\} \\
&\geq e^{-\beta N}\left\{\mathbb{P}\left(\frac{R_N}{N} < \pi-\alpha\right)\right. \\
&+ e^{\beta N(\pi-\alpha)}\mathbb{P}\left(\frac{R_N}{N} \in [\pi-\alpha,\pi+\alpha]\right) \\
&+ \left. e^{\beta N(\pi+\alpha)}\mathbb{P}\left(\frac{R_N}{N} > \pi+\alpha\right)\right\} \\
&= e^{-\beta N}\left\{1 + e^{\beta N(\pi-\alpha)} + e^{\beta N(\pi+\alpha)}e^{-\zeta(\pi+\alpha)N}\right\}
\end{aligned}$$

Let us consider β small, more precisely $\beta \leq \beta_c^Q \doteqdot \frac{\zeta(\pi+\alpha)}{2\alpha}$.
Then $\max\{0, \beta(\pi-\alpha), \beta(\pi+\alpha)-\zeta(\pi+\alpha)\} = \beta(\pi-\alpha)$. Therefore for $\beta > 0$ but small enough,

$$Q_\beta(N) \geq e^{-\beta(1-\pi+\alpha)N}.$$

$\square$

Proposition 40.
For $\beta > 0$ small enough we have
$$\eta_\beta < 0.$$

Proof.
Using the two bounds obtained in lemmas 38 and 39 we obtain

$$A_\beta(N) \leq e^{-\beta N(\gamma-\alpha)}.$$

Therefore choosing $\gamma > \alpha > 0$ we have, for $\beta > 0$ and $\beta \leq \beta_c \doteqdot \min\{\beta_c^Q, \beta_c^P\}$

$$\eta_\beta = -\frac{\beta(\gamma-\alpha)}{\ln 2d} < 0.$$

□

We proved that under hypothesis 37 for random walks with a penalty of $e^{-\beta}$ to each intersection, when $\beta > 0$ small, we have $\eta_\beta < 0$ in three or higher dimensions ($\beta = 0$ corresponds to simple random walk with uniform probability distribution). Therefore for the generalized number of matrices we have an asymptotic growth factor $\bar{\gamma} < 1$.

8 Summary

In this work we mainly study the exponential growth factor of the number of different contact matrices. We find for random walk on a strip that the number of contact matrices has the same behavior of the one-dimensional case. In two dimensions, because of the recurrence of the random walks, the growth factor is equal to one. Surprisingly also in dimension three or higher, for simple random walks on a cubic lattice, we have the same result ($\bar{\gamma} = 1$). For bond-self-avoiding walks it is no more the case. In fact $\bar{\gamma} < 1$ (the same holds for self-avoiding walks with a different definition of contact matrices). Finally we consider random walks with a penalty of $e^{-\beta}$ for each intersection. For weighted number of contact matrices we obtain, under an assumption, $\bar{\gamma} < 1$ also when $\beta > 0$ is very small.
We also have some additional results on the degeneracy of contact matrices in two and three dimensions and on the end-to-end distance in two dimensions.

9 Appendix: tables and graphics

N	$W(N)-1$	$A(N)=\frac{W(N)}{2^N}$
5	1	$6.25\cdot 10^{-2}$
6	3	$6.25\cdot 10^{-2}$
7	7	$6.25\cdot 10^{-2}$
8	15	$6.25\cdot 10^{-2}$
9	32	$6.45\cdot 10^{-2}$
10	66	$6.54\cdot 10^{-2}$
11	143	$7.03\cdot 10^{-2}$
12	291	$7.13\cdot 10^{-2}$
13	603	$7.37\cdot 10^{-2}$
14	$1'203$	$7.35\cdot 10^{-2}$
15	$2'457$	$7.50\cdot 10^{-2}$
16	$4'865$	$7.42\cdot 10^{-2}$
17	$9'906$	$7.56\cdot 10^{-2}$
18	$19'616$	$7.48\cdot 10^{-2}$
19	$39'813$	$7.59\cdot 10^{-2}$
20	$78'970$	$7.53\cdot 10^{-2}$
21	$160'092$	$7.63\cdot 10^{-2}$
22	$317'954$	$7.58\cdot 10^{-2}$
23	$643'712$	$7.67\cdot 10^{-2}$
24	$1'279'887$	$7.63\cdot 10^{-2}$
∞	∞	$8\cdot 10^{-2}$

Table 4: $W(N)$ and $A(N)$ for a random walk on a 2-strip without immediate return. The last one is the expected value of $\lim_{N\to\infty} A(N)$.

