

# CD<sub>loc</sub>(K, N) IMPLIES MCP(K, N)

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ABSTRACT. We prove that for non-branching metric measure spaces the local curvature condition  $CD_{loc}(K, N)$  implies the global version of  $MCP(K, N)$ . The curvature condition  $CD(K, N)$  introduced by the second author and also studied by Lott & Villani is the generalization to metric measure space of lower bounds on Ricci curvature together with upper bounds on the dimension. This paper is the following step of [1] where it is shown that  $CD_{loc}(K, N)$  is equivalent to a global condition  $CD^*(K, N)$ , slightly weaker than the usual  $CD(K, N)$ . It is worth pointing out that our result implies sharp Bishop-Gromov volume growth inequality and sharp Poincaré inequality.

## 1. INTRODUCTION

An important class of singular spaces is the one of metric measure spaces with generalized lower bounds on the Ricci curvature formulated in terms of optimal transportation. This class of spaces together with the condition on lower bounds on curvature have been introduced by the second author in [7, 8] and independently by Lott and Villani in [5].

The condition called *curvature-dimension condition*  $CD(K, N)$  depends on two parameters  $K$  and  $N$ , playing the role of a curvature and dimension bound, respectively. We recall two important properties of the condition  $CD(K, N)$ :

- the curvature-dimension condition is stable under convergence of metric measure spaces with respect to the  $L^2$ -transportation distance  $\mathbb{D}$  introduced in [7];
- a complete Riemannian manifold satisfies  $CD(K, N)$  if and only if its Ricci curvature is bounded from below by  $K$  and its dimension from above by  $N$ .

Moreover a broad variety of geometric and functional analytic properties can be deduced from the curvature-dimension condition  $CD(K, N)$ : the Brunn-Mikowski inequality, the Bishop-Gromov volume comparison theorem, the Bonnet-Myers theorem, the doubling property and local Poincaré inequalities on balls. All this listed results are quantitative results (volume of intermediate points, volume growth, upper bound on the diameter and so on) depending on  $K, N$ .

A variant of  $CD(K, N)$  is the *measure-contraction property*,  $MCP(K, N)$ , introduced in [6] and [8]. In the setting of non-branching metric measure spaces it is proven that condition  $CD(K, N)$  implies  $MCP(K, N)$ . While  $CD(K, N)$  is a condition on the optimal transport between any pair of absolutely continuous (w.r.t.  $m$ ) probability measure on  $M$ ,  $MCP(K, N)$  is a condition on the optimal transport between Dirac masses and the uniform distribution  $m$  on  $M$ . Nevertheless a great part of the geometric and functional analytic properties verified by spaces satisfying the condition  $CD(K, N)$  are also verified by spaces satisfying the  $MCP(K, N)$ :

- generalized Bishop-Gromov volume growth inequality;
- doubling property;
- a bound on the Hausdorff dimension;
- generalized Bonnet-Myers theorem.

Again this results are in a quantitative form depending on  $K, N$ . For a complete list of analytic consequences of the measure contraction property see [8].

Among the relevant questions on  $CD(K, N)$  that are still open, we are interested in studying the following one: can we say that a metric measure space  $(M, d, m)$  satisfies  $CD(K, N)$  provided  $CD(K, N)$  holds true locally on a family of sets  $M_i$  covering  $M$ ?

In other words it is still not known whether  $CD(K, N)$  verifies the globalization property (or the local-to-global property).

A partial answer to this problem is contained in the work by Bacher and the second author [1]: they proved that if a metric measure space  $(M, d, m)$  verifies the local curvature-dimension condition

$\text{CD}_{loc}(K, N)$  then it verifies the global reduced curvature-dimension condition  $\text{CD}^*(K, N)$ . The latter is strictly weaker than  $\text{CD}(K, N)$  and a converse implication can be obtained only changing the value of the lower bound on the curvature: condition  $\text{CD}^*(K, N)$  implies  $\text{CD}(K^*, N)$  where  $K^* = K(N - 1)/N$ . Therefore  $\text{CD}^*(K, N)$  gives worse geometric and analytic information than  $\text{CD}(K, N)$ .

In this paper we prove that if  $(M, d, m)$  is a non-branching metric measure space that verifies  $\text{CD}_{loc}(K, N)$  then  $(M, d, m)$  verifies  $\text{MCP}(K, N)$ .

Hence our result implies that from the local condition  $\text{CD}_{loc}(K, N)$  one can obtain all the global geometric and functional analytic consequences implied by  $\text{MCP}(K, N)$  and therefore the geometric and functional analytic consequences are obtained in the sharp quantitative version.

We now present our approach to the problem.

As already pointed out, the curvature-dimension condition  $\text{CD}(K, N)$  prescribes how the volume of a given set is affected by curvature when it is moved via optimal transportation. Condition  $\text{CD}(K, N)$  impose that the distortion is ruled by the coefficient  $\tau_{K, N}^{(t)}(\theta)$  depending on the curvature  $K$ , on the dimension  $N$ , on the time of the evolution  $t$  and on the point  $\theta$ .

The main feature of the coefficient  $\tau_{K, N}^{(t)}(\theta)$  is that it is obtained mixing two different information on how the volume should evolve: an  $(N - 1)$ -dimensional distortion depending on the curvature  $K$  by and a one dimensional evolution that doesn't feel the curvature. To be more precise

$$\tau_{K, N}^{(t)}(\theta) = t^{1/N} \sigma_{K, N-1}^{(t)}(\theta)^{(N-1)/N},$$

where  $\sigma_{K, N-1}^{(t)}(\theta)^{(N-1)/N}$  contains the information on the  $(N - 1)$ -dimensional volume distortion and the evolution in the remaining direction is ruled just by  $t^{1/N}$ . This is a clear similarity with the Riemannian case.

Our aim is, starting from  $\text{CD}_{loc}(K, N)$ , to isolate a local  $(N - 1)$ -dimensional condition ruled by the coefficient  $\sigma_{K, N-1}^{(t)}(\theta)$  and then, using the easier structure of  $\sigma_{K, N-1}^{(t)}(\theta)$ , obtain a global  $(N - 1)$ -dimensional condition with coefficient  $\sigma_{K, N-1}^{(t)}(\theta)$ . At that point, using Hölder inequality and the linear behavior of the other direction, it is not difficult to pass from the  $(N - 1)$ -dimensional version to the full-dimensional version with coefficient  $\tau_{K, N}^{(t)}(\theta)$ .

However to detect a local  $(N - 1)$ -dimensional condition it is necessary to decompose the whole evolution into a family of  $(N - 1)$ -dimensional evolutions. Considering the optimal transport between a Dirac mass in  $o$  and the uniform distribution  $m$ , the family of spheres around  $x_0$  immediately provides the correct  $(N - 1)$ -dimensional evolutions. This motivates why we obtain  $\text{MCP}(K, N)$  and not  $\text{CD}(K, N)$ .

We state the main result of this paper.

