Localization and Tensorization Properties of the Curvature-Dimension Condition for Metric Measure Spaces

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Abstract

This paper is devoted to the analysis of metric measure spaces satisfying locally the curvature-dimension condition $\mathsf{CD}(K,N)$ introduced by the second author and also studied by Lott & Villani. We prove that the local version of $\mathsf{CD}(K,N)$ is equivalent to a global condition $\mathsf{CD}^*(K,N)$, slightly weaker than the (usual, global) curvature-dimension condition. This so-called reduced curvature-dimension condition $\mathsf{CD}^*(K,N)$ has the *local-to-global property*. We also prove the *tensorization property* for $\mathsf{CD}^*(K,N)$.

As an application we conclude that the fundamental group $\pi_1(\mathsf{M},x_0)$ of a metric measure space $(\mathsf{M},\mathsf{d},\mathsf{m})$ is finite whenever it satisfies locally the curvature-dimension condition $\mathsf{CD}(K,N)$ with positive K and finite N.

1 Introduction

In two similar but independent approaches, the second author [Stu06a, Stu06b] and Lott & Villani [LV07] presented a concept of generalized lower Ricci bounds for metric measure spaces $(\mathsf{M},\mathsf{d},\mathsf{m})$. The full strength of this concept appears if the condition $\mathsf{Ric}(\mathsf{M},\mathsf{d},\mathsf{m})\geq K$ is combined with a kind of upper bound N for the dimension. This leads to the so-called *curvature-dimension condition* $\mathsf{CD}(K,N)$ which makes sense for each pair of numbers $K\in\mathbb{R}$ and $N\in[1,\infty]$.

The condition $\mathsf{CD}(K,N)$ for a given metric measure space $(\mathsf{M},\mathsf{d},\mathsf{m})$ is formulated in terms of optimal transportation. For general (K,N) this condition is quite involved. There are two cases which lead to significant simplifications: $N=\infty$ and $K=0$.

→ The condition $\mathsf{CD}(K,\infty)$, also formulated as $\mathsf{Ric}(\mathsf{M},\mathsf{d},\mathsf{m})\geq K$, states that for each pair $\nu_0,\nu_1\in\mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$ there exists a geodesic $\nu_t=\rho_t\, m$ in $\mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$ connecting them such that the relative (Shannon) entropy

$$\mathsf{Ent}(\nu_t|\mathsf{m}):=\int_{\mathsf{M}}\rho_t\log\rho_t\,dm$$

is K-convex in $t\in[0,1]$.

Here $\mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$ denotes the space of m-absolutely continuous measures $\nu=\rho\mathsf{m}$ on M with bounded support. It is equipped with the L_2-Wasserstein distance $\mathsf{d_W}$, see below.

→ The condition $\mathsf{CD}(0,N)$ for $N\in(1,\infty)$ states that for each pair $\nu_0,\nu_1\in\mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$ there exists a geodesic $\nu_t=\rho_t\,\mathsf{m}$ in $\mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$ connecting them such that the Rényi entropy functional

$$\mathsf{S}_{N'}(\nu_t|\mathsf{m}):=-\int_{\mathsf{M}}\rho_t^{1-1/N'}d\mathsf{m}$$

is convex in $t\in[0,1]$ for each $N'\geq N$.

For general $K \in \mathbb{R}$ and $N \in (1,\infty)$ the condition $\mathsf{CD}(K,N)$ states that for each pair $\nu_0, \nu_1 \in \mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$ there exist an optimal coupling q of $\nu_0 = \rho_0 \mathsf{m}$ and $\nu_1 = \rho_1 \mathsf{m}$ and a geodesic $\nu_t = \rho_t\, m$ in $\mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$ connecting them such that

$$\mathsf{S}_{N'}(\nu_t|\mathsf{m}) \leq -\int_{\mathsf{M}\times\mathsf{M}} \left[\tau_{K,N'}^{(1-t)}(\mathsf{d}(x_0,x_1))\rho_0^{-1/N'}(x_0) + \tau_{K,N'}^{(t)}(\mathsf{d}(x_0,x_1))\rho_1^{-1/N'}(x_1)\right] d\mathsf{q}(x_0,x_1) \quad (1.1)$$

for all $t \in [0,1]$ and all $N' \geq N$. In order to define the *volume distortion coefficients* $\tau_{K,N}^{(t)}(\cdot)$ we introduce for $\theta \in \mathbb{R}_+$,

$$\mathfrak{S}_k(\theta) := \begin{cases} \frac{\sin(\sqrt{k}\theta)}{\sqrt{k}\theta} & \text{if } k > 0 \\ 1 & \text{if } k = 0 \\ \frac{\sinh(\sqrt{-k}\theta)}{\sqrt{-k}\theta} & \text{if } k < 0 \end{cases}$$

and set for $t \in [0,1]$,

$$\sigma_{K,N}^{(t)}(\theta) := \begin{cases} \infty & \text{if } K\theta^2 \geq N\pi^2 \\ t\frac{\mathfrak{S}_{K/N}(t\theta)}{\mathfrak{S}_{K/N}(\theta)} & \text{else} \end{cases}$$

as well as $\tau_{K,N}^{(t)}(\theta) := t^{1/N}\sigma_{K,N-1}^{(t)}(\theta)^{1-1/N}$.

The definitions of the condition $\mathsf{CD}(K,N)$ in [Stu06a, Stu06b] and [LV07] slightly differ. We follow the notation of [Stu06a, Stu06b], – except that all probability measures under consideration are now assumed to have bounded support (instead of merely having finite second moments). For non-branching spaces, all these concepts coincide. In this case, it indeed suffices to verify (1.1) for $N' = N$ since this already implies (1.1) for all $N' \geq N$. To simplify the presentation, we will assume for the remaining parts of the introduction that all metric measure spaces under consideration are non-branching.

Examples of metric measure spaces satisfying the condition $\mathsf{CD}(K,N)$ include

- Riemannian manifolds and weighted Riemannian spaces [OV00], [CMS01], [RS05], [Stu05]
- Finsler spaces [Oht]
- Alexandrov spaces of generalized nonnegative sectional curvature [Pet09]
- Finite or infinite dimensional Gaussian spaces [Stu06a], [LV09].

Slightly modified versions are satisfied for

- Infinite dimensional spaces, like the Wiener space [FSS], as well as for
- Discrete spaces [BS09], [Oll09].

Numerous important geometric and functional analytic estimates can be deduced from the curvature-dimension condition $\mathsf{CD}(K,N)$. Among them the Brunn-Minkowski inequality, the Bishop-Gromov volume growth estimate, the Bonnet-Myers diameter bound, and the Lichnerowicz bound on the spectral gap. Moreover, the condition $\mathsf{CD}(K,N)$ is stable under convergence. However, two questions remained open:

▷ whether the curvature-dimension condition $\mathsf{CD}(K,N)$ for general (K,N) is a *local property*, i.e. whether $\mathsf{CD}(K,N)$ for all subsets M_i, $i \in I$, of a covering of M implies $\mathsf{CD}(K,N)$ for a given space $(\mathsf{M},\mathsf{d},\mathsf{m})$;

▷ whether the curvature-dimension condition $\mathsf{CD}(K,N)$ has the *tensorization property*, i.e. whether $\mathsf{CD}(K,N_i)$ for each factor M_i with $i \in I$ implies $\mathsf{CD}(K,\sum_{i\in I} N_i)$ for the product space $\mathsf{M} = \bigotimes_{i\in I} \mathsf{M}_i$.

Both properties are known to be true – or easy to verify – in the particular cases $K = 0$ and $N = \infty$. Locality of $\mathsf{CD}(K, \infty)$ was proved in [Stu06a] and, analogously, locality of $\mathsf{CD}(0, N)$ by Villani [Vil09]. The tensorization property of $\mathsf{CD}(K, \infty)$ was proved in [Stu06a].

The goal of this paper is to study metric measure spaces satisfying the local version of the curvature-dimension condition $\mathsf{CD}(K, N)$. We prove that the *local* version of $\mathsf{CD}(K, N)$ is equivalent to a *global* condition $\mathsf{CD}^*(K, N)$, slightly weaker than the (usual, global) curvature-dimension condition. More precisely,

$$\mathsf{CD}_{\mathsf{loc}}(K-, N) \Leftrightarrow \mathsf{CD}^*_{\mathsf{loc}}(K-, N) \Leftrightarrow \mathsf{CD}^*(K, N).$$

This so-called *reduced curvature-dimension condition* $\mathsf{CD}^*(K, N)$ is obtained from $\mathsf{CD}(K, N)$ by replacing the volume distortion coefficients $\tau^{(t)}_{K,N}(\cdot)$ by the slightly smaller coefficients $\sigma^{(t)}_{K,N}(\cdot)$.

Again the reduced curvature-dimension condition turns out to be *stable* under convergence. Moreover, we prove the *tensorization property* for $\mathsf{CD}^*(K, N)$. Finally, also the reduced curvature-dimension condition allows to deduce all the geometric and functional analytic inequalities mentioned above (Bishop-Gromov, Bonnet-Myers, Lichnerowicz, etc), – however, with slightly worse constants. Actually, this can easily be seen from the fact that for $K > 0$

$$\mathsf{CD}(K, N) \Rightarrow \mathsf{CD}^*(K, N) \Rightarrow \mathsf{CD}(K^*, N)$$

with $K^* = \frac{N-1}{N} K$.

As an interesting application of these results we prove that the fundamental group $\pi_1(\mathsf{M}, x_0)$ of a metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ is finite whenever it satisfies the local curvature-dimension condition $\mathsf{CD}_{\mathsf{loc}}(K, N)$ with positive K and finite N. Indeed, the local curvature-dimension condition for a given metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ carries over to its universal cover $(\hat{\mathsf{M}}, \hat{\mathsf{d}}, \hat{\mathsf{m}})$. The global version of the reduced curvature-dimension condition then implies a Bonnet-Myers theorem (with non-sharp constants) and thus compactness of $\hat{\mathsf{M}}$.

For the purpose of comparison we point out that a similar, but slightly weaker condition than $\mathsf{CD}(K, N)$ – the measure contraction property $\mathsf{MCP}(K, N)$ introduced in [Oht07a] and [Stu06b] – satisfies the tensorization property due to [Oht07b] (where no assumption of non-branching metric measure spaces is used), but does not fulfill the local-to-global property according to [Stu06b, Remark 5.6].

2 Reduced Curvature-Dimension Condition $\mathsf{CD}^*(K, N)$

Throughout this paper, $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ always denotes a *metric measure space* consisting of a complete separable metric space (M, d) and a locally finite measure m on $(\mathsf{M}, \mathcal{B}(\mathsf{M}))$, that is, the volume $\mathsf{m}(B_r(x))$ of balls centered at x is finite for all $x \in \mathsf{M}$ and all sufficiently small $r > 0$. The metric space (M, d) is called *proper* if and only if every bounded closed subset of M is compact. It is called a *length space* if and only if $\mathsf{d}(x, y) = \inf \mathsf{Length}(\gamma)$ for all $x, y \in \mathsf{M}$, where the infimum runs over all curves γ in M connecting x and y. Finally, it is called a *geodesic space* if and only if every two points $x, y \in \mathsf{M}$ are connected by a curve γ with $\mathsf{d}(x, y) = \mathsf{Length}(\gamma)$. Such a curve is called *geodesic*. We denote by $\mathcal{G}(\mathsf{M})$ the space of geodesics $\gamma : [0, 1] \to \mathsf{M}$ equipped with the topology of uniform convergence.