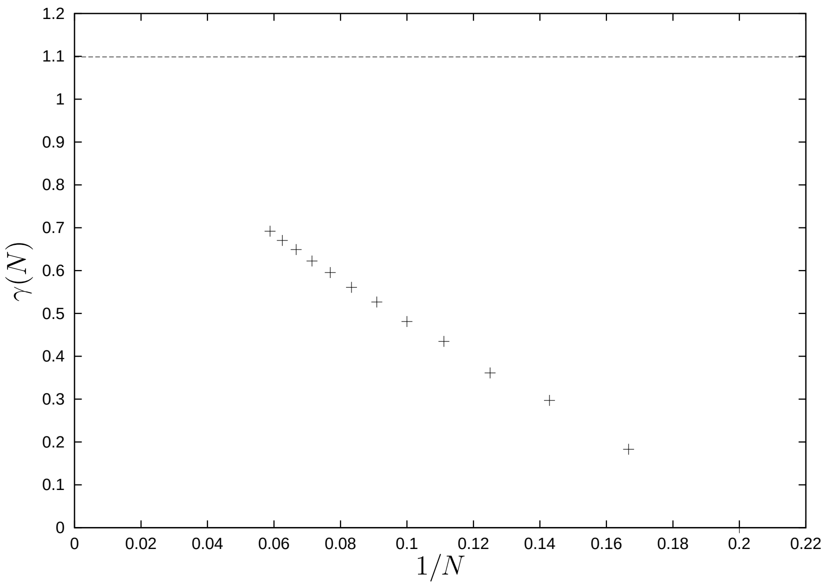


Figure 8: $\gamma(N)$ as a function of $1/N$ for random walk on $\mathbb{Z}^2$ that cannot come back immediately.

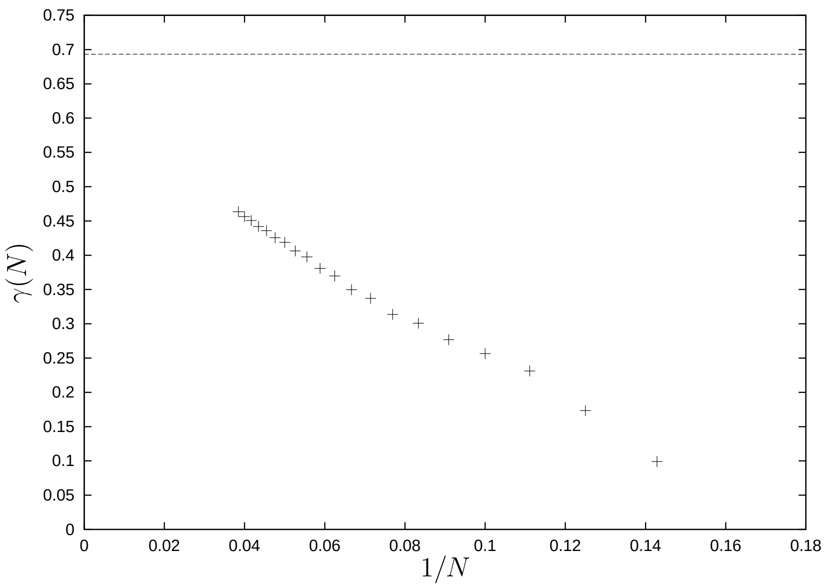


Figure 9: $\gamma(N)$ as a function of $1/N$ for random walk on honeycomb lattice that cannot return immediately.

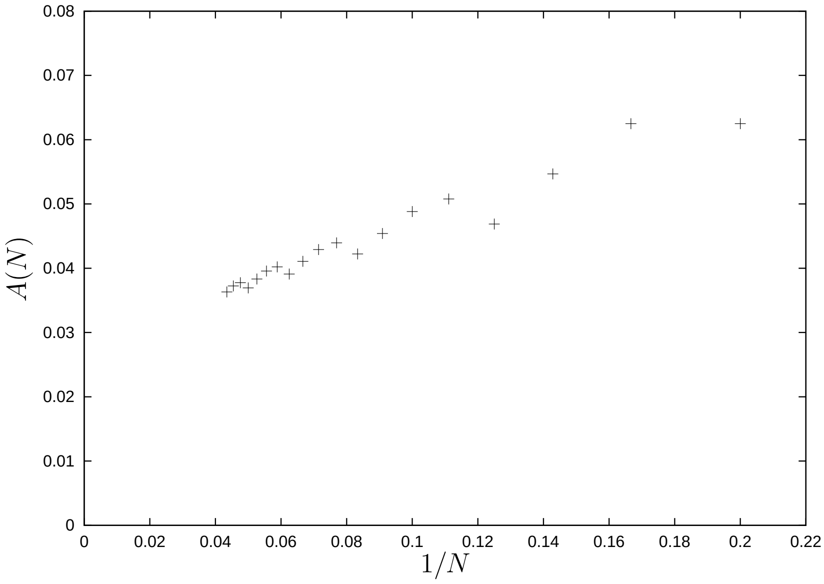


Figure 10: $A(N)$ as a function of $1/N$ for random walk on $\mathbb{Z}^2$ that turns 90 degrees at each step.