**Theorem 1.1** (Theorem 6.2). *Let  $(M, d, m)$  be a non-branching metric measure space. Assume that  $(M, d, m)$  satisfies  $\text{CD}_{loc}(K, N)$ . Then  $(M, d, m)$  satisfies  $\text{MCP}(K, N)$ .*

## 2. PRELIMINARIES

Let  $(M, d)$  be a metric space. The length  $\mathbf{L}(\gamma)$  of a continuous curve  $\gamma : [0, 1] \rightarrow M$  is defined as

$$\mathbf{L}(\gamma) := \sup \sum_{k=1}^n d(\gamma(t_{k-1}), \gamma(t_k))$$

where the supremum runs over  $n \in \mathbb{N}$  and over all partitions  $0 = t_0 < t_1 < \dots < t_n = 1$ . Clearly  $\mathbf{L}(\gamma) \geq d(\gamma(0), \gamma(1))$ . The curve is called *geodesic* if and only if  $\mathbf{L}(\gamma) = d(\gamma(0), \gamma(1))$ . In this case we always assume that  $\gamma$  has constant speed, i.e.  $\mathbf{L}(\gamma|_{[s, t]}) = |s - t|\mathbf{L}(\gamma) = |s - t|d(\gamma(0), \gamma(1))$  for every  $0 \leq s \leq t \leq 1$ .

With  $\mathcal{G}(M)$  we denote the space of geodesic  $\gamma : [0, 1] \rightarrow M$  in  $M$ , regarded as subset of  $\text{Lip}([0, 1], M)$  of Lipschitz functions equipped with the topology of uniform convergence.

$(M, d)$  is said a *length space* if and only if for all  $x, y \in M$ ,

$$d(x, y) = \inf \mathbf{L}(\gamma)$$

where the infimum runs over all continuous curves connecting  $x$  to  $y$ . It is said to be a *geodesic space* if and only if every  $x, y \in M$  are connected by a geodesic.

**Definition 2.1.** A geodesic space  $(M, d)$  is *non-branching* if and only if for all  $r \geq 0$  and  $x, y \in M$  such that  $d(x, y) = r/2$  the set

$$\{z \in M : d(x, z) = r\} \cap \{z \in M : d(y, z) = r/2\}$$

is a singleton.

A *metric measure space* will always be a triple  $(M, d, m)$  where  $(M, d)$  is a complete separable metric space and  $m$  is a locally finite measure (i.e.  $m(B_r(x)) < \infty$  for all  $x \in M$  and all sufficiently small  $r > 0$ ) on  $M$  equipped with its Borel  $\sigma$ -algebra. We exclude the case  $m(M) = 0$ . A *non-branching* metric measure space will be a metric measure space  $(M, d, m)$  such that  $(M, d)$  is a non-branching geodesic space. Throughout the following we will use the notation  $B_p(z) = \{x : d(x, z) < p\}$ .

**2.1. Geometry of metric measure spaces.**  $\mathcal{P}_2(M, d)$  denotes the  $L^2$ -Wasserstein space of probability measures on  $M$  and  $d_W$  the corresponding  $L^2$ -Wasserstein distance. The subspace of  $m$ -absolutely continuous measures is denoted by  $\mathcal{P}_2(M, d, m)$ . A point  $z$  will be called  $t$ -intermediate point of points  $x$  and  $y$  if  $d(x, z) = td(x, y)$  and  $d(z, y) = (1 - t)d(x, y)$ .

The following are well-known results in optimal transportation and are valid for general metric measure spaces.

**Lemma 2.2.** *Let  $(M, d, m)$  be a metric measure space. For each geodesic  $\Gamma : [0, 1] \rightarrow \mathcal{P}_2(M)$  there exists a probability measure  $\Xi$  on  $\mathcal{G}(M)$  such that*

- $e_{t\sharp}\Xi = \Gamma(t)$  for all  $t \in [0, 1]$ ;
- for each pair  $(s, t)$  the transference plan  $(\gamma_s, \gamma_t)\sharp\Xi$  is an optimal coupling.

The curvature-dimension condition  $\text{CD}(K, N)$  is defined in terms of convexity properties of the lower semi-continuous Rényi entropy functional

$$(2.1) \quad \mathcal{S}_N(\mu|m) := - \int_M \varrho^{-1/N}(x) \mu(dx)$$

on  $\mathcal{P}_2(M, d)$  where  $\varrho$  denotes the density of the absolutely continuous part  $\mu^c$  in the Lebesgue decomposition  $\mu = \mu^c + \mu^s = \varrho m + \mu^s$ .

Given two numbers  $K, N \in \mathbb{R}$  with  $N \geq 1$ , we put for  $(t, \theta) \in [0, 1] \times \mathbb{R}_+$ ,

$$(2.2) \quad \tau_{K,N}^{(t)}(\theta) := \begin{cases} \infty, & \text{if } K\theta^2 \geq (N-1)\pi^2, \\ t^{1/N} \left( \frac{\sin(t\theta\sqrt{K/(N-1)})}{\sin(\theta\sqrt{K/(N-1)})} \right)^{1-1/N} & \text{if } K\theta^2 \leq (N-1)\pi^2, \\ t & \text{if } K\theta^2 < 0 \text{ or} \\ & \text{if } K\theta^2 = 0 \text{ and } N = 1, \\ t^{1/N} \left( \frac{\sinh(t\theta\sqrt{-K/(N-1)})}{\sinh(\theta\sqrt{-K/(N-1)})} \right)^{1-1/N} & \text{if } K\theta^2 \leq 0 \text{ and } N > 1. \end{cases}$$

That is,  $\tau_{K,N}^{(t)}(\theta) := t^{1/N} \sigma_{K,N-1}^{(t)}(\theta)^{(N-1)/N}$  where

$$\sigma_{K,N}^{(t)}(\theta) = \frac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})},$$

if  $0 < K\theta^2 < N\pi^2$  and with appropriate interpretation otherwise. Moreover we put

$$\varsigma_{K,N}^{(t)}(\theta) := \tau_{K,N}^{(t)}(\theta)^N.$$

The coefficients  $\tau_{K,N}^{(t)}(\theta)$ ,  $\sigma_{K,N}^{(t)}(\theta)$  and  $\varsigma_{K,N}^{(t)}(\theta)$  are all volume distortion coefficients depending on the curvature  $K$  and on the dimension  $N$ .

**Definition 2.3** (Curvature-Dimension condition). Let two number  $K, N \in \mathbb{R}$  with  $N \geq 1$  be given. We say that  $(M, d, m)$  satisfies the curvature-dimension condition - denoted by  $\text{CD}(K, N)$  - if and only if for

each pair  $\nu_0, \nu_1 \in \mathcal{P}_2(M, d, m)$  there exists an optimal coupling  $\pi$  of  $\nu_0 = \varrho_0 m$  and  $\nu_1 = \varrho_1 m$ , and a geodesic  $\Gamma : [0, 1] \rightarrow \mathcal{P}_2(M, d, m)$  connecting  $\nu_0$  and  $\nu_1$  with

$$(2.3) \quad \mathcal{S}_{N'}(\Gamma(t)|m) \leq - \int_{M \times M} \left[ \tau_{K, N'}^{(1-t)}(d(x_0, x_1)) \varrho_0^{-1/N'}(x_0) \right. \\ \left. + \tau_{K, N'}^{(t)}(d(x_0, x_1)) \varrho_1^{-1/N'}(x_1) \right] \pi(dx_0 dx_1),$$

for all  $t \in [0, 1]$  and all  $N' \geq N$ .