A *non-branching* metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ consists of a geodesic metric space (M, d) such that for every tuple (z, x_0, x_1, x_2) of points in M for which z is a midpoint of x_0 and x_1 as well as of x_0 and x_2, it follows that $x_1 = x_2$.

The *diameter* $\mathsf{diam}(\mathsf{M}, \mathsf{d}, \mathsf{m})$ of a metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ is defined as the diameter of its support, namely, $\mathsf{diam}(\mathsf{M}, \mathsf{d}, \mathsf{m}) := \sup\{\mathsf{d}(x, y) : x, y \in \mathsf{supp}(\mathsf{m})\}$.

We denote by $(\mathcal{P}_2(\mathsf{M}, \mathsf{d}), \mathsf{d}_{\mathsf{W}})$ the L_2-*Wasserstein space* of probability measures ν on $(\mathsf{M}, \mathcal{B}(\mathsf{M}))$ with finite second moments which means that $\int_{\mathsf{M}} \mathsf{d}^2(x_0, x) d\nu(x) < \infty$ for some (hence all) $x_0 \in \mathsf{M}$.

The L_2-*Wasserstein distance* $\mathsf{d}_\mathsf{W}(\mu,\nu)$ between two probability measures $\mu,\nu \in \mathcal{P}_2(\mathsf{M},\mathsf{d})$ is defined as

$$\mathsf{d}_\mathsf{W}(\mu,\nu) = \inf\left\{ \left(\int_{\mathsf{M}\times\mathsf{M}} \mathsf{d}^2(x,y) d\mathsf{q}(x,y) \right)^{1/2} : \mathsf{q} \text{ coupling of } \mu \text{ and } \nu \right\}.$$

Here the infimum ranges over all *couplings* of μ and ν which are probability measures on $\mathsf{M}\times\mathsf{M}$ with marginals μ and ν.

The L_2-Wasserstein space $\mathcal{P}_2(\mathsf{M},\mathsf{d})$ is a complete separable metric space. The subspace of m-absolutely continuous measures is denoted by $\mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$ and the subspace of m-absolutely continuous measures with bounded support by $\mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$.

The L_2-*transportation distance* $\mathbb{D}$ is defined for two metric measure spaces $(\mathsf{M},\mathsf{d},\mathsf{m}), (\mathsf{M}',\mathsf{d}',\mathsf{m}')$ by

$$\mathbb{D}((\mathsf{M},\mathsf{d},\mathsf{m}),(\mathsf{M}',\mathsf{d}',\mathsf{m}')) = \inf \left(\int_{\mathsf{M}\times\mathsf{M}'} \hat{\mathsf{d}}^2(x,y') d\mathsf{q}(x,y') \right)^{1/2}.$$

The infimum is taken over all couplings q of m and m' and over all couplings $\hat{\mathsf{d}}$ of d and d'. Given two metric measure spaces $(\mathsf{M},\mathsf{d},\mathsf{m})$ and $(\mathsf{M}',\mathsf{d}',\mathsf{m}')$, we say that a measure q on the product space $\mathsf{M}\times\mathsf{M}'$ is a *coupling* of m and m' if and only if

$$\mathsf{q}(A\times\mathsf{M}') = \mathsf{m}(A) \quad \text{and} \quad \mathsf{q}(\mathsf{M}\times A') = \mathsf{m}'(A')$$

for all $A \in \mathcal{B}(\mathsf{M})$ and all $A' \in \mathcal{B}(\mathsf{M}')$. We say that a pseudo-metric $\hat{\mathsf{d}}$ – meaning that $\hat{\mathsf{d}}$ may vanish outside the diagonal – on the disjoint union $\mathsf{M}\sqcup\mathsf{M}'$ is a *coupling* of d and d' if and only if

$$\hat{\mathsf{d}}(x,y) = \mathsf{d}(x,y) \quad \text{and} \quad \hat{\mathsf{d}}(x',y') = \mathsf{d}'(x',y')$$

for all $x,y \in \mathsf{supp}(\mathsf{m}) \subseteq \mathsf{M}$ and all $x',y' \in \mathsf{supp}(\mathsf{m}') \subseteq \mathsf{M}'$.

The L_2-transportation distance $\mathbb{D}$ defines a complete separable length metric on the family of isomorphism classes of normalized metric measure spaces $(\mathsf{M},\mathsf{d},\mathsf{m})$ satisfying $\int_\mathsf{M} \mathsf{d}^2(x_0,x)d\mathsf{m}(x) < \infty$ for some $x_0 \in \mathsf{M}$.

Before we give the precise definition of the reduced curvature-dimension condition $\mathsf{CD}^*(K,N)$, we summarize two properties of the coefficients $\sigma_{K,N}^{(t)}(\cdot)$. These statements can be found in [Stu06b].

Lemma 2.1. *For all $K,K' \in \mathbb{R}$, all $N,N' \in [1,\infty)$ and all $(t,\theta) \in [0,1]\times\mathbb{R}_+$,*

$$\sigma_{K,N}^{(t)}(\theta)^N \cdot \sigma_{K',N'}^{(t)}(\theta)^{N'} \geq \sigma_{K+K',N+N'}^{(t)}(\theta)^{N+N'}.$$

Remark 2.2. *For fixed $t \in (0,1)$ and $\theta \in (0,\infty)$ the function $(K,N) \mapsto \sigma_{K,N}^{(t)}(\theta)$ is continuous, non-decreasing in K and non-increasing in N.*

Definition 2.3. *Let two numbers $K,N \in \mathbb{R}$ with $N \geq 1$ be given.*

(i) We say that a metric measure space $(\mathsf{M},\mathsf{d},\mathsf{m})$ satisfies the reduced curvature-dimension condition $\mathsf{CD}^*(K,N)$ (globally) if and only if for all $\nu_0,\nu_1 \in \mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$ there exist an optimal coupling q of $\nu_0 = \rho_0\mathsf{m}$ and $\nu_1 = \rho_1\mathsf{m}$ and a geodesic $\Gamma : [0,1] \to \mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$ connecting ν_0 and ν_1 such that

$$\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m}) \leq -\int_{\mathsf{M}\times\mathsf{M}} \left[\sigma_{K,N'}^{(1-t)}(\mathsf{d}(x_0,x_1))\rho_0^{-1/N'}(x_0) + \sigma_{K,N'}^{(t)}(\mathsf{d}(x_0,x_1))\rho_1^{-1/N'}(x_1) \right] d\mathsf{q}(x_0,x_1) \tag{2.1}$$

for all $t \in [0,1]$ and all $N' \geq N$.

(ii) We say that $(\mathsf{M},\mathsf{d},\mathsf{m})$ satisfies the reduced curvature-dimension condition $\mathsf{CD}^*(K,N)$ locally - denoted by $\mathsf{CD}^*_{\mathsf{loc}}(K,N)$ - if and only if each point x of M has a neighborhood $M(x)$ such that for each pair $\nu_0,\nu_1 \in \mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$ supported in $M(x)$ there exist an optimal coupling q of ν_0 and ν_1 and a geodesic $\Gamma : [0,1] \to \mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$ connecting ν_0 and ν_1 satisfying (2.1) for all $t \in [0,1]$ and all $N' \geq N$.

Remark 2.4. *(i) For non-branching spaces, the curvature-dimension condition* $\mathsf{CD}^*(K,N)$ *– which is formulated as a condition on probability measures with bounded support – implies property (2.1) for all measures* $\nu_0,\nu_1 \in \mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$. *We refer to Lemma 2.11. An analogous assertion holds for the condition* $\mathsf{CD}(K,N)$.

(ii) In the case $K=0$*, the reduced curvature-dimension condition* $\mathsf{CD}^*(0,N)$ *coincides with the usual one* $\mathsf{CD}(0,N)$ *simply because* $\sigma^{(t)}_{0,N}(\theta) = t = \tau^{(t)}_{0,N}(\theta)$ *for all* $\theta \in \mathbb{R}_+$.

(iii) Note that we do not require that $\Gamma(t)$ *is supported in* $M(x)$ *for* $t \in \,]0,1[$ *in part (ii) of Definition 2.3.*

(iv) Theorem 6.2 will imply that a metric measure space $(\mathsf{M},\mathsf{d},\mathsf{m})$ *satisfying* $\mathsf{CD}^*(K,N)$ *has a proper support. In particular, the support of a metric measure space* $(\mathsf{M},\mathsf{d},\mathsf{m})$ *fulfilling* $\mathsf{CD}^*_{\mathsf{loc}}(K,N)$ *is locally compact.*

Proposition 2.5. *(i)* $\mathsf{CD}(K,N) \Rightarrow \mathsf{CD}^*(K,N)$*: For each metric measure space* $(\mathsf{M},\mathsf{d},\mathsf{m})$*, the curvature-dimension condition* $\mathsf{CD}(K,N)$ *for given numbers* $K,N \in \mathbb{R}$ *implies the reduced curvature-dimension condition* $\mathsf{CD}^*(K,N)$.

(ii) $\mathsf{CD}^*(K,N) \Rightarrow \mathsf{CD}(K^*,N)$*: Assume that* $(\mathsf{M},\mathsf{d},\mathsf{m})$ *satisfies the reduced curvature-dimension condition* $\mathsf{CD}^*(K,N)$ *for some* $K>0$ *and* $N \geq 1$. *Then* $(\mathsf{M},\mathsf{d},\mathsf{m})$ *satisfies* $\mathsf{CD}(K^*,N)$ *for* $K^* = \frac{K(N-1)}{N}$.

Proof. (i) Due to Lemma 2.1 we have for all $K',N' \in \mathbb{R}$ with $N' \geq 1$ and all $(t,\theta) \in [0,1]\times\mathbb{R}_+$,

$$\tau^{(t)}_{K',N'}(\theta)^{N'} = t \cdot \sigma^{(t)}_{K',N'-1}(\theta)^{N'-1} = \sigma^{(t)}_{0,1}(\theta)\cdot \sigma^{(t)}_{K',N'-1}(\theta)^{N'-1} \geq \sigma^{(t)}_{K',N'}(\theta)^{N'}$$

which means

$$\tau^{(t)}_{K',N'}(\theta) \geq \sigma^{(t)}_{K',N'}(\theta).$$

Now we consider two probability measures $\nu_0,\nu_1 \in \mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$. Due to $\mathsf{CD}(K,N)$ there exist an optimal coupling q of $\nu_0 = \rho_0\mathsf{m}$ and $\nu_1 = \rho_1\mathsf{m}$ and a geodesic $\Gamma : [0,1] \to \mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$ connecting ν_0 and ν_1 such that

$$\begin{aligned}\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m}) &\leq -\int_{\mathsf{M}\times\mathsf{M}} \left[\tau^{(1-t)}_{K,N'}(\mathsf{d}(x_0,x_1))\rho_0^{-1/N'}(x_0) + \tau^{(t)}_{K,N'}(\mathsf{d}(x_0,x_1))\rho_1^{-1/N'}(x_1)\right] d\mathsf{q}(x_0,x_1)\\ &\leq -\int_{\mathsf{M}\times\mathsf{M}} \left[\sigma^{(1-t)}_{K,N'}(\mathsf{d}(x_0,x_1))\rho_0^{-1/N'}(x_0) + \sigma^{(t)}_{K,N'}(\mathsf{d}(x_0,x_1))\rho_1^{-1/N'}(x_1)\right] d\mathsf{q}(x_0,x_1)\end{aligned}$$

for all $t \in [0,1]$ and all $N' \geq N$.