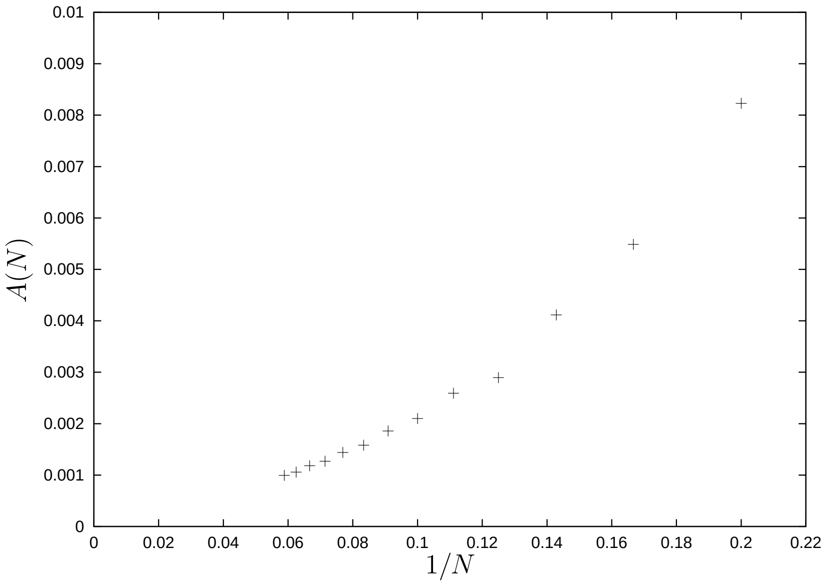


Figure 11: $A(N)$ as a function of $1/N$ for random walk on $\mathbb{Z}^2$ with immediate return avoided.

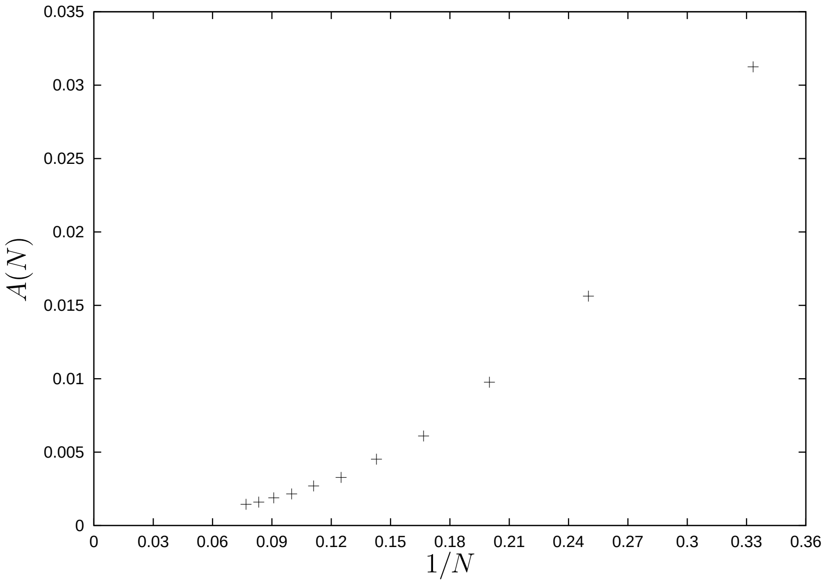


Figure 12: $A(N)$ as a function of $1/N$ for simple random walk on $\mathbb{Z}^2$.