We recall also the definition of the reduced curvature-dimension condition  $\text{CD}^*(K, N)$  introduced in [1] as well as the definition of  $\text{CD}_{loc}(K, N)$ .

**Definition 2.4** (Reduced Curvature-Dimension condition). Let two number  $K, N \in \mathbb{R}$  with  $N \geq 1$  be given. We say that  $(M, d, m)$  satisfies the reduced curvature-dimension condition - denoted by  $\text{CD}^*(K, N)$  - if and only if for each pair  $\nu_0, \nu_1 \in \mathcal{P}_2(M, d, m)$  there exists an optimal coupling  $\pi$  of  $\nu_0 = \varrho_0 m$  and  $\nu_1 = \varrho_1 m$ , and a geodesic  $\Gamma : [0, 1] \rightarrow \mathcal{P}_2(M, d, m)$  connecting  $\nu_0$  and  $\nu_1$  such that (2.3) holds true for all  $t \in [0, 1]$  and all  $N' \geq N$  with the coefficients  $\tau_{K, N'}^{(t)}(d(x_0, x_1))$  and  $\tau_{K, N'}^{(1-t)}(d(x_0, x_1))$  replaced by  $\sigma_{K, N}^{(t)}(d(x_0, x_1))$  and  $\sigma_{K, N}^{(1-t)}(d(x_0, x_1))$ , respectively.

**Definition 2.5** (Local Curvature-Dimension condition). Let two number  $K, N \in \mathbb{R}$  with  $N \geq 1$  be given. We say that  $(M, d, m)$  satisfies the curvature-dimension condition locally - denoted by  $\text{CD}_{loc}(K, N)$  - if and only if each point  $x \in M$  has a neighborhood  $M(x)$  such that for each pair  $\nu_0, \nu_1 \in \mathcal{P}_2(M, d, m)$  supported in  $M(x)$  there exists an optimal coupling  $\pi$  of  $\nu_0 = \varrho_0 m$  and  $\nu_1 = \varrho_1 m$ , and a geodesic  $\Gamma : [0, 1] \rightarrow \mathcal{P}_2(M, d, m)$  connecting  $\nu_0$  and  $\nu_1$  with

$$(2.4) \quad \mathcal{S}_{N'}(\Gamma(t)|m) \leq - \int_{M \times M} \left[ \tau_{K, N'}^{(1-t)}(d(x_0, x_1)) \varrho_0^{-1/N'}(x_0) \right. \\ \left. + \tau_{K, N'}^{(t)}(d(x_0, x_1)) \varrho_1^{-1/N'}(x_1) \right] \pi(dx_0 dx_1),$$

for all  $t \in [0, 1]$  and all  $N' \geq N$ .

Notice that the geodesic  $\Gamma$  of the above definition can exit from the neighborhood  $M(x)$ .

As already emphasized in the introduction, in [1] it is proved that  $\text{CD}_{loc}(K, N)$  implies  $\text{CD}^*(K, N)$ .

If a non-branching metric measure space  $(M, d, m)$  satisfies  $\text{CD}(K, N)$  then the uniqueness of geodesics can be proven. The next result is taken from [8].

**Lemma 2.6.** *Assume that  $(M, d, m)$  is non-branching and satisfies  $\text{CD}(K, N)$  for some pair  $(K, N)$ . Then for every  $x \in \text{supp}[m]$  and  $m$ -a.e.  $y \in M$  (with the exceptional set depending on  $x$ ) there exists a unique geodesic between  $x$  and  $y$ .*

*Moreover, there exists a measurable map  $\gamma : M^2 \rightarrow \mathcal{G}(M)$  such that for  $m \otimes m$ -a.e.  $(x, y) \in M^2$  the curve  $t \mapsto \gamma_t(x, y)$  is the unique geodesic connecting  $x$  and  $y$ .*

In the setting of non-branching metric measure space  $\text{CD}(K, N)$  has an equivalent point-wise formulation:  $(M, d, m)$  satisfies  $\text{CD}(K, N)$  if and only if for each pair  $\nu_0, \nu_1 \in \mathcal{P}_2(M, d, m)$  and each optimal coupling  $\pi$  of them

$$(2.5) \quad \varrho_t(\gamma_t(x_0, x_1)) \leq \left[ \tau_{K, N'}^{(1-t)}(d(x_0, x_1)) \varrho_0^{-1/N'}(x_0) + \tau_{K, N'}^{(t)}(d(x_0, x_1)) \varrho_1^{-1/N'}(x_1) \right]^{-N},$$

for all  $t \in [0, 1]$ , and  $\pi$ -a.e.  $(x_0, x_1) \in M \times M$ . Here  $\varrho_t$  is the density of the push-forward of  $\pi$  under the map  $(x_0, x_1) \mapsto \gamma_t(x_0, x_1)$ .

We recall the definition of the measure contraction property.

A Markov kernel on  $M$  is a map  $Q : M \times \mathcal{B}(M) \rightarrow [0, 1]$  (where  $\mathcal{B}(M)$  denotes the Borel  $\sigma$ -algebra of  $M$ ) with the following properties:

- (i) for each  $x \in M$  the map  $Q(x, \cdot) : \mathcal{B}(M) \rightarrow [0, 1]$  is a probability measure on  $M$ ;
- (ii) for each  $A \in \mathcal{B}(M)$  the function  $Q(\cdot, A) : M \rightarrow [0, 1]$  is  $m$ -measurable.

**Definition 2.7** (Measure contraction property). Let two number  $K, N \in \mathbb{R}$  with  $N \geq 1$  be given. We say that  $(M, d, m)$  satisfies the *measure contraction property*  $\text{MCP}(K, N)$  if and only if for each  $0 < t < 1$  there exists a Markov kernel  $Q_t$  from  $M^2$  to  $M$  such that for  $m^2$ -a.e.  $(x, y) \in M^2$  and for  $Q_t(x, y, \cdot)$ -a.e.  $z$

the point  $z$  is a  $t$ -intermediate point of  $x$  and  $y$ , and such that for  $m$ -a.e.  $x \in M$  and for every measurable  $B \subset M$ ,

$$(2.6) \quad \begin{aligned} \int_M \varsigma_{K,N}^{(t)}(d(x,y)) Q_t(x,y;B) m(dy) &\leq m(B), \\ \int_M \varsigma_{K,N}^{(1-t)}(d(x,y)) Q_t(y,x;B) m(dy) &\leq m(B). \end{aligned}$$

**2.2. Disintegration of measures.** Given a measurable space  $(R, \mathcal{R})$  and a function  $r : R \rightarrow S$ , with  $S$  generic set, we can endow  $S$  with the *push forward  $\sigma$ -algebra*  $\mathcal{S}$  of  $\mathcal{R}$ :

$$Q \in \mathcal{S} \iff r^{-1}(Q) \in \mathcal{R},$$

which could be also defined as the biggest  $\sigma$ -algebra on  $S$  such that  $r$  is measurable. Moreover given a measure space  $(R, \mathcal{R}, \rho)$ , the *push forward measure*  $\eta$  is then defined as  $\eta := (r_# \rho)$ .