(ii) Put $K^* := \frac{K(N-1)}{N}$ and note that $K^* \leq \frac{K(N'-1)}{N'}$ for all $N' \geq N$. Comparing the relevant coefficients $\tau^{(t)}_{K^*,N'}(\theta)$ and $\sigma^{(t)}_{K,N'}(\theta)$, yields

$$\tau^{(t)}_{K^*,N'}(\theta) = \tau^{(t)}_{\frac{K(N'-1)}{N'},N'}(\theta) = t^{1/N'}\left(\frac{\sin(t\theta\sqrt{K/N'})}{\sin(\theta\sqrt{K/N'})}\right)^{1-1/N'} \leq \sigma^{(t)}_{K,N'}(\theta) \tag{2.2}$$

for all $\theta \in \mathbb{R}_+$, $t \in [0,1]$ and $N' \geq N$.

According to our curvature assumption, for every $\nu_0,\nu_1 \in \mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$ there exist an optimal coupling q of $\nu_0 = \rho_0\mathsf{m}$ and $\nu_1 = \rho_1\mathsf{m}$ and a geodesic $\Gamma : [0,1] \to \mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$ from ν_0 to ν_1 with property (2.1). From (2.2) we deduce

$$\begin{aligned}\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m}) &\leq -\int_{\mathsf{M}\times\mathsf{M}} \left[\sigma^{(1-t)}_{K,N'}(\mathsf{d}(x_0,x_1))\rho_0^{-1/N'}(x_0) + \sigma^{(t)}_{K,N'}(\mathsf{d}(x_0,x_1))\rho_1^{-1/N'}(x_1)\right] d\mathsf{q}(x_0,x_1)\\ &\leq -\int_{\mathsf{M}\times\mathsf{M}} \left[\tau^{(1-t)}_{K^*,N'}(\mathsf{d}(x_0,x_1))\rho_0^{-1/N'}(x_0) + \tau^{(t)}_{K^*,N'}(\mathsf{d}(x_0,x_1))\rho_1^{-1/N'}(x_1)\right] d\mathsf{q}(x_0,x_1)\end{aligned}$$

for all $t \in [0,1]$ and all $N' \geq N$. This proves property $\mathsf{CD}(K^*, N)$.

$\square$

A crucial property on non-branching spaces is that a mutually singular decomposition of terminal measures leads to mutually singular decompositions of t-midpoints. This fact was already repeatedly used in [Stu06b, LV09]. Following the advice of the referee, we include a complete proof for the readers convenience.

Lemma 2.6. *Let $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ be a non-branching geodesic metric measure space. Let $\nu_0, \nu_1 \in \mathcal{P}_\infty(\mathsf{M}, \mathsf{d}, \mathsf{m})$ and let ν_t be a t-midpoint of ν_0 and ν_1 with $t \in [0,1]$. Assume that for $n \in \mathbb{N}$ or $n = \infty$*

$$\nu_i = \sum_{k=1}^{n} \alpha_k \nu_i^k$$

for $i = 0, t, 1$ and suitable $\alpha_k > 0$ where ν_i^k are probability measures such that ν_t^k is a t-midpoint of ν_0^k and ν_1^k for every k. If the family $\left(\nu_0^k\right)_{k=1,\ldots,n}$ is mutually singular, then $\left(\nu_t^k\right)_{k=1,\ldots,n}$ is mutually singular as well.

Proof. We set $t_1 = 0$, $t_2 = t$ and $t_3 = 1$. For $k = 1, \ldots, n$ we consider probability measures q^k on M^3 with the following properties:

* the projection on the i-th factor is $\nu_{t_i}^k$ for $i = 1, 2, 3$
* for q^k-almost every $(x_1, x_2, x_3) \in \mathsf{M}^3$ and every $i, j = 1, 2, 3$

$$\mathsf{d}(x_i, x_j) = |t_i - t_j| \mathsf{d}(x_1, x_3).$$

We define $\mathsf{q} := \sum_{k=1}^{n} \alpha_k \mathsf{q}^k$. Then the projection of q on the first and the third factor is an optimal coupling of ν_0 and ν_1 due to [Stu06b, Lemma 2.11(ii)]. Assume that there exist $i, j \in \{1, \ldots, n\}$ with $i \neq j$ and $(x_i, z, y_i), (x_j, z, y_j) \in \mathsf{M}^3$ such that

$$z \in \mathsf{supp}(\nu_t^i) \cap \mathsf{supp}(\nu_t^j)$$

and

$$\underbrace{(x_i, z, y_i)}_{\in \mathsf{supp}(\mathsf{q}^i)}, \underbrace{(x_j, z, y_j)}_{\in \mathsf{supp}(\mathsf{q}^j)} \in \mathsf{supp}(\mathsf{q}).$$

Hence, $x_i \neq x_j$. Since every optimal coupling is d^2-cyclically monotone according to [Vil09, Theorem 5.10], we have

$$\begin{aligned}
&\mathsf{d}^2(x_i, y_i) + \mathsf{d}^2(x_j, y_j) \\
&\quad \leq \mathsf{d}^2(x_i, y_j) + \mathsf{d}^2(x_j, y_i) \\
&\quad \leq \left[\mathsf{d}(x_i, z) + \mathsf{d}(z, y_j)\right]^2 + \left[\mathsf{d}(x_j, z) + \mathsf{d}(z, y_i)\right]^2 \\
&\quad = \mathsf{d}^2(x_i, z) + \mathsf{d}^2(z, y_j) + 2\mathsf{d}(x_i, z)\mathsf{d}(z, y_j) \\
&\qquad\qquad + \mathsf{d}^2(x_j, z) + \mathsf{d}^2(z, y_i) + 2\mathsf{d}(x_j, z)\mathsf{d}(z, y_i) \\
&\quad = \left(t^2 + (1-t)^2\right)\left[\mathsf{d}^2(x_i, y_i) + \mathsf{d}^2(x_j, y_j)\right] \\
&\qquad\qquad + 4t(1-t)\mathsf{d}(x_i, y_i)\mathsf{d}(x_j, y_j) \\
&\quad \leq \left(t^2 + (1-t)^2 + 2t(1-t)\right)\left[\mathsf{d}^2(x_i, y_i) + \mathsf{d}^2(x_j, y_j)\right] \\
&\quad = \mathsf{d}^2(x_i, y_i) + \mathsf{d}^2(x_j, y_j).
\end{aligned}$$

Thus, all inequalities have to be equalities. In particular,

$$\mathsf{d}(x_j, y_i) = \mathsf{d}(x_j, z) + \mathsf{d}(z, y_i),$$

meaning that z is an s-midpoint of x_j and y_i for an appropriately choosen $s \in [0,1]$. Hence, there exists a tupel $(z, a_0, a_1, a_2) \in \mathsf{M}^4$ – a_1 lying on the geodesic connecting x_i and z, a_2 on the one conneting x_j and z, a_0 on the one from z to y_i – such that z is a midpoint of a_0 and a_1 as well as of a_0 and a_2. This contradicts our assumption of non-branching metric measure spaces. □

We summarize two properties of the reduced curvature-dimension condition $\mathsf{CD}^*(K,N)$. The analogous results for metric measure spaces $(\mathsf{M},\mathsf{d},\mathsf{m})$ satisfying the "original" curvature-dimension condition $\mathsf{CD}(K,N)$ of Lott, Villani and Sturm are formulated and proved in [Stu06b].

The first result states the uniqueness of geodesics:

Proposition 2.7 (Geodesics). *Let $(\mathsf{M},\mathsf{d},\mathsf{m})$ be a non-branching metric measure space satisfying the condition $\mathsf{CD}^*(K,N)$ for some numbers $K,N \in \mathbb{R}$. Then for every $x \in \mathsf{supp}(\mathsf{m}) \subseteq \mathsf{M}$ and m-almost every $y \in \mathsf{M}$ - with exceptional set depending on x - there exists a unique geodesic between x and y.*

Moreover, there exists a measurable map $\gamma : \mathsf{M} \times \mathsf{M} \to \mathcal{G}(\mathsf{M})$ such that for $\mathsf{m} \otimes \mathsf{m}$-almost every $(x,y) \in \mathsf{M} \times \mathsf{M}$ the curve $t \longmapsto \gamma_t(x,y)$ is the unique geodesic connecting x and y.

The second one provides equivalent characterizations of the curvature-dimension condition $\mathsf{CD}^*(K,N)$:

Proposition 2.8 (Equivalent characterizations). *For each proper non-branching metric measure space $(\mathsf{M},\mathsf{d},\mathsf{m})$, the following statements are equivalent:*

(i) $(\mathsf{M},\mathsf{d},\mathsf{m})$ *satisfies* $\mathsf{CD}^*(K,N)$.

(ii) For all $\nu_0, \nu_1 \in \mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$ there exists a geodesic $\Gamma : [0,1] \to \mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$ connecting ν_0 and ν_1 such that for all $t \in [0,1]$ and all $N' \geq N$,

$$\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m}) \leq \sigma_{K,N'}^{(1-t)}(\theta)\mathsf{S}_{N'}(\nu_0|\mathsf{m}) + \sigma_{K,N'}^{(t)}(\theta)\mathsf{S}_{N'}(\nu_1|\mathsf{m}), \tag{2.3}$$

where

$$\theta := \begin{cases} \inf_{x_0 \in \mathcal{S}_0, x_1 \in \mathcal{S}_1} \mathsf{d}(x_0,x_1), & \text{if } K \geq 0, \\ \sup_{x_0 \in \mathcal{S}_0, x_1 \in \mathcal{S}_1} \mathsf{d}(x_0,x_1), & \text{if } K < 0, \end{cases} \tag{2.4}$$

denoting by $\mathcal{S}_0$ and $\mathcal{S}_1$ the supports of ν_0 and ν_1, respectively.

(iii) For all $\nu_0, \nu_1 \in \mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$ there exists an optimal coupling q of $\nu_0 = \rho_0\mathsf{m}$ and $\nu_1 = \rho_1\mathsf{m}$ such that

$$\rho_t^{-1/N}(\gamma_t(x_0,x_1)) \geq \sigma_{K,N}^{(1-t)}(\mathsf{d}(x_0,x_1))\rho_0^{-1/N}(x_0) + \sigma_{K,N}^{(t)}(\mathsf{d}(x_0,x_1))\rho_1^{-1/N}(x_1) \tag{2.5}$$

for all $t \in [0,1]$ and q-almost every $(x_0,x_1) \in \mathsf{M} \times \mathsf{M}$. Here for all $t \in [0,1]$, ρ_t denotes the density with respect to m of the push-forward measure of q under the map $(x_0,x_1) \longmapsto \gamma_t(x_0,x_1)$.

Proof. (i) ⇒ (ii): This implication follows from the fact that

$$\sigma_{K,N'}^{(t)}(\theta_\alpha) \geq \sigma_{K,N'}^{(t)}(\theta_\beta)$$

for all $t \in [0,1]$, all N' and all $\theta_\alpha, \theta_\beta \in \mathbb{R}_+$ with $K\theta_\alpha \geq K\theta_\beta$.