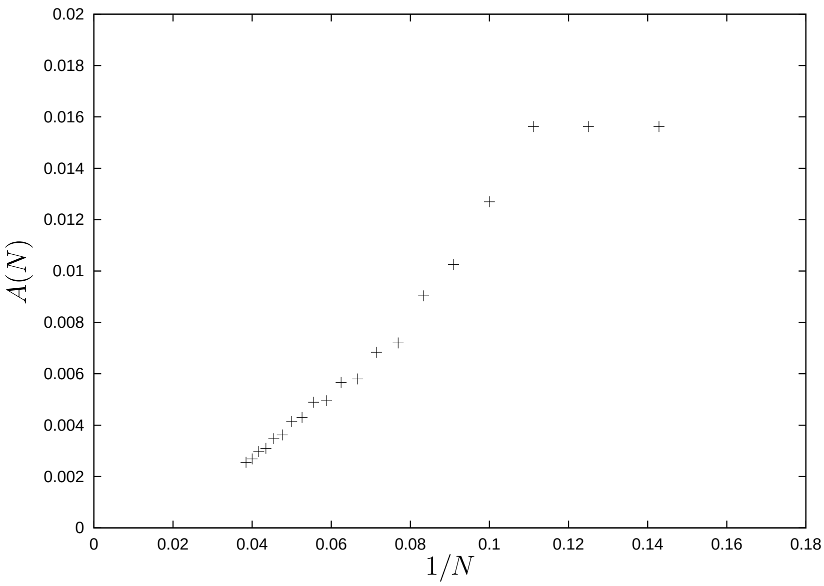


Figure 13: $A(N)$ as a function of $1/N$ for random walk on honeycomb lattice with immediate return avoided.

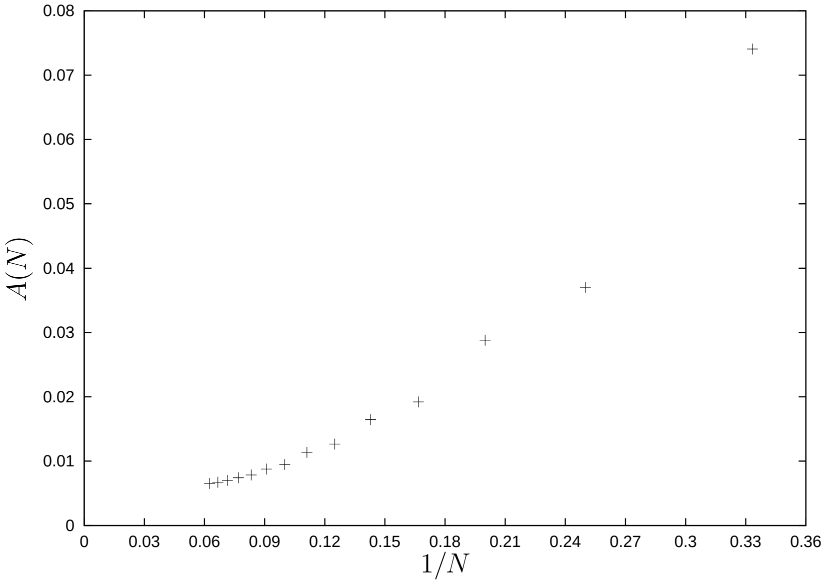


Figure 14: $A(N)$ as a function of $1/N$ for random walk on honeycomb lattice without constraints.

References

[1] T. Haliloglu, I. Bahar and B. Erman, *Gaussian Dynamics of Folded Proteins*, Phys. Rev. Lett. **79,16**:3090-3093 (1997).

[2] Y. Hamana and H. Kesten, *A large deviation result for the range of random walk and for the Wiener sausage*, preprint (2000).

[3] R. van der Hofstad and W. König, *A survey of one-dimensional random polymers*, to appear in J. Stat. Phys., preprint (2000).

[4] I. Kanter, M. Vendruscolo, B. Subramanian, E. Domany and J. Lebowitz, *Statistical Properties of Contact Maps*, Phys. Rev. E, **59**:977-984 (1999).

[5] H. Kesten, *On the Number of Self-Avoiding Walks*, J. Math. Phys. **4**:960 (1963).

[6] J.-F. Le Gall, *Propriétés d'intersection des marches aléatoires. I et II*, Comm. Math. Phys. **104,3**:471-528 (1963).

[7] N.Madras and G. Slade, *The Self-Avoiding Walk*, Probability and Its Applications, Birkhäuser Boston ed. (1996).

[8] E.W. Montroll and G.H. Weiss, *Random Walks on Lattices. II*, J. Math. Phys. **6**:167 (1965).

[9] Th. M. Nieuwenhuizen, *Trapping and Lifshitz Tails in Random Media, Self-Attracting Polymers and the Number of Distinct Sites Visited: A Renormalized Instanton Approach in Three Dimensions*, Phys. Rev. Lett. **62,4**:357:360 (1989).

[10] S. Revathi and V. Balakrishnan, *Analytic calculation of the diffusion coefficient for random walks on strips of finite width*, Phys. Rev. E **47**:916 (1993).

[11] F. Spitzer, *Principles of Random Walk*, second edition, Springer (1976).

[12] F. van Wijland, S. Caser and H.J. Hilhorst, *Topology of the support of the two-dimensional random walk*, J. Phys. A **30**:507 (1997).

[13] F. van Wijland and H.J. Hilhorst, *Universal Fluctuations in the Support of the Random Walk*, J. Stat. Phys. **89**:119 (1997).

[14] F. van Wijland, *Thèse*, Université de Paris-Sud, U.F.R. Scientifique d'Orsay, (1998).