Consider a probability space  $(R, \mathcal{R}, \rho)$  and its push forward measure space  $(S, \mathcal{S}, \eta)$  induced by a map  $r$ . From the above definition the map  $r$  is clearly measurable and inverse measure preserving.

**Definition 2.8.** A *disintegration* of  $\rho$  consistent with  $r$  is a map  $\rho : \mathcal{R} \times S \rightarrow [0, 1]$  such that

- (1)  $\rho_s(\cdot)$  is a probability measure on  $(R, \mathcal{R})$  for all  $s \in S$ ,
- (2)  $\rho(\cdot)$  is  $\eta$ -measurable for all  $B \in \mathcal{R}$ ,

and satisfies for all  $B \in \mathcal{R}, C \in \mathcal{S}$  the consistency condition

$$\rho(B \cap r^{-1}(C)) = \int_C \rho_s(B) \eta(ds).$$

A disintegration is *strongly consistent with respect to  $r$*  if for all  $s$  we have  $\rho_s(r^{-1}(s)) = 1$ .

The measures  $\rho_s$  are called *conditional probabilities*.

We say that a  $\sigma$ -algebra  $\mathcal{H}$  is *essentially countably generated* with respect to a measure  $m$  if there exists a countably generated  $\sigma$ -algebra  $\hat{\mathcal{H}}$  such that for all  $A \in \mathcal{H}$  there exists  $\hat{A} \in \hat{\mathcal{H}}$  such that  $m(A \Delta \hat{A}) = 0$ .

We recall the following version of the disintegration theorem that can be found on [4], Section 452 (see [2] for a direct proof).

**Theorem 2.9** (Disintegration of measures). *Assume that  $(R, \mathcal{R}, \rho)$  is a countably generated probability space,  $R = \cup_{s \in S} R_s$  a partition of  $R$ ,  $r : R \rightarrow S$  the quotient map and  $(S, \mathcal{S}, \eta)$  the quotient measure space. Then  $\mathcal{S}$  is essentially countably generated w.r.t.  $\eta$  and there exists a unique disintegration  $s \mapsto \rho_s$  in the following sense: if  $\rho_1, \rho_2$  are two consistent disintegration then  $\rho_{1,s}(\cdot) = \rho_{2,s}(\cdot)$  for  $\eta$ -a.e.  $s$ .*

If  $\{S_n\}_{n \in \mathbb{N}}$  is a family essentially generating  $\mathcal{S}$  define the equivalence relation:

$$s \sim s' \iff \{s \in S_n \iff s' \in S_n, \forall n \in \mathbb{N}\}.$$

Denoting with  $p$  the quotient map associated to the above equivalence relation and with  $(L, \mathcal{L}, \lambda)$  the quotient measure space, the following properties hold:

- $R_l := \cup_{s \in p^{-1}(l)} R_s = (p \circ r)^{-1}(l)$  is  $\rho$ -measurable and  $R = \cup_{l \in L} R_l$ ;
- the disintegration  $\rho = \int_L \rho_l \lambda(dl)$  satisfies  $\rho_l(R_l) = 1$ , for  $\lambda$ -a.e.  $l$ . In particular there exists a strongly consistent disintegration w.r.t.  $p \circ r$ ;
- the disintegration  $\rho = \int_S \rho_s \eta(ds)$  satisfies  $\rho_s = \rho_{p(s)}$  for  $\eta$ -a.e.  $s$ .

In particular we will use the following corollary.

**Corollary 2.10.** *If  $(S, \mathcal{S}) = (X, \mathcal{B}(X))$  with  $X$  Polish space, then the disintegration is strongly consistent.*

### 3. POLAR COORDINATES

From now on we will assume  $(M, d, m)$  to be a non-branching metric measure space satisfying CD<sub>loc</sub>(K, N) for some  $K, N \in \mathbb{R}$  and  $N \geq 1$ . Since we want to prove that  $(M, d, m)$  satisfies MCP(K, N) we also fix once forever  $o \in M$ .

Decompose  $M = \cup_{r \geq 0} M_r$  with  $M_r := \partial B_r(o)$  and, accordingly to this decomposition,  $m$  can be disintegrated in the following way

$$m = \int \bar{m}_r q(dr), \quad q(A) = m(\{x : d(x, 0) \in A\}).$$

It is fairly easy to prove that the disintegration is strongly consistent. Indeed restrict  $m$  to  $B_R(o)$ , with  $R > 0$ , and consider any constant speed geodesic  $\gamma$  going from  $o$  to  $M_R$  and take  $[0, R]$  as the quotient set. It follows that the quotient space is a Polish space and then by Corollary 2.10 the disintegration is strongly consistent. Letting  $R \nearrow +\infty$ , we obtain the strong consistency for the whole measure and  $q$  will be a locally finite measure, therefore:

$$\bar{m}_r(\{x : d(x, o) = r\}) = 1, \quad \text{for } q - \text{a.e. } r \in [0, \bar{R}].$$

**Proposition 3.1.** *The quotient measure  $q \ll \mathcal{L}^1$ .*

*Proof.* Since  $(M, d, m)$  satisfies  $\text{CD}_{\text{loc}}(K, N)$ , from [1]  $(M, d, m)$  verifies  $\text{CD}^*(K, N)$ , then defining

$$v(r) := m(\bar{B}_r(o)), \quad s(r) := \limsup_{\delta \rightarrow 0} \frac{1}{\delta} m(\bar{B}_{r+\delta}(o) \setminus B_r(o)),$$

the map  $r \mapsto v(r)$  is locally Lipschitz with  $s$  as weak derivative, Theorem 2.3. of [8]. Being  $s$  the density of  $q$  w.r.t.  $\mathcal{L}^1$ , it follows that  $q \ll \mathcal{L}^1$ .  $\square$

With a slight abuse of notation  $q(dr) = q(r)\mathcal{L}^1$ . Let  $m_r := q(r)\bar{m}_r$  so we have

$$m = \int m_r dr.$$

Let  $s_r := m_r(M) = m_r(M_r) = \frac{d^+}{dr} m(B_r(o))$  and note that reduced Bishop-Gromov inequality, see [1], implies that for all  $0 < r \leq R \leq \pi\sqrt{(N-1)/K^*}$

$$(3.1) \quad \frac{s_r}{s_R} \geq \left( \frac{\sin(r\sqrt{K^*/(N-1)})}{\sin(R\sqrt{K^*/(N-1)})} \right)^N,$$

where  $K^* = K(N-1)/N$ .