(ii) ⇒ (i): We consider two measures $\nu_0 = \rho_0\mathsf{m}$, $\nu_1 = \rho_1\mathsf{m} \in \mathcal{P}(B_R(o),\mathsf{d},\mathsf{m}) \subseteq \mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$ for some $o \in \mathsf{M}$ and $R > 0$ and choose an arbitrary optimal coupling $\tilde{\mathsf{q}}$ of them. For each $\epsilon > 0$, there exists a finite covering $(C_i)_{i=1,\ldots,n \in \mathbb{N}}$ of $M_c := \overline{B_{2R}(o)}$ by disjoint sets $C_1, \ldots, C_n$ with diameter $\leq \epsilon/2$ due to the compactness of M_c which is ensured by the properness of M. Now, we define probability measures ν_0^{ij} and ν_1^{ij} for $i,j = 1,\ldots,n$ on (M_c,d) by

$$\nu_0^{ij}(A) := \frac{1}{\alpha_{ij}}\tilde{\mathsf{q}}((A \cap C_i) \times C_j) \quad \text{and} \quad \nu_1^{ij}(A) := \frac{1}{\alpha_{ij}}\tilde{\mathsf{q}}(C_i \times (A \cap C_j)),$$

provided that $\alpha_{ij} := \tilde{\mathsf{q}}(C_i \times C_j) \neq 0$. Then

$$\mathsf{supp}(\nu_0^{ij}) \subseteq \overline{C_i} \quad \text{and} \quad \mathsf{supp}(\nu_1^{ij}) \subseteq \overline{C_j}.$$

By assumption there exists a geodesic $\Gamma^{ij} : [0,1] \to \mathcal{P}(M_c, \mathsf{d}, \mathsf{m})$ connecting $\nu_0^{ij} = \rho_0^{ij}\mathsf{m}$ and $\nu_1^{ij} = \rho_1^{ij}\mathsf{m}$ and satisfying

$$\begin{aligned} &\mathsf{S}_{N'}(\Gamma^{ij}(t)|\mathsf{m}) \\ &\quad \leq -\int_{\mathsf{M}\times\mathsf{M}} \Big[\sigma_{K,N'}^{(1-t)}(\max\{\mathsf{d}(x_0,x_1) \mp \epsilon, 0\})\rho_0^{ij}(x_0)^{-1/N'} + \\ &\qquad\qquad + \sigma_{K,N'}^{(t)}(\max\{\mathsf{d}(x_0,x_1) \mp \epsilon, 0\})\rho_1^{ij}(x_1)^{-1/N'}\Big] d\mathsf{q}^{ij}(x_0,x_1) \end{aligned}$$

for all $t \in [0,1]$ and all $N' \geq N$, with $\mp$ depending on the sign of K and with q^{ij} being an optimal coupling of ν_0^{ij} and ν_1^{ij}. We define for each $\epsilon > 0$ and all $t \in [0,1]$,

$$\mathsf{q}^{(\epsilon)} := \sum_{i,j=1}^{n} \alpha_{ij}\mathsf{q}^{ij} \quad \text{and} \quad \Gamma^{(\epsilon)}(t) := \sum_{i,j=1}^{n} \alpha_{ij}\Gamma^{ij}(t).$$

Then $\mathsf{q}^{(\epsilon)}$ is an optimal coupling of ν_0 and ν_1 and $\Gamma^{(\epsilon)}$ defines a geodesic connecting them. Furthermore, since $\Gamma^{ij}(t)$ is a t-midpoint of ν_0^{ij} and ν_1^{ij}, since the $\nu_0^{ij} \otimes \nu_1^{ij}$ are mutually singular for different choices of $(i,j) \in \{1,\dots,n\}^2$ and since $(M_c, \mathsf{d}, \mathsf{m})$ is non-branching, the $\Gamma^{ij}(t)$ are as well mutually singular for different choices of $(i,j) \in \{1,\dots,n\}^2$ and for each fixed $t \in [0,1]$ due to Lemma 2.6. Hence, for all N',

$$\mathsf{S}_{N'}(\Gamma^{(\epsilon)}(t)|\mathsf{m}) = \sum_{ij} \alpha_{ij}^{1-1/N'} \mathsf{S}_{N'}(\Gamma^{ij}(t)|\mathsf{m}).$$

Compactness of (M_c, d) implies that there exists a sequence $(\epsilon(k))_{k\in\mathbb{N}}$ converging to 0 such that $(\mathsf{q}^{(\epsilon(k))})_{k\in\mathbb{N}}$ converges to some q and such that $(\Gamma^{(\epsilon(k))})_{k\in\mathbb{N}}$ converges to some geodesic Γ in $\mathcal{P}_\infty(M_c, \mathsf{d}, \mathsf{m})$. Therefore, for fixed $\varepsilon > 0$, all $t \in [0,1]$ and all $N' \geq N$,

$$\begin{aligned} &\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m}) \\ &\leq \liminf_{k\to\infty} \mathsf{S}_{N'}(\Gamma^{(\epsilon(k))}(t)|\mathsf{m}) \\ &\leq -\limsup_{k\to\infty} \int \Big[\sigma_{K,N'}^{(1-t)}(\max\{\mathsf{d}(x_0,x_1) \mp \varepsilon, 0\})\rho_0^{-\frac{1}{N'}}(x_0) + \\ &\qquad\qquad + \sigma_{K,N'}^{(t)}(\max\{\mathsf{d}(x_0,x_1) \mp \varepsilon, 0\})\rho_1^{-\frac{1}{N'}}(x_1)\Big] d\mathsf{q}^{(\epsilon(k))}(x_0,x_1) \\ &\leq -\int_{\mathsf{M}\times\mathsf{M}} \Big[\sigma_{K,N'}^{(1-t)}(\max\{\mathsf{d}(x_0,x_1) \mp \varepsilon, 0\})\rho_0^{-\frac{1}{N'}}(x_0) + \\ &\qquad\qquad + \sigma_{K,N'}^{(t)}(\max\{\mathsf{d}(x_0,x_1) \mp \varepsilon, 0\})\rho_1^{-\frac{1}{N'}}(x_1)\Big] d\mathsf{q}(x_0,x_1) \end{aligned}$$

where the proof of the last inequality is similar to the proof of [Stu06b, Lemma 3.3]. In the limit $\varepsilon \to 0$ the claim follows due to the theorem of monotone convergence.

The equivalence (i) $\Leftrightarrow$ (iii) is obtained by following the arguments of the proof of [Stu06b, Proposition 4.2] replacing the coefficients $\tau_{K,N}^{(t)}(\cdot)$ by $\sigma_{K,N}^{(t)}(\cdot)$. □

Remark 2.9. *To be honest, we suppressed an argument in the proof of Proposition 2.8, (ii) $\Rightarrow$ (i): In fact, the compactness of (M_c, d) implies the compactness of $\mathcal{P}(M_c, \mathsf{d})$ and therefore, we can deduce the existence of a limit Γ of $(\Gamma^{(\epsilon(k))})_{k\in\mathbb{N}}$ - using the same notation as in the above proof - in*

$\mathcal{P}(M_c, \mathsf{d})$! A further observation ensures that Γ is not only in $\mathcal{P}(M_c, \mathsf{d})$ but also in $\mathcal{P}_\infty(M_c, \mathsf{d}, \mathsf{m})$ - as claimed in the above proof: The characterizing inequality of $\mathsf{CD}^*(K,N)$ implies the characterizing inequality of the property $\mathsf{Curv}(\mathsf{M}, \mathsf{d}, \mathsf{m}) \geq K$ (at this point we refer to [Stu06a],[Stu06b]). Thus, the geodesic Γ satisfies

$$\mathsf{Ent}(\Gamma(t)|\mathsf{m}) \leq (1-t)\mathsf{Ent}(\Gamma(0)|\mathsf{m}) + t\mathsf{Ent}(\Gamma(1)|\mathsf{m}) - \frac{K}{2} t(1-t)\mathsf{d}_\mathsf{W}^2(\Gamma(0), \Gamma(1))$$

for all $t \in [0,1]$. This implies that $\mathsf{Ent}(\Gamma(t)|\mathsf{m}) < +\infty$ and consequently, $\Gamma(t) \in \mathcal{P}_\infty(M_c, \mathsf{d}, \mathsf{m})$ for all $t \in [0,1]$. In the sequel, we will use similar arguments from time to time without emphasizing on them explicitly.

Proposition 2.10 (Midpoints)**.** *A proper non-branching metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ satisfies $\mathsf{CD}^*(K,N)$ if and only if for all $\nu_0, \nu_1 \in \mathcal{P}_\infty(\mathsf{M}, \mathsf{d}, \mathsf{m})$ there exists a midpoint $\eta \in \mathcal{P}_\infty(\mathsf{M}, \mathsf{d}, \mathsf{m})$ of ν_0 and ν_1 satisfying*

$$\mathsf{S}_{N'}(\eta|\mathsf{m}) \leq \sigma_{K,N'}^{(1/2)}(\theta)\mathsf{S}_{N'}(\nu_0|\mathsf{m}) + \sigma_{K,N'}^{(1/2)}(\theta)\mathsf{S}_{N'}(\nu_1|\mathsf{m}), \tag{2.6}$$

for all $N' \geq N$ where θ is defined as in (2.4).

Proof. We only consider the case $K > 0$. The general case requires analogous calculations. Due to Proposition 2.8, we have to prove that the existence of midpoints with property (2.6) for all $N' \geq N$ implies the existence of geodesics satisfying property (2.3) for all $N' \geq N$. Given $\Gamma(0) := \nu_0$ and $\Gamma(1) := \nu_1$, we define $\Gamma(\frac{1}{2})$ as a midpoint of $\Gamma(0)$ and $\Gamma(1)$ with property (2.6) for all $N' \geq N$. Then we define $\Gamma(\frac{1}{4})$ as a midpoint of $\Gamma(0)$ and $\Gamma(\frac{1}{2})$ satisfying (2.6) for all $N' \geq N$ and accordingly, $\Gamma(\frac{3}{4})$ as a midpoint of $\Gamma(\frac{1}{2})$ and $\Gamma(1)$ with (2.6) for all $N' \geq N$. By iterating this procedure, we obtain $\Gamma(t)$ for all dyadic $t = l2^{-k} \in [0,1]$ for $k \in \mathbb{N}$ and odd $l = 0, \ldots, 2^k$ with

$$\begin{aligned}
&\mathsf{S}_{N'}\left(\Gamma\left(l2^{-k}\right)|\mathsf{m}\right) \leq \\
&\qquad \leq \sigma_{K,N'}^{(1/2)}\left(2^{-k+1}\theta\right)\mathsf{S}_{N'}\left(\Gamma\left((l-1)2^{-k}\right)|\mathsf{m}\right) + \sigma_{K,N'}^{(1/2)}\left(2^{-k+1}\theta\right)\mathsf{S}_{N'}\left(\Gamma\left((l+1)2^{-k}\right)|\mathsf{m}\right),
\end{aligned}$$

for all $N' \geq N$ where θ is defined as above.

Now, we consider $k > 0$. By induction, we are able to pass from level $k-1$ to level k: Assuming that $\Gamma(t)$ satisfies property (2.3) for all $t = l2^{-k+1} \in [0,1]$ and all $N' \geq N$, we have for an odd number $l \in \{0, \ldots, 2^{-k}\}$,

$$\begin{aligned}
&\mathsf{S}_{N'}\left(\Gamma\left(l2^{-k}\right)|\mathsf{m}\right) \leq \\
&\qquad \leq \sigma_{K,N'}^{(1/2)}\left(2^{-k+1}\theta\right)\mathsf{S}_{N'}\left(\Gamma\left((l-1)2^{-k}\right)|\mathsf{m}\right) + \sigma_{K,N'}^{(1/2)}\left(2^{-k+1}\theta\right)\mathsf{S}_{N'}\left(\Gamma\left((l+1)2^{-k}\right)|\mathsf{m}\right) \\
&\qquad \leq \sigma_{K,N'}^{(1/2)}\left(2^{-k+1}\theta\right)\left[\sigma_{K,N'}^{\left(1-(l-1)2^{-k}\right)}(\theta)\mathsf{S}_{N'}(\Gamma(0)|\mathsf{m}) + \sigma_{K,N'}^{\left((l-1)2^{-k}\right)}(\theta)\mathsf{S}_{N'}(\Gamma(1)|\mathsf{m})\right] + \\
&\qquad + \sigma_{K,N'}^{(1/2)}\left(2^{-k+1}\theta\right)\left[\sigma_{K,N'}^{\left(1-(l+1)2^{-k}\right)}(\theta)\mathsf{S}_{N'}(\Gamma(0)|\mathsf{m}) + \sigma_{K,N'}^{\left((l+1)2^{-k}\right)}(\theta)\mathsf{S}_{N'}(\Gamma(1)|\mathsf{m})\right]
\end{aligned}$$

for all $N' \geq N$. Calculating the prefactor of $\mathsf{S}_{N'}(\Gamma(0)|\mathsf{m})$ yields

$$\sigma_{K,N'}^{(1/2)}\left(2^{-k+1}\theta\right)\sigma_{K,N'}^{\left(1-(l-1)2^{-k}\right)}(\theta)+\sigma_{K,N'}^{(1/2)}\left(2^{-k+1}\theta\right)\sigma_{K,N'}^{\left(1-(l+1)2^{-k}\right)}(\theta)=$$

$$=\frac{\sin\left(2^{-k}\theta\sqrt{K/N'}\right)\cdot\left[\sin\left(\left(1-(l-1)2^{-k}\right)\theta\sqrt{K/N'}\right)+\sin\left(\left(1-(l+1)2^{-k}\right)\theta\sqrt{K/N'}\right)\right]}{\sin\left(2^{-k+1}\theta\sqrt{K/N'}\right)\sin\left(\theta\sqrt{K/N'}\right)}$$

$$=\frac{2\sin\left(\left(1-l2^{-k}\right)\theta\sqrt{K/N'}\right)\cos\left(2^{-k}\theta\sqrt{K/N'}\right)}{2\cos\left(2^{-k}\theta\sqrt{K/N'}\right)\sin\left(\theta\sqrt{K/N'}\right)}=$$

$$=\frac{\sin\left(\left(1-l2^{-k}\right)\theta\sqrt{K/N'}\right)}{\sin\left(\theta\sqrt{K/N'}\right)}=\sigma_{K,N'}^{\left(1-l2^{-k}\right)}(\theta),$$

and calculating the one of $\mathsf{S}_{N'}(\Gamma(1)|\mathsf{m})$ gives

$$\sigma_{K,N'}^{(1/2)}\left(2^{-k+1}\theta\right)\sigma_{K,N'}^{\left((l-1)2^{-k}\right)}(\theta)+\sigma_{K,N'}^{(1/2)}\left(2^{-k+1}\theta\right)\sigma_{K,N'}^{\left((l+1)2^{-k}\right)}(\theta)=$$

$$=\frac{\sin\left(2^{-k}\theta\sqrt{K/N'}\right)\cdot\left[\sin\left((l-1)2^{-k}\theta\sqrt{K/N'}\right)+\sin\left((l+1)2^{-k}\theta\sqrt{K/N'}\right)\right]}{\sin\left(2^{-k+1}\theta\sqrt{K/N'}\right)\sin\left(\theta\sqrt{K/N'}\right)}$$

$$=\frac{2\sin\left(l2^{-k}\theta\sqrt{K/N'}\right)\cos\left(2^{-k}\theta\sqrt{K/N'}\right)}{2\cos\left(2^{-k}\theta\sqrt{K/N'}\right)\sin\left(\theta\sqrt{K/N'}\right)}=$$

$$=\frac{\sin\left(l2^{-k}\theta\sqrt{K/N'}\right)}{\sin\left(\theta\sqrt{K/N'}\right)}=\sigma_{K,N'}^{\left(l2^{-k}\right)}(\theta).$$

Combining the above results leads to property (2.3),

$$\mathsf{S}_{N'}\left(\Gamma\left(l2^{-k}\right)|\mathsf{m}\right)\leq\sigma_{K,N'}^{\left(1-l2^{-k}\right)}(\theta)\mathsf{S}_{N'}(\Gamma(0)|\mathsf{m})+\sigma_{K,N'}^{\left(l2^{-k}\right)}(\theta)\mathsf{S}_{N'}(\Gamma(1)|\mathsf{m})$$

for all $N' \geq N$. The continuous extension of $\Gamma(t)$ – t dyadic – yields the desired geodesic due to the lower semi-continuity of the Rényi entropy. ☐

Lemma 2.11. *Fix two real parameters K and $N \geq 1$. If $(\mathsf{M},\mathsf{d},\mathsf{m})$ is non-branching then the reduced curvature-dimension condition $\mathsf{CD}^*(K,N)$ implies that for all $\nu_0,\nu_1 \in \mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$ there exist an optimal coupling q of $\nu_0=\rho_0\mathsf{m}$ and $\nu_1=\rho_1\mathsf{m}$ and a geodesic $\Gamma:[0,1]\to\mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$ connecting ν_0 and ν_1 and satisfying (2.1) for all $N'\geq N$.*

Proof. We assume that $(\mathsf{M},\mathsf{d},\mathsf{m})$ satisfies $\mathsf{CD}^*(K,N)$. Fix a covering of M by mutual disjoint, bounded sets $L_i, i\in\mathbb{N}$. Let $\nu_0=\rho_0\mathsf{m},\nu_1=\rho_1\mathsf{m}\in\mathcal{P}_2(\mathsf{M},\mathsf{d},\mathsf{m})$ and an optimal coupling $\tilde{q}$ of ν_0 and ν_1 be given. Define probability measures $\nu_0^{ij},\nu_1^{ij}\in\mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$ for $i,j\in\mathbb{N}$ by

$$\nu_0^{ij}(A):=\frac{1}{\alpha_{ij}}\tilde{q}((A\cap L_i)\times L_j)\quad\text{ and }\quad\nu_1^{ij}(A):=\frac{1}{\alpha_{ij}}\tilde{q}(L_i\times(A\cap L_j))$$

provided $\alpha_{ij}:=\tilde{q}(L_i\times L_j)\neq 0$. According to $\mathsf{CD}^*(K,N)$, for each pair $i,j\in\mathbb{N}$, there exist an optimal coupling q_{ij} of $\nu_0^{ij}=\rho_0^{ij}\mathsf{m}$ and $\nu_1^{ij}=\rho_1^{ij}\mathsf{m}$ and a geodesic $\Gamma^{ij}:[0,1]\to\mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$ joining them such that

$$\mathsf{S}_{N'}(\Gamma^{ij}(t)|\mathsf{m})\leq$$

$$\leq-\int_{\mathsf{M}\times\mathsf{M}}\left[\sigma_{K,N'}^{(1-t)}(\mathsf{d}(x_0,x_1))\rho_0^{ij}(x_0)^{-1/N'}+\sigma_{K,N'}^{(t)}(\mathsf{d}(x_0,x_1))\rho_1^{ij}(x_1)^{-1/N'}\right]dq_{ij}(x_0,x_1)$$

for all $t \in [0,1]$ and all $N' \geq N$. Define

$$q := \sum_{i,j=1}^{\infty} \alpha_{ij} q^{ij}, \quad \Gamma_t := \sum_{i,j=1}^{\infty} \alpha_{ij} \Gamma_t^{ij}.$$

Then q is an optimal coupling of ν_0 and ν_1 and Γ is a geodesic connecting them. Moreover, since the $\nu_0^{ij} \otimes \nu_1^{ij}$ for different choices of $(i,j) \in \mathbb{N}^2$ are mutually singular and since M is non-branching, also the Γ_t^{ij} for different choices of $(i,j) \in \{1,\ldots,n\}^2$ are mutually singular, Lemma 2.6 (for each fixed $t \in [0,1]$). Hence,

$$S_{N'}(\Gamma_t|m) = \sum_{i,j=1}^{\infty} \alpha_{ij}^{1-1/N'} \cdot S_{N'}(\Gamma_t^{ij}|m)$$

and one simply may sum up both sides of the previous inequality – multiplied by $\alpha_{ij}^{1-1/N'}$ – to obtain the claim. □

Remark 2.12. *Let us point out that the same arguments prove that on non-branching spaces the curvature-dimension condition* $\mathsf{CD}(K,N)$ *as formulated in this paper – which requires only conditions on probability measures with bounded support – implies the analogous condition in the second author's previous paper [Stu06b] (where conditions on all probability measures with finite second moments had been imposed).*

Remark 2.13. *The curvature-dimension condition* $\mathsf{CD}(K,N)$ *does not imply the non-branching property. For instance, Banach spaces satisfy* $\mathsf{CD}(0,N)$ *whereas they are not always non-branching. Moreover, even in the special case of limits of Riemannian manifolds with uniform lower Ricci curvature bounds, it is not known whether they are non-branching or not.*

3 Stability under Convergence

Theorem 3.1. *Let* $((\mathsf{M}_n, \mathsf{d}_n, \mathsf{m}_n))_{n\in\mathbb{N}}$ *be a sequence of normalized metric measure spaces with the property that for each* $n \in \mathbb{N}$ *the space* $(\mathsf{M}_n, \mathsf{d}_n, \mathsf{m}_n)$ *satisfies the reduced curvature-dimension condition* $\mathsf{CD}^*(K_n, N_n)$. *Assume that for* $n \to \infty$,

$$(\mathsf{M}_n, \mathsf{d}_n, \mathsf{m}_n) \xrightarrow{\mathbb{D}} (\mathsf{M}, \mathsf{d}, \mathsf{m})$$

as well as $(K_n, N_n) \to (K, N)$ *for some* $(K,N) \in \mathbb{R}^2$. *Then the space* $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *fulfills* $\mathsf{CD}^*(K,N)$.

Proof. The proof essentially follows the line of argumentation in [Stu06b, Theorem 3.1] with two modifications:

- The coefficients $\tau_{K,N}^{(t)}(\cdot)$ will be replaced by $\sigma_{K,N}^{(t)}(\cdot)$.
- The assumption of a uniform upper bound $L_0 < L_{\max}$ on the diameters will be removed. (Here $L_{\max}$ will be $\pi\sqrt{\frac{N}{K}}$ for $K > 0$, previously it was $\pi\sqrt{\frac{N-1}{K}}$.)

(i) Let us firstly observe that $\mathsf{CD}^*(K_n, N_n)$ with $K_n \to K$ and $N_n \to N$ implies that the spaces $(\mathsf{M}_n, \mathsf{d}_n, \mathsf{m}_n)$ have the 'doubling property' with a common doubling constant C on subsets $M_n' \subseteq \mathsf{supp}(\mathsf{m}_n)$ with uniformly bounded diameter θ (see [Stu06b, Corollary 2.4] and also Theorem 6.2). This version of the doubling property is stable under $\mathbb{D}$-convergence due to [Stu06a, Theorem 3.15] and thus also holds on bounded sets $M' \subseteq \mathsf{supp}(\mathsf{m})$. Therefore, $\mathsf{supp}(\mathsf{m})$ is proper.