Fix  $R > 0$  with  $s_R > 0$  and let  $(p_r)_{r \in [0, R]}$  denote the geodesic in  $\mathcal{P}(M)$  connecting the probability measures  $p_0 = \delta_{x_0}$  and  $p_R = \frac{1}{s_R} m_R$ . Note that for each  $r$  the measure  $p_r$  is supported on  $M_r$ . The next lemma follows straightforwardly from (3.1).

**Lemma 3.2.** *The measure  $p_r$  is absolute continuous with respect to the surface measure  $m_r$ .*

Let  $\hat{h}_r(x) := \frac{dp_r}{dm_r}(x)$  denote the density. Clearly  $\hat{h}_r$  can be defined arbitrarily outside  $M_r$ . Therefore for  $\mathcal{L}^1$ -a.e.  $p_r = \hat{h}_r m_r$ .

**Remark 3.3.** Let us consider the set of geodesic

$$\mathcal{G}_{[0, R]}(M) := \{\gamma : [0, R] \rightarrow M, \text{ constant speed geodesic}\}.$$

Let  $\nu \in \mathcal{P}(\mathcal{G}_{[0, R]}(M))$  such that for  $\mathcal{L}^1$ -a.e.  $r \in [0, R]$ ,  $e_{r\#}\nu = p_r$ . Neglecting a set of arbitrarily small  $\nu$ -measure, we assume w.l.o.g. that

$$G := \text{supp}[\nu] \subset \mathcal{G}_{[0, R]}(M), \quad \hat{G}_r := e_r(G) \subset M_r, \quad \hat{G} := \cup_{r \in [0, R]} \hat{G}_r \subset M,$$

with  $G$  compact and the maps  $e_r : G \rightarrow \hat{G}_r$  and

$$e : (0, R) \times G \rightarrow \hat{G} \\ (r, \gamma) \mapsto e_r(\gamma) := \gamma_r$$

are both homeomorphisms. We also prefer to think of  $\hat{h}_r$  as a function defined on  $G$  rather than on  $\hat{G}$ , hence define  $h_r : G \rightarrow [0, \infty]$  by  $h_r(\gamma) := \hat{h}_r(\gamma_r)$ .

#### 4. THE $(N-1)$ -DIMENSIONAL ESTIMATE

Consider  $H \subset G$ ,  $\nu$ -measurable with  $\nu(H) > 0$  and numbers  $R_0, L_0, R_1, L_1 > 0$  with  $R_0 < R_1$  such that  $R_t + L_t < R$  for all  $t \in [0, 1]$  where  $R_t := (1-t)R_0 + tR_1$  and  $L_t := (1-t)L_0 + tL_1$ , then the following holds.

**Lemma 4.1.** *The curve*

$$(4.1) \quad t \mapsto \mu_t := \frac{1}{L_t \nu(H)} \int_0^R 1_{(R_t, R_t+L_t)} \times_H (e^{-1}(x)) p_r(dx) \mathcal{L}^1(dr) \in \mathcal{P}(M)$$

is a geodesic.

*Proof.* Observe that coupling each  $\gamma_{R_s+\lambda L_s}$  with  $\gamma_{R_t+\lambda L_t}$  for  $\lambda \in [0, 1], \gamma \in H$  we obtain a  $d^2$ -cyclically monotone coupling of  $\mu_s$  with  $\mu_t$ . The property then follows straightforwardly.  $\square$

Hence, the optimal transport is achieved by not changing the ‘‘angular’’ parts and coupling radial parts according to optimal coupling on  $\mathbb{R}$ . Observe that for each  $t \in [0, 1]$  the density  $\varrho_t(x)$  of  $\mu_t$  w.r.t.  $m$  is given by

$$(4.2) \quad \varrho_t(\gamma_r) = \begin{cases} \frac{1}{L_t \nu(H)} h_r(\gamma), & (r, \gamma) \in [R_t, R_t + L_t] \times H, \\ 0, & \text{otherwise.} \end{cases}$$

The following regularity result for densities holds true.

**Lemma 4.2.** *For  $\nu$ -a.e.  $\gamma \in G$ , the function  $r \mapsto h_r^{-1/N}(\gamma)$  is semi-concave on  $(0, R)$  and satisfies in distributional sense*

$$\partial_r^2 h_r^{-1/N}(\gamma) \leq -\frac{K}{N} h_r^{-1/N}(\gamma).$$

*Proof.* Recall that CD<sub>loc</sub>(K, N) implies CD\*(K, N). Consider the geodesic  $\mu_t$  defined in (4.1) with  $L_0 = L_1 = 1$  and apply the definition of CD\*(K, N) to get

$$(4.3) \quad h_s^{-1/N}(\gamma) \geq \frac{\sin(t-s)\sqrt{K/N}}{\sin(t-r)\sqrt{K/N}} h_r^{-1/N}(\gamma) + \frac{\sin(s-r)\sqrt{K/N}}{\sin(t-r)\sqrt{K/N}} h_t^{-1/N}(\gamma),$$

for all  $0 < r < s < t < R$  and  $\nu$ -a.e.  $\gamma \in G$ . The claim is equivalent to (4.3).  $\square$

Now fix an open set  $H \subset G$  and  $[a, b] \subset [0, R]$  such that the curvature dimension condition CD(K, N) holds true for all measures  $\mu_0, \mu_1$  supported in  $e([a, b] \times \bar{H})$ . For each  $R_0, R_1 \in (a, b)$  choose  $L_0, L_1$  such that  $R_0 + L_0, R_1 + L_1 \leq b$  and define  $(\mu_t)_{t \in [0, 1]}$  as before in (4.1). Moreover we have to consider the following map  $\Phi : \mathcal{G}_{[0, R]}(M) \times [0, 1] \rightarrow \mathcal{G}_{[0, 1]}(M)$  with  $\Phi(\gamma, s)$  being the geodesic  $t \mapsto \eta_t = \gamma_{(1-t)(R_0+sL_0)+t(R_1+sL_1)}$ . Consider

$$\tilde{\nu} := \Phi_{\#} \left( \frac{1}{\nu(H)} \nu_{\perp H} \otimes \mathcal{L}^1_{\perp [0, 1]} \right),$$

then  $\mu_t = e_{t\#} \tilde{\nu}$ .

**Theorem 4.3.** *For  $\nu$ -a.e.  $\gamma \in H$  and for sufficiently close  $R_0 < R_1$  the following holds true:*

$$(4.4) \quad h_{R_{1/2}}^{-\frac{1}{N-1}}(\gamma) \geq \sigma_{K, N-1}^{(1/2)}(R_1 - R_0) \left\{ h_{R_0}^{-\frac{1}{N-1}}(\gamma) + h_{R_1}^{-\frac{1}{N-1}}(\gamma) \right\}.$$

*Proof.* Consider the measures  $\mu_0$  and  $\mu_1$ , the corresponding measure on the space of geodesics  $\tilde{\nu}$  and recall that  $\mu_t = e_{t\#} \tilde{\nu}$ .