(ii) Choose $\bar{N} > N$ and $\bar{K} < K$ and put $\bar{L} := \pi\sqrt{\frac{\bar{N}}{\bar{K}}}$ as well as $L := \pi\sqrt{\frac{N}{K}}$ provided that $\bar{K} > 0$ and $K > 0$. Otherwise, $\bar{L} = \infty$, $L = \infty$. Then

$$\max\left\{\frac{\partial}{\partial\theta}\sigma^{(s)}_{K',N'}(\theta) : s \in [0,1], K' \leq \bar{K}, N' \geq \bar{N}, \theta \in \left[0, \frac{L+\bar{L}}{2}\right]\right\}$$

is bounded.

(iii) For each $n \in \mathbb{N}$, $\mathsf{diam}(\mathsf{supp}(\mathsf{m}_n)) \leq L_n := \pi\sqrt{\frac{N_n}{K_n}}$ due to Corollary 6.3. In particular, given $\bar{K}$, $\bar{N}$ as above

$$\mathsf{diam}(\mathsf{supp}(\mathsf{m}_n)) \leq \frac{L+\bar{L}}{2}$$

for all sufficiently large $n \in \mathbb{N}$. The latter implies

$$\mathsf{diam}(\mathsf{supp}(\mathsf{m})) \leq \frac{L+\bar{L}}{2}$$

according to [Stu06a, Theorem 3.16].

(iv) Let us now follow the proof in [Stu06b, Theorem 3.1]. In short, we consider $\nu_0, \nu_1 \in \mathcal{P}_\infty(\mathsf{M}, \mathsf{d}, \mathsf{m})$ and approximate them by probability measures $\nu_{0,n}$ and $\nu_{1,n}$ in $\mathcal{P}_\infty(\mathsf{M}_n, \mathsf{d}_n, \mathsf{m}_n)$ satisfying the relevant equation (2.1) with an optimal coupling q_n and a geodesic $\Gamma_{t,n}$ due to the reduced curvature-dimension condition on $(\mathsf{M}_n, \mathsf{d}_n, \mathsf{m}_n)$. Via a map $\mathcal{Q} : \mathcal{P}_2(\mathsf{M}_n, \mathsf{d}_n, \mathsf{m}_n) \to \mathcal{P}_2(\mathsf{M}, \mathsf{d}, \mathsf{m})$ introduced in [Stu06a, Lemma 4.19] we define an 'ε-approximative' geodesic $\Gamma^\varepsilon_t := \mathcal{Q}(\Gamma_{t,n})$ from ν_0 to ν_1 satisfying (2.1) for an 'ε-approximative' coupling q^ε of ν_0 and ν_1.

(v) The properness of $\mathsf{supp}(\mathsf{m})$ implies that Γ^ε_t and q^ε are tight (i.e. essentially supported on compact sets – uniformly in ε) which yields the existence of accumulation points $\bar{\Gamma}_t$ and $\bar{\mathsf{q}}$ satisfying (2.1) – with K' in the place of K – for all $K' \leq \bar{K}$ and all $N' \geq \bar{N}$.

(vi) Choosing sequences $\bar{N}_l \searrow N$ and $\bar{K}_l \nearrow K$ and again passing to the limits $\Gamma_t = \lim_l \bar{\Gamma}^l_t$ and $\mathsf{q} = \lim_l \bar{\mathsf{q}}^l$ we obtain an optimal coupling q and a geodesic Γ satisfying (2.1) for all $K' < K$ and all $N' > N$. Finally, continuity of all the involved terms in K' and N' proves the claim.

□

Remark 3.2. *The previous proof demonstrates that in the analogous formulation of the stability result for* $\mathsf{CD}(K,N)$ *in [Stu06b, Theorem 3.1] the assumption*

$$\limsup_{n\to\infty} \frac{K_n L_n^2}{N_n - 1} < \pi$$

is unnecessary.

4 Tensorization

Theorem 4.1 (Tensorization)**.** *Let* $(\mathsf{M}_i, \mathsf{d}_i, \mathsf{m}_i)$ *be non-branching metric measure spaces satisfying the reduced curvature-dimension condition* $\mathsf{CD}^*(K, N_i)$ *with two real parameters* K *and* $N_i \geq 1$ *for* $i = 1, \ldots, k$ *with* $k \in \mathbb{N}$*. Then*

$$(\mathsf{M}, \mathsf{d}, \mathsf{m}) := \bigotimes_{i=1}^{k} (\mathsf{M}_i, \mathsf{d}_i, \mathsf{m}_i)$$

fulfills $\mathsf{CD}^*\left(K, \sum_{i=1}^k N_i\right)$.

Proof. Without restriction we assume that $k=2$. We consider $\nu_0=\rho_0\mathsf{m},\nu_1=\rho_1\mathsf{m}\in\mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$. In the first step, we treat the special case

$$\nu_0=\nu_0^{(1)}\otimes\nu_0^{(2)}\quad\text{and}\quad\nu_1=\nu_1^{(1)}\otimes\nu_1^{(2)}$$

with $\nu_0^{(i)}=\rho_0^{(i)}\mathsf{m}_i,\nu_1^{(i)}=\rho_1^{(i)}\mathsf{m}_i\in\mathcal{P}_\infty(\mathsf{M}_i,\mathsf{d}_i,\mathsf{m}_i)$ for $i=1,2$. According to our curvature assumption, there exists an optimal coupling q_i of $\nu_0^{(i)}$ and $\nu_1^{(i)}$ such that

$$\begin{aligned}&\rho_t^{(i)}\left(\gamma_t^{(i)}\left(x_0^{(i)},x_1^{(i)}\right)\right)^{-1/N_i}\geq\\&\qquad\geq\sigma_{K,N_i}^{(1-t)}\left(\mathsf{d}_i\left(x_0^{(i)},x_1^{(i)}\right)\right)\rho_0^{(i)}\left(x_0^{(i)}\right)^{-1/N_i}+\sigma_{K,N_i}^{(t)}\left(\mathsf{d}_i\left(x_0^{(i)},x_1^{(i)}\right)\right)\rho_1^{(i)}\left(x_1^{(i)}\right)^{-1/N_i}\end{aligned}$$

for all $t\in[0,1]$ and q_i-almost every $\left(x_0^{(i)},x_1^{(i)}\right)\in\mathsf{M}_i\times\mathsf{M}_i$ with $i=1,2$. As in Proposition 2.8, for all $t\in[0,1]$, $\rho_t^{(i)}$ denotes the density with respect to m_i of the push-forward measure of q_i under the map $\left(x_0^{(i)},x_1^{(i)}\right)\mapsto\gamma_t^{(i)}\left(x_0^{(i)},x_1^{(i)}\right)$ for $i=1,2$. We introduce the map

$$\begin{aligned}\mathsf{T}:\mathsf{M}_1\times\mathsf{M}_1\times\mathsf{M}_2\times\mathsf{M}_2&\to\mathsf{M}_1\times\mathsf{M}_2\times\mathsf{M}_1\times\mathsf{M}_2=\mathsf{M}\times\mathsf{M}\\\left(x_0^{(1)},x_1^{(1)},x_0^{(2)},x_1^{(2)}\right)&\mapsto\left(x_0^{(1)},x_0^{(2)},x_1^{(1)},x_1^{(2)}\right),\end{aligned}$$

we put $\tilde{\mathsf{q}}:=\mathsf{q}_1\otimes\mathsf{q}_2$ and define q as the push-forward measure of $\tilde{\mathsf{q}}$ under the map T, that means $\mathsf{q}:=\mathsf{T}_*\tilde{\mathsf{q}}$. Then q is an optimal coupling of ν_0 and ν_1 and for all $t\in[0,1]$, $\rho_t(x,y):=\rho_t^{(1)}(x)\cdot\rho_t^{(2)}(y)$ is the density with respect to m of the push-forward measure of q under the map

$$\begin{aligned}\gamma_t:\mathsf{M}\times\mathsf{M}&\to\mathsf{M}=\mathsf{M}_1\times\mathsf{M}_2\\\left(x_0^{(1)},x_0^{(2)},x_1^{(1)},x_1^{(2)}\right)&\mapsto\left(\gamma_t^{(1)}\left(x_0^{(1)},x_1^{(1)}\right),\gamma_t^{(2)}\left(x_0^{(2)},x_1^{(2)}\right)\right).\end{aligned}$$

Moreover, for q-almost every $x_0=\left(x_0^{(1)},x_0^{(2)}\right),x_1=\left(x_1^{(1)},x_1^{(2)}\right)\in\mathsf{M}$ and all $t\in[0,1]$, it holds that

$$\begin{aligned}&\sigma_{K,N_1+N_2}^{(1-t)}(\mathsf{d}(x_0,x_1))\rho_0(x_0)^{-1/(N_1+N_2)}+\sigma_{K,N_1+N_2}^{(t)}(\mathsf{d}(x_0,x_1))\rho_1(x_1)^{-1/(N_1+N_2)}=\\&=\sigma_{K,N_1+N_2}^{(1-t)}(\mathsf{d}(x_0,x_1))\rho_0^{(1)}\left(x_0^{(1)}\right)^{-1/(N_1+N_2)}\cdot\rho_0^{(2)}\left(x_0^{(2)}\right)^{-1/(N_1+N_2)}\quad+\\&\qquad+\sigma_{K,N_1+N_2}^{(t)}(\mathsf{d}(x_0,x_1))\rho_1^{(1)}\left(x_1^{(1)}\right)^{-1/(N_1+N_2)}\cdot\rho_1^{(2)}\left(x_1^{(2)}\right)^{-1/(N_1+N_2)}\\&\leq\prod_{i=1}^2\sigma_{K,N_i}^{(1-t)}\left(\mathsf{d}_i\left(x_0^{(i)},x_1^{(i)}\right)\right)^{N_i/(N_1+N_2)}\rho_0^{(i)}\left(x_0^{(i)}\right)^{-1/(N_1+N_2)}\quad+\\&\qquad+\prod_{i=1}^2\sigma_{K,N_i}^{(t)}\left(\mathsf{d}_i\left(x_0^{(i)},x_1^{(i)}\right)\right)^{N_i/(N_1+N_2)}\rho_1^{(i)}\left(x_1^{(i)}\right)^{-1/(N_1+N_2)}\\&\leq\prod_{i=1}^2\Bigg[\sigma_{K,N_i}^{(1-t)}\left(\mathsf{d}_i\left(x_0^{(i)},x_1^{(i)}\right)\right)\rho_0^{(i)}\left(x_0^{(i)}\right)^{-1/N_i}+\\&\qquad+\sigma_{K,N_i}^{(t)}\left(\mathsf{d}_i\left(x_0^{(i)},x_1^{(i)}\right)\right)\rho_1^{(i)}\left(x_1^{(i)}\right)^{-1/N_i}\Bigg]^{N_i/(N_1+N_2)}\\&\leq\prod_{i=1}^2\rho_t^{(i)}\left(\gamma_t^{(i)}\left(x_0^{(i)},x_1^{(i)}\right)\right)^{-1/(N_1+N_2)}\\&=\rho_t\left(\gamma_t^{(1)}\left(x_0^{(1)},x_1^{(1)}\right),\gamma_t^{(2)}\left(x_0^{(2)},x_1^{(2)}\right)\right)^{-1/(N_1+N_2)}=\rho_t\left(\gamma_t(x_0,x_1)\right)^{-1/(N_1+N_2)}.\end{aligned}$$

In this chain of inequalities, the second one follows from Lemma 2.1 and the third one from Hölder's inequality.

In the second step, we consider $o \in \mathsf{supp}(\mathsf{m})$ and $R > 0$ and set $M_b := B_R(o) \cap \mathsf{supp}(\mathsf{m})$ as well as $M_c := \overline{B_{2R}(o)} \cap \mathsf{supp}(\mathsf{m})$. We consider arbitrary probability measures $\nu_0, \nu_1 \in \mathcal{P}_\infty(M_b, \mathsf{d}, \mathsf{m})$ and $\varepsilon > 0$. There exist

$$\nu_0^\varepsilon = \rho_0^\varepsilon m = \frac{1}{n} \sum_{j=1}^n \nu_{0,j}^\varepsilon$$

with mutually singular product measures $\nu_{0,j}^\varepsilon$ and

$$\nu_1^\varepsilon = \rho_1^\varepsilon m = \frac{1}{n} \sum_{j=1}^n \nu_{1,j}^\varepsilon$$

with mutually singular product measures $\nu_{1,j}^\varepsilon$ for $j = 1, \ldots, n$ and $n \in \mathbb{N}$ such that

$$\mathsf{S}_{N_1+N_2}\left(\nu_0^\varepsilon|\mathsf{m}\right) \leq \mathsf{S}_{N_1+N_2}(\nu_0|\mathsf{m}) + \varepsilon,$$
$$\mathsf{S}_{N_1+N_2}\left(\nu_1^\varepsilon|\mathsf{m}\right) \leq \mathsf{S}_{N_1+N_2}(\nu_1|\mathsf{m}) + \varepsilon$$

as well as

$$\mathsf{d}_\mathsf{W}\left(\nu_0, \nu_0^\varepsilon\right) \leq \varepsilon, \quad \mathsf{d}_\mathsf{W}\left(\nu_1, \nu_1^\varepsilon\right) \leq \varepsilon$$

and

$$\mathsf{d}_\mathsf{W}\left(\nu_0^\varepsilon, \nu_1^\varepsilon\right) \geq \left[\frac{1}{n} \sum_{j=1}^n \mathsf{d}_\mathsf{W}^2\left(\nu_{0,j}^\varepsilon, \nu_{1,j}^\varepsilon\right)\right]^{1/2} - \varepsilon.$$

Moreover,

$$\theta := \begin{cases} \inf\limits_{\substack{x_0 \in \mathrm{supp}(\nu_0), \\ x_1 \in \mathrm{supp}(\nu_1)}} \mathsf{d}(x_0, x_1) \leq \inf\limits_{\substack{x_0 \in \mathrm{supp}\left(\nu_{0,j}^\varepsilon\right), \\ x_1 \in \mathrm{supp}\left(\nu_{1,j}^\varepsilon\right)}} \mathsf{d}(x_0, x_1), & \text{if } K \geq 0, \\ \sup\limits_{\substack{x_0 \in \mathrm{supp}(\nu_0), \\ x_1 \in \mathrm{supp}(\nu_1)}} \mathsf{d}(x_0, x_1) \geq \sup\limits_{\substack{x_0 \in \mathrm{supp}\left(\nu_{0,j}^\varepsilon\right), \\ x_1 \in \mathrm{supp}\left(\nu_{1,j}^\varepsilon\right)}} \mathsf{d}(x_0, x_1), & \text{if } K < 0. \end{cases}$$

Since ν_0^ε is the sum of mutually singular measures $\nu_{0,j}^\varepsilon$ for $j = 1, \ldots, n$,

$$\mathsf{S}_{N_1+N_2}\left(\nu_0^\varepsilon|\mathsf{m}\right) = \left(\frac{1}{n}\right)^{1-1/(N_1+N_2)} \sum_{j=1}^n \mathsf{S}_{N_1+N_2}\left(\nu_{0,j}^\varepsilon|\mathsf{m}\right)$$

and analogously,

$$\mathsf{S}_{N_1+N_2}\left(\nu_1^\varepsilon|\mathsf{m}\right) = \left(\frac{1}{n}\right)^{1-1/(N_1+N_2)} \sum_{j=1}^n \mathsf{S}_{N_1+N_2}\left(\nu_{1,j}^\varepsilon|\mathsf{m}\right).$$

Due to the first step, for each $j = 1, \ldots, n$ there exists a midpoint $\eta_j^\varepsilon \in \mathcal{P}_\infty(M_c, \mathsf{d}, \mathsf{m})$ of $\nu_{0,j}^\varepsilon$ and $\nu_{1,j}^\varepsilon$ satisfying

$$\mathsf{S}_{N_1+N_2}\left(\eta_j^\varepsilon|\mathsf{m}\right) \leq \sigma_{K,N_1+N_2}^{(1/2)}(\theta)\mathsf{S}_{N_1+N_2}\left(\nu_{0,j}^\varepsilon|\mathsf{m}\right) + \sigma_{K,N_1+N_2}^{(1/2)}(\theta)\mathsf{S}_{N_1+N_2}\left(\nu_{1,j}^\varepsilon|\mathsf{m}\right).$$

Since M is non-branching and since the measures $\nu_{0,j}^\varepsilon$ for $j = 1, \ldots, n$ are mutually singular, also the η_j^ε are mutually singular for $j = 1, \ldots, n$ – we refer to Lemma 2.6. Therefore,

$$\eta^\varepsilon := \frac{1}{n} \sum_{j=1}^n \eta_j^\varepsilon$$

satisfies

$$\mathsf{S}_{N_1+N_2}\left(\eta^\varepsilon|\mathsf{m}\right) = \left(\frac{1}{n}\right)^{1-1/(N_1+N_2)} \sum_{j=1}^n \mathsf{S}_{N_1+N_2}\left(\eta_j^\varepsilon|\mathsf{m}\right)$$

and consequently,

$$\begin{aligned}\mathsf{S}_{N_1+N_2}\left(\eta^\varepsilon|\mathsf{m}\right) &\leq \sigma_{K,N_1+N_2}^{(1/2)}(\theta)\mathsf{S}_{N_1+N_2}\left(\nu_0^\varepsilon|\mathsf{m}\right) + \sigma_{K,N_1+N_2}^{(1/2)}(\theta)\mathsf{S}_{N_1+N_2}\left(\nu_1^\varepsilon|\mathsf{m}\right)\\ &\leq \sigma_{K,N_1+N_2}^{(1/2)}(\theta)\mathsf{S}_{N_1+N_2}\left(\nu_0|\mathsf{m}\right) + \sigma_{K,N_1+N_2}^{(1/2)}(\theta)\mathsf{S}_{N_1+N_2}\left(\nu_1|\mathsf{m}\right) + 2\varepsilon.\end{aligned}$$

Moreover, η^ε is an approximate midpoint of ν_0 and ν_1,

$$\begin{aligned}\mathsf{d}_\mathsf{W}\left(\nu_0, \eta^\varepsilon\right) \leq \mathsf{d}_\mathsf{W}\left(\nu_0^\varepsilon, \eta^\varepsilon\right) + \varepsilon &\leq \left[\frac{1}{n}\sum_{j=1}^n \mathsf{d}_\mathsf{W}^2\left(\nu_{0,j}^\varepsilon, \eta_j^\varepsilon\right)\right]^{1/2} + \varepsilon\\ &\leq \frac{1}{2}\mathsf{d}_\mathsf{W}\left(\nu_0^\varepsilon, \nu_1^\varepsilon\right) + 2\varepsilon \leq \frac{1}{2}\mathsf{d}_\mathsf{W}\left(\nu_0, \nu_1\right) + 3\varepsilon,\end{aligned}$$

a similar calculation holds true for $\mathsf{d}_\mathsf{W}\left(\eta^\varepsilon, \nu_1\right)$. According to the compactness of (M_c, d), the family $\{\eta^\varepsilon : \varepsilon > 0\}$ of approximate midpoints is tight. Hence, there exists a suitable subsequence $(\eta^{\varepsilon_k})_{k\in\mathbb{N}}$ converging to some $\eta \in \mathcal{P}_\infty(M_c, \mathsf{d}, \mathsf{m})$. Continuity of the Wasserstein distance d_W and lower semi-continuity of the Rényi entropy functional $\mathsf{S}_{N_1+N_2}(\cdot|\mathsf{m})$ imply that η is a midpoint of ν_0 and ν_1 and that

$$\mathsf{S}_{N_1+N_2}\left(\eta|\mathsf{m}\right) \leq \sigma_{K,N_1+N_2}^{(1/2)}(\theta)\mathsf{S}_{N_1+N_2}\left(\nu_0|\mathsf{m}\right) + \sigma_{K,N_1+N_2}^{(1/2)}(\theta)\mathsf{S}_{N_1+N_2}\left(\nu_1|\mathsf{m}\right).$$

Applying Proposition 2.10 finally yields the claim. □

5 From Local to Global

Theorem 5.1 ($\mathsf{CD}^*_{\mathsf{loc}}(K,N) \Leftrightarrow \mathsf{CD}^*(K,N)$)**.** *Let $K, N \in \mathbb{R}$ with $N \geq 1$ and let $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ be a non-branching metric measure space. We assume additionally that $\mathcal{P}_\infty(\mathsf{M}, \mathsf{d}, \mathsf{m})$ is a geodesic space. Then $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ satisfies $\mathsf{CD}^*(K,N)$ globally if and only if it satisfies $\mathsf{CD}^*(K,N)$ locally.*

Proof. Note that in any case, according to a generalized version of the Hopf-Rinow theorem (see e.g. [Ba], section I.2) $\mathsf{supp}(\mathsf{m})$ will be proper: The fact that $\mathcal{P}_\infty(\mathsf{M}, \mathsf{d}, \mathsf{m})$ is a geodesic space implies that $\mathsf{supp}(\mathsf{m})$ is a length space. Combined with its local compactness due to Remark 2.4(iv), this yields the properness of $\mathsf{supp}(\mathsf{m})$.

We confine ourselves to treating the case $K > 0$. The general one follows by analogous calculations.

For each number $k \in \mathbb{N} \cup \{0\}$ we define a set I_k of points in time,

$$I_k := \{l2^{-k} : l = 0, \ldots, 2^k\}.$$

For a given geodesic $\Gamma : [0,1] \to \mathcal{P}_\infty(\mathsf{M}, \mathsf{d}, \mathsf{m})$ we denote by $\mathcal{G}_k^\Gamma$ the set of all geodesics $[x] := (x_t)_{0\leq t\leq 1}$ in M satisfying $x_t \in \mathsf{supp}(\Gamma(t)) =: \mathcal{S}_t$ for all $t \in I_k$.