*Step 1.* Condition CD<sub>loc</sub>(K, N) for  $t = 1/2$  and the assumptions on  $R_0, L_0$  and  $R_1, L_1$  imply that for  $\tilde{\nu}$ -a.e.  $\eta \in \mathcal{G}_{[0, 1]}(M)$

$$\varrho_{1/2}^{-1/N}(\eta_{1/2}) \geq \tau_{K, N}^{(1/2)}(d(\eta_0, \eta_1)) \left\{ \varrho_0^{-1/N}(\eta_0) + \varrho_1^{-1/N}(\eta_1) \right\},$$

that can be formulated also in the following way: for  $\mathcal{L}^1$ -a.e.  $s \in [0, 1]$  and  $\nu$ -a.e.  $\gamma \in H$

$$\varrho_{1/2}^{-1/N}(\gamma_{R_{1/2}+sL_{1/2}}) \geq \tau_{K, N}^{(1/2)}(R_1 - R_0 + s|L_1 - L_0|) \left\{ \varrho_0^{-1/N}(\gamma_{R_0+sL_0}) + \varrho_1^{-1/N}(\gamma_{R_1+sL_1}) \right\}.$$

Then using (4.2) and the continuity of  $r \mapsto h_r(\gamma)$  (Proposition 4.2), letting  $s \searrow 0$ , it follows that

$$(4.5) \quad (L_0 + L_1)^{1/N} h_{R_{1/2}}^{-1/N}(\gamma) \geq \sigma_{K, N}^{(1/2)}(R_1 - R_0)^{\frac{N-1}{N}} \left\{ L_0^{1/N} h_{R_0}^{-1/N}(\gamma) + L_1^{1/N} h_{R_1}^{-1/N}(\gamma) \right\}$$

for all  $R_0 < R_1 \in (a, b)$ , all sufficiently small  $L_0, L_1$  and  $\nu$ -a.e.  $\gamma \in H$ , with exceptional set depending on  $R_0, R_1, L_0, L_1$ .

*Step 2.* Note that all the involved quantities in (4.5) are continuous w.r.t.  $R_0, R_1, L_0, L_1$ , therefore there exists a common exceptional set  $H' \subset H$  of zero  $\nu$ -measure such that (4.5) holds true for all  $R_0 < R_1 \in (a, b)$ , all sufficiently small  $L_0, L_1$  and all  $\gamma \in H \setminus H'$ .

For fixed  $R_0 < R_1 \in (a, b)$  and fixed  $\gamma \in H \setminus H'$ , varying  $L_0, L_1$  in (4.5) yields

$$h_{R_{1/2}}^{-\frac{1}{N-1}}(\gamma) \geq \sigma_{K, N-1}^{(1/2)}(R_1 - R_0) \left\{ h_{R_0}^{-\frac{1}{N-1}}(\gamma) + h_{R_1}^{-\frac{1}{N-1}}(\gamma) \right\}.$$

Indeed the optimal choice is

$$L_0 = L \frac{h_{R_0}^{-1/(N-1)}(\gamma)}{h_{R_0}^{-1/(N-1)}(\gamma) + h_{R_1}^{-1/(N-1)}(\gamma)}, \quad L_1 = L \frac{h_{R_1}^{-1/(N-1)}(\gamma)}{h_{R_0}^{-1/(N-1)}(\gamma) + h_{R_1}^{-1/(N-1)}(\gamma)}$$

for sufficiently small  $L > 0$ .  $\square$

## 5. THE GLOBAL ESTIMATES

From Theorem 4.3 we have that for every fixed  $\gamma \in G \setminus H'$ : for every  $0 < R_0 < R$  there exists  $\varepsilon > 0$  such that for all  $R_0 < R_1 < R_0 + \varepsilon$  it holds

$$h_{R_{1/2}}^{-\frac{1}{N-1}}(\gamma) \geq \sigma_{K, N-1}^{(1/2)}(R_1 - R_0) \left\{ h_{R_0}^{-\frac{1}{N-1}}(\gamma) + h_{R_1}^{-\frac{1}{N-1}}(\gamma) \right\}.$$

We prove that mid-points inequality is equivalent to the complete inequality.

**Lemma 5.1** (Midpoints). *Inequality (4.4) holds true if and only if*

$$(5.1) \quad h_{R_t}^{-\frac{1}{N-1}}(\gamma) \geq \sigma_{K, N-1}^{(1-t)}(R_1 - R_0) h_{R_0}^{-\frac{1}{N-1}}(\gamma) + \sigma_{K, N-1}^{(t)}(R_1 - R_0) h_{R_1}^{-\frac{1}{N-1}}(\gamma)$$

for all  $t \in [0, 1]$ .

*Proof.* We only consider the case  $K > 0$ . The general case requires analogous calculations. Fix  $0 \leq R_0 \leq R_1 \leq R$ , put  $\theta := R_1 - R_0$  and  $h(s) := h_s(\gamma) = h(\gamma(s))$ .

*Step 1.* For every  $k \in \mathbb{N}$  we have

$$\begin{aligned} h^{-\frac{1}{N-1}}(R_0 + l2^{-k}\theta) &\geq \sigma_{K, N-1}^{(1/2)}(2^{-k+1}\theta) h^{-\frac{1}{N-1}}(R_0 + (l-1)2^{-k}\theta) \\ &\quad + \sigma_{K, N-1}^{(1/2)}(2^{-k+1}\theta) h^{-\frac{1}{N-1}}(R_0 + (l+1)2^{-k}\theta), \end{aligned}$$

for every odd  $l = 0, \dots, 2^k$ .

*Step 2.* We perform an induction argument on  $k$ : suppose that inequality (5.1) is satisfied for all  $t = l2^{-k+1} \in [0, 1]$  with  $l$  odd, then (5.1) is verified by every  $t = l2^{-k} \in [0, 1]$  with  $l$  odd:

$$\begin{aligned} &h^{-\frac{1}{N-1}}(R_0 + l2^{-k}\theta) \\ &\geq \sigma_{K, N-1}^{(1/2)}(2^{-k+1}\theta) h^{-\frac{1}{N-1}}(R_0 + (l-1)2^{-k}\theta) \\ &\quad + \sigma_{K, N-1}^{(1/2)}(2^{-k+1}\theta) h^{-\frac{1}{N-1}}(R_0 + (l+1)2^{-k}\theta) \\ &\geq \sigma_{K, N-1}^{(1/2)}(2^{-k+1}\theta) \left[ h^{-\frac{1}{N-1}}(R_0) \sigma_{K, N-1}^{(1-(l-1)2^{-k})}(\theta) + h^{-\frac{1}{N-1}}(R_1) \sigma_{K, N-1}^{((l-1)2^{-k})}(\theta) \right] \\ &\quad + \sigma_{K, N-1}^{(1/2)}(2^{-k+1}\theta) \left[ h^{-\frac{1}{N-1}}(R_0) \sigma_{K, N-1}^{(1-(l+1)2^{-k})}(\theta) + h^{-\frac{1}{N-1}}(R_1) \sigma_{K, N-1}^{((l+1)2^{-k})}(\theta) \right]. \end{aligned}$$

Following the calculation of the proof of Proposition 2.10 of [1], one obtain that

$$h^{-\frac{1}{N-1}}(R_0 + l2^{-k}\theta) \geq \sigma_{K, N-1}^{(1-l2^{-k})}(\theta) h^{-\frac{1}{N-1}}(R_0) + \sigma_{K, N-1}^{(l2^{-k})}(\theta) h^{-\frac{1}{N-1}}(R_1).$$

The claim is easily proved by the continuity of  $h$  and  $\sigma$ .  $\square$

We prove that (5.1) satisfies a local-to-global property.

**Theorem 5.2** (Local to Global). *Suppose that for every  $R_0 \in [0, R]$  there exists  $\varepsilon > 0$  such that whenever  $R_0 < R_1 < R_0 + \varepsilon$  then (5.1) holds true for all  $t \in [0, 1]$ . Then (5.1) holds true for all  $0 < R_0 < R_1 \leq R$  and  $t \in [0, 1]$ .*



*Proof.* We only consider the case  $K > 0$ . The general case requires analogous calculations. Fix  $0 < R_0 < R_1 \leq R$ ,  $\theta := R_1 - R_0$  and  $h(s) := h_s(\gamma) = h(\gamma(s))$ .

*Step 1.* According to our assumption, every point  $R_0 \in [0, R]$  has a neighborhood  $(R_0 - \varepsilon(R_0), R_0 + \varepsilon(R_0))$  such that if  $R_1$  belongs to that neighborhood then (5.1) is verified. By compactness of  $[0, R]$  there exist  $x_1, \dots, x_n$  such that the family  $\{B_{\varepsilon(x_i)/2}(x_i)\}_{i=1, \dots, n}$  is a covering of  $[0, R]$ . Let  $\lambda := \min\{\varepsilon(x_i)/2 : i = 1, \dots, n\}$ . Possibly taking a lower value for  $\lambda$ , we assume that  $\lambda = 2^{-k}\theta$ . Hence we have

$$\begin{aligned} h^{-\frac{1}{N-1}}(R_0 + \frac{1}{2}\theta) &\geq \sigma_{K, N-1}^{(1/2)}(2^{-k+1}\theta)h^{-\frac{1}{N-1}}(R_0 + \frac{1}{2}\theta - 2^{-k}\theta) \\ &\quad + \sigma_{K, N-1}^{(1/2)}(2^{-k+1}\theta)h^{-\frac{1}{N-1}}(R_0 + \frac{1}{2}\theta + 2^{-k}\theta). \end{aligned}$$

*Step 2.* We iterate the above inequality:

$$\begin{aligned} h^{-\frac{1}{N-1}}(R_0 + \frac{1}{2}\theta) &\geq \sigma_{K, N-1}^{(1/2)}(2^{-k+1}\theta)h^{-\frac{1}{N-1}}(R_0 + \frac{1}{2}\theta - 2^{-k}\theta) \\ &\quad + \sigma_{K, N-1}^{(1/2)}(2^{-k+1}\theta)h^{-\frac{1}{N-1}}(R_0 + \frac{1}{2}\theta + 2^{-k}\theta) \\ &\geq \sigma_{K, N-1}^{(1/2)}(2^{-k+1}\theta) \left[ \sigma_{K, N-1}^{(1/2)}(2^{-k+1}\theta)h^{-\frac{1}{N-1}}(R_0 + \frac{1}{2}\theta - 2^{-k+1}\theta) \right. \\ &\quad \left. + \sigma_{K, N-1}^{(1/2)}(2^{-k+1}\theta)h^{-\frac{1}{N-1}}(R_0 + \frac{1}{2}\theta) \right] \\ &\quad + \sigma_{K, N-1}^{(1/2)}(2^{-k+1}\theta) \left[ \sigma_{K, N-1}^{(1/2)}(2^{-k+1}\theta)h^{-\frac{1}{N-1}}(R_0 + \frac{1}{2}\theta) \right. \\ &\quad \left. + \sigma_{K, N-1}^{(1/2)}(2^{-k+1}\theta)h^{-\frac{1}{N-1}}(R_0 + \frac{1}{2}\theta + 2^{-k+1}\theta) \right] \\ &\geq \sigma_{K, N-1}^{(1/2)}(2^{-k+1}\theta)^2 h^{-\frac{1}{N-1}}(R_0 + \frac{1}{2}\theta - 2^{-k+1}\theta) \\ &\quad + \sigma_{K, N-1}^{(1/2)}(2^{-k+1}\theta)^2 h^{-\frac{1}{N-1}}(R_0 + \frac{1}{2}\theta + 2^{-k+1}\theta). \end{aligned}$$

Observing that  $\sigma_{K, N-1}^{(1/2)}(\alpha)^2 \geq \sigma_{K, N-1}^{(1/2)}(2\alpha)$ , it is fairly easy to obtain:

$$\begin{aligned} h^{-\frac{1}{N-1}}(R_0 + \frac{1}{2}\theta) &\geq \sigma_{K, N-1}^{(1/2)}(2^{-k+i+1}\theta)h^{-\frac{1}{N-1}}(R_0 + \frac{1}{2}\theta - 2^{-k+i}\theta) \\ &\quad + \sigma_{K, N-1}^{(1/2)}(2^{-k+i+1}\theta)h^{-\frac{1}{N-1}}(R_0 + \frac{1}{2}\theta + 2^{-k+i}\theta), \end{aligned}$$

for every  $i = 0, \dots, k$ . For  $i = k - 1$  Lemma 5.1 implies the claim.  $\square$

## 6. FROM LOCAL CD(K, N) TO MCP(K, N)

So we have proved that for any  $0 < R_0 < R_1 < R$  the density  $h_r$ , of  $p_r$  w.r.t.  $m_r$ , satisfies the following inequality:

$$(6.1) \quad h_{R_t}^{-\frac{1}{N-1}}(\gamma) \geq \sigma_{K, N-1}^{(1-t)}(R_1 - R_0)h_{R_0}^{-\frac{1}{N-1}}(\gamma) + \sigma_{K, N-1}^{(t)}(R_1 - R_0)h_{R_1}^{-\frac{1}{N-1}}(\gamma).$$

for  $\nu$ -a.e.  $\gamma \in G$  and all  $t \in [0, 1]$ .

Consider  $0 < r_0 < r_1 \leq R$  and the following probability measure

$$\mu_0 := \frac{1}{r_1 - r_0} \int_{(r_0, r_1)} \frac{m_r}{s_r} dr.$$

Let  $[0, 1] \ni t \mapsto \mu_t \in \mathcal{P}_2(M, d, m)$  be the geodesic connecting  $\mu_0$  to  $\mu_1 = \delta_{x_0}$  with  $\mu_t = \varrho_t m$ . Let moreover  $\pi_t \in \Pi(\mu_0, \mu_t)$  the corresponding optimal coupling.

**Proposition 6.1.** *Fix  $t \in [0, 1]$ . Then for  $\pi_t$ -a.e.  $(z_0, z_1) \in M^2$  the following holds true*

$$\varrho_{ts}(\gamma_s(z_0, z_1))^{-1/N} \geq \varrho_0(z_0)^{-1/N} \tau_{K, N}^{(1-s)}(d(z_0, z_1)) + \varrho_t(z_1)^{-1/N} \tau_{K, N}^{(s)}(d(z_0, z_1)),$$

for every  $s \in [0, 1]$ , where  $\gamma_s(z_0, z_1)$  is the  $s$ -intermediate point on the geodesic  $\gamma$  connecting  $z_0$  to  $z_1$ .

*Proof.* We use the following notation: for a given  $R$  consider the geodesic  $(p_{R,r})_{s \in [0,R]}$  with  $p_{R,0} = \delta_{x_0}$  and  $p_{R,R} = m_R$ . The same rule will apply to densities  $h_{R,r}$ .

Let  $[0, 1] \ni s \mapsto \Gamma_{st} := \mu_{st}$  and observe that

$$(6.2) \quad \Gamma_s = \mu_{st} = \frac{1}{(1-st)(r_1-r_0)} \int_{(1-st)(r_0, r_1)} h_{r/(1-t), r} m_r dr.$$

Consider  $x_0 \in M_{\bar{r}}$  with  $r_0 \leq \bar{r} \leq r_1$ . Then the unique  $x_1$  such that  $(x_0, x_1)$  is in the support of the optimal plan  $\pi_t$ , belongs to  $M_{(1-t)\bar{r}}$ . Then from Theorem 4.3 and (6.2)

$$\begin{aligned} ((r_1-r_0)\varrho_{st}(\gamma_s(x_0, x_1)))^{-1/N} &= \left( \frac{1}{1-st} h_{\bar{r}, (1-st)\bar{r}}(\gamma) \right)^{-1/N} \\ &= \left( \frac{1}{(1-t)s + 1-s} \right)^{-\frac{1}{N}} \left( h_{\bar{r}, (1-t)s\bar{r} + (1-s)\bar{r}}(\gamma) \right)^{\frac{N-1}{N}} \\ &\geq (1-s)^{1/N} \left( \sigma_{K, N-1}^{(1-s)}(t\bar{r}) h_{\bar{r}, \bar{r}}^{-\frac{1}{N-1}}(\gamma) \right)^{\frac{N-1}{N}} \\ &\quad + ((1-t)s)^{1/N} \left( \sigma_{K, N-1}^{(s)}(t\bar{r}) h_{\bar{r}, (1-t)\bar{r}}^{-\frac{1}{N-1}}(\gamma) \right)^{\frac{N-1}{N}} \\ &= \tau_{K, N}^{(1-s)}(d(z_0, z_1)) ((r_1-r_0)\varrho_0(x_0))^{-1/N} \\ &\quad + \tau_{K, N}^{(s)}(d(z_0, z_1)) ((r_1-r_0)\varrho_t(x_1))^{-1/N} \end{aligned}$$

The claim follows.  $\square$

So far we have proven that given  $\mu_0 := m(A)^{-1} m_{\perp A}$ ,  $x_0 \in \text{supp}[m]$  and the unique geodesic  $[0, 1] \ni t \mapsto \Gamma(t)$  such that  $\Gamma(0) = \mu_0$ ,  $\Gamma(1) = \delta_{x_0}$  and  $\Gamma(t) = \varrho_t m$  for  $t \in [0, 1]$  we have for any  $t \in [0, 1]$ :

$$(6.3) \quad \begin{aligned} \mathcal{S}_{N'}(\Gamma(ts)|m) &\leq - \int_{M \times M} \left[ \tau_{K, N'}^{(1-s)}(d(x_0, x_1)) \varrho_0^{-1/N'}(x_0) \right. \\ &\quad \left. + \tau_{K, N'}^{(s)}(d(x_0, x_1)) \varrho_t^{-1/N'}(x_1) \right] \pi_t(dx_0 dx_1), \end{aligned}$$

for all  $s \in [0, 1]$  and all  $N' \geq N$ , where  $\pi_t = (P_0, P_t)_{\#} \Xi$ .

We are ready to prove the main theorem of this chapter.

**Theorem 6.2.** *Let  $(M, d, m)$  be a non-branching metric measure spaces satisfying  $\text{CD}_{loc}(K, N)$ . Then  $(M, d, m)$  satisfies  $\text{MCP}(K, N)$ .*

*Proof. Step 1.* Let  $\gamma : M^2 \rightarrow \mathcal{G}(M)$  be the map introduced in Lemma 2.6 and define for each  $t \in [0, 1]$  a Markov kernel  $Q_t$  from  $M^2$  to  $M$  by

$$Q_t(x, y; B) := 1_B(\gamma_t(x, y))$$

and for each pair  $t, x$  a measure  $m_{t,x} = \int Q_t(x, y; \cdot) m(dy)$ .

For each  $x \in M$  let  $M_x$  denote the set of all  $y \in M$  for which there exists a unique geodesic connecting  $x$  and  $y$  and let  $M_0$  be the set of  $x$  such that  $m(M \setminus M_x) = 0$ . By assumption  $m(M \setminus M_0) = 0$ .

*Step 2.* Fix  $x_0 \in M_0$  and  $B \subset M$ . Put  $A_0 := \gamma_t(x_0, \cdot)^{-1}(B)$  and  $\mu_0 := m(A_0)^{-1} m_{\perp A_0}$ . Considering  $s = 1$  in (6.3) it follows that

$$m(B)^{1/N} \geq \inf_{y \in A_0} \tau_{K, N}^{(t)}(d(y, x_0)) m(A_0)^{1/N},$$

or equivalently

$$m(B) \geq \inf_{y \in \gamma_t(x_0, \cdot)^{-1}(B)} \varsigma_{K, N}^{(t)}(d(y, x_0)) m(\gamma_t(x_0, \cdot)^{-1}(B)) = \inf_{z \in B} \varsigma_{K, N}^{(t)} \left( \frac{d(z, x_0)}{t} \right) m_{t, x_0}(B).$$

Decomposing  $B$  into a disjoint union  $\cup_i B_i$  with  $B_i = B \cap (\bar{B}_{\varepsilon_i}(x_0) \setminus \bar{B}_{\varepsilon_{i-1}}(x_0))$ , and applying the previous estimate to each of the  $B_i$  we obtain as  $\varepsilon \rightarrow 0$

$$m(B) \geq \int_B \varsigma_{K, N}^{(t)} \left( \frac{d(z, x_0)}{t} \right) m_{t, x_0}(dz)$$

or equivalently

$$m(B) \geq \int_B \varsigma_{K,N}^{(t)}(d(z, x_0)) Q_t(x_0, y; B) m(dy).$$

□

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