We consider $o \in \mathsf{supp}(\mathsf{m})$ and $R > 0$ and set $M_b := B_R(o) \cap \mathsf{supp}(\mathsf{m})$ as well as $M_c := \overline{B_{2R}(o)} \cap \mathsf{supp}(\mathsf{m})$. Now, we formulate a property $\mathsf{C(k)}$ for every $k \in \mathbb{N} \cup \{0\}$:

$\mathsf{C(k)}$: For each geodesic $\Gamma : [0,1] \to \mathcal{P}_\infty(\mathsf{M}, \mathsf{d}, \mathsf{m})$ satisfying $\Gamma(0), \Gamma(1) \in \mathcal{P}_\infty(M_b, \mathsf{d}, \mathsf{m})$ and for each pair $s, t \in I_k$ with $t - s = 2^{-k}$ there exists a midpoint $\eta(s,t) \in \mathcal{P}_\infty(\mathsf{M}, \mathsf{d}, \mathsf{m})$ of $\Gamma(s)$ and $\Gamma(t)$ such that

$$\mathsf{S}_{N'}(\eta(s,t)|\mathsf{m}) \leq \sigma_{K,N'}^{(1/2)}(\theta_{s,t})\mathsf{S}_{N'}(\Gamma(s)|\mathsf{m}) + \sigma_{K,N'}^{(1/2)}(\theta_{s,t})\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m}),$$

for all $N' \geq N$ where

$$\theta_{s,t} := \inf_{[x] \in \mathcal{G}_k^\Gamma} \mathsf{d}(x_s, x_t).$$

Our first claim is:

Claim 5.2. *For each $k \in \mathbb{N}$, $\mathsf{C(k)}$ implies $\mathsf{C(k-1)}$.*

In order to prove this claim, let $k \in \mathbb{N}$ with property $\mathsf{C(k)}$ be given. Moreover, let a geodesic Γ in $\mathcal{P}_\infty(\mathsf{M}, \mathsf{d}, \mathsf{m})$ satisfying $\Gamma(0), \Gamma(1) \in \mathcal{P}_\infty(M_b, \mathsf{d}, \mathsf{m})$ and numbers $s, t \in I_{k-1}$ with $t - s = 2^{1-k}$ be given. We put $\theta := \inf_{[x] \in \mathcal{G}_{k-1}^\Gamma} \mathsf{d}(x_s, x_t)$, and we define iteratively a sequence $(\Gamma^{(i)})_{i \in \mathbb{N} \cup \{0\}}$ of geodesics in $\mathcal{P}_\infty(M_c, \mathsf{d}, \mathsf{m})$ coinciding with Γ on $[0, s] \cup [t, 1]$ as follows:

Start with $\Gamma^{(0)} := \Gamma$. Assuming that $\Gamma^{(2i)}$ is already given, let $\Gamma^{(2i+1)}$ be any geodesic in $\mathcal{P}_\infty(M_c, \mathsf{d}, \mathsf{m})$ which coincides with Γ on $[0, s] \cup [t, 1]$, for which $\Gamma^{(2i+1)}\left(s + 2^{-(k+1)}\right)$ is a midpoint of $\Gamma(s) = \Gamma^{(2i)}(s)$ and $\Gamma^{(2i)}\left(s + 2^{-k}\right)$ and for which $\Gamma^{(2i+1)}\left(s + 3 \cdot 2^{-(k+1)}\right)$ is a midpoint of $\Gamma^{(2i)}\left(s + 2^{-k}\right)$ and $\Gamma(t) = \Gamma^{(2i)}(t)$ satisfying

$$\begin{aligned} &\mathsf{S}_{N'}\left(\Gamma^{(2i+1)}\left(s + 2^{-(k+1)}\right)|\mathsf{m}\right) \leq \\ &\quad \leq \sigma_{K,N'}^{(1/2)}\left(\theta^{(2i+1)}\right)\mathsf{S}_{N'}(\Gamma(s)|\mathsf{m}) + \sigma_{K,N'}^{(1/2)}\left(\theta^{(2i+1)}\right)\mathsf{S}_{N'}\left(\Gamma^{(2i)}\left(s + 2^{-k}\right)|\mathsf{m}\right) \end{aligned}$$

for all $N' \geq N$ where

$$\theta^{(2i+1)} := \inf_{[x] \in \mathcal{G}_k^{\Gamma^{(2i)}}} \mathsf{d}\left(x_s, x_{s+2^{-k}}\right) \geq \tfrac{1}{2}\theta,$$

that is,

$$\begin{aligned} &\mathsf{S}_{N'}\left(\Gamma^{(2i+1)}\left(s + 2^{-(k+1)}\right)|\mathsf{m}\right) \leq \\ &\quad \leq \sigma_{K,N'}^{(1/2)}\left(\tfrac{1}{2}\theta\right)\mathsf{S}_{N'}(\Gamma(s)|\mathsf{m}) + \sigma_{K,N'}^{(1/2)}\left(\tfrac{1}{2}\theta\right)\mathsf{S}_{N'}\left(\Gamma^{(2i)}\left(s + 2^{-k}\right)|\mathsf{m}\right) \end{aligned}$$

for all $N' \geq N$ and accordingly,

$$\begin{aligned} &\mathsf{S}_{N'}\left(\Gamma^{(2i+1)}\left(s + 3 \cdot 2^{-(k+1)}\right)|\mathsf{m}\right) \leq \\ &\quad \leq \sigma_{K,N'}^{(1/2)}\left(\tfrac{1}{2}\theta\right)\mathsf{S}_{N'}\left(\Gamma^{(2i)}\left(s + 2^{-k}\right)|\mathsf{m}\right) + \sigma_{K,N'}^{(1/2)}\left(\tfrac{1}{2}\theta\right)\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m}) \end{aligned}$$

for all $N' \geq N$. Such midpoints exist due to $\mathsf{C(k)}$.

Now let $\Gamma^{(2i+2)}$ be any geodesic in $\mathcal{P}_\infty(M_c, \mathsf{d}, \mathsf{m})$ which coincides with $\Gamma^{(2i+1)}$ on $[0, s+2^{-(k+1)}] \cup [s + 3 \cdot 2^{-(k+1)}, 1]$ and for which $\Gamma^{(2i+2)}\left(s + 2^{-k}\right)$ is a midpoint of $\Gamma^{(2i+1)}\left(s + 2^{-(k+1)}\right)$ and $\Gamma^{(2i+1)}\left(s + 3 \cdot 2^{-(k+1)}\right)$ satisfying

$$\begin{aligned} &\mathsf{S}_{N'}\left(\Gamma^{(2i+2)}\left(s + 2^{-k}\right)|\mathsf{m}\right) \leq \\ &\quad \leq \sigma_{K,N'}^{(1/2)}\left(\tfrac{1}{2}\theta\right)\mathsf{S}_{N'}\left(\Gamma^{(2i+1)}\left(s + 2^{-(k+1)}\right)|\mathsf{m}\right) + \\ &\qquad + \sigma_{K,N'}^{(1/2)}\left(\tfrac{1}{2}\theta\right)\mathsf{S}_{N'}\left(\Gamma^{(2i+1)}\left(s + 3 \cdot 2^{-(k+1)}\right)|\mathsf{m}\right) \end{aligned}$$

for all $N' \geq N$. Again such a midpoint exists according to $\mathsf{C(k)}$. This yields a sequence $(\Gamma^{(i)})_{i \in \mathbb{N} \cup \{0\}}$ of geodesics. Combining the above inequalities yields

$$\begin{aligned} &\mathsf{S}_{N'}\left(\Gamma^{(2i+2)}\left(s + 2^{-k}\right)|\mathsf{m}\right) \leq \\ &\quad \leq 2\sigma_{K,N'}^{(1/2)}\left(\tfrac{1}{2}\theta\right)^2\mathsf{S}_{N'}\left(\Gamma^{(2i)}\left(s + 2^{-k}\right)|\mathsf{m}\right) + \\ &\qquad + \sigma_{K,N'}^{(1/2)}\left(\tfrac{1}{2}\theta\right)^2\mathsf{S}_{N'}\left(\Gamma(s)|\mathsf{m}\right) + \sigma_{K,N'}^{(1/2)}\left(\tfrac{1}{2}\theta\right)^2\mathsf{S}_{N'}\left(\Gamma(t)|\mathsf{m}\right) \end{aligned}$$

and by iteration,

$$\begin{aligned}&\mathsf{S}_{N'}\left(\Gamma^{(2i)}\left(s+2^{-k}\right)|\mathsf{m}\right)\le\\&\quad\le 2^i\sigma^{(1/2)}_{K,N'}\left(\tfrac12\theta\right)^{2i}\mathsf{S}_{N'}\left(\Gamma\left(s+2^{-k}\right)|\mathsf{m}\right)+\\&\qquad\qquad+\tfrac12\sum_{k=1}^{i}\left(2\sigma^{(1/2)}_{K,N'}\left(\tfrac12\theta\right)^2\right)^k\left[\mathsf{S}_{N'}\left(\Gamma(s)|\mathsf{m}\right)+\mathsf{S}_{N'}\left(\Gamma(t)|\mathsf{m}\right)\right]\end{aligned}$$

for all $N'\ge N$.

By compactness of $\mathcal{P}(M_c,\mathsf{d})$, there exists a suitable subsequence of $\left(\Gamma^{(2i)}\left(s+2^{-k}\right)\right)_{i\in\mathbb{N}\cup\{0\}}$ converging to some $\eta\in\mathcal{P}(M_c,\mathsf{d})$. Continuity of the distance implies that η is a midpoint of $\Gamma(s)$ and $\Gamma(t)$ and the lower semi-continuity of the Rényi entropy functional implies

$$\mathsf{S}_{N'}(\eta|\mathsf{m})\le\sigma^{(1/2)}_{K,N'}(\theta)\mathsf{S}_{N'}(\Gamma(s)|\mathsf{m})+\sigma^{(1/2)}_{K,N'}(\theta)\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m})$$

for all $N'\ge N$. This proves property $\mathsf{C}(\mathsf{k}-1)$. At this point, we do not want to suppress the calculations leading to this last implication: For all $N'\ge N$, we have

$$\begin{aligned}\sigma^{(1/2)}_{K,N'}\left(\tfrac12\theta\right)&=\frac{\sin\left(\frac14\theta\sqrt{K/N'}\right)}{\sin\left(\frac12\theta\sqrt{K/N'}\right)}=\frac{\sin\left(\frac14\theta\sqrt{K/N'}\right)}{2\sin\left(\frac14\theta\sqrt{K/N'}\right)\cos\left(\frac14\theta\sqrt{K/N'}\right)}\\&=\frac{1}{2\cos\left(\frac14\theta\sqrt{K/N'}\right)}.\end{aligned}$$

In the case $2\sigma^{(1/2)}_{K,N'}\left(\frac12\theta\right)^2<1$,

$$\begin{aligned}\tfrac12\lim_{i\to\infty}\sum_{k=1}^{i}\left(2\sigma^{(1/2)}_{K,N'}\left(\tfrac12\theta\right)^2\right)^k&=\tfrac12\left[\left(1-2\sigma^{(1/2)}_{K,N'}\left(\tfrac12\theta\right)^2\right)^{-1}-1\right]\\&=\tfrac12\left[\left(\frac{2\cos^2\left(\frac14\theta\sqrt{K/N'}\right)-1}{2\cos^2\left(\frac14\theta\sqrt{K/N'}\right)}\right)^{-1}-1\right]\\&=\tfrac12\left[\frac{2\cos^2\left(\frac14\theta\sqrt{K/N'}\right)}{\cos\left(\frac12\theta\sqrt{K/N'}\right)}-1\right]\\&=\tfrac12\left[\frac{\cos\left(\frac12\theta\sqrt{K/N'}\right)+1-\cos\left(\frac12\theta\sqrt{K/N'}\right)}{\cos\left(\frac12\theta\sqrt{K/N'}\right)}\right]\\&=\frac{1}{2\cos\left(\frac12\theta\sqrt{K/N'}\right)}=\sigma^{(1/2)}_{K,N'}(\theta).\end{aligned}$$

The case $2\sigma^{(1/2)}_{K,N'}\left(\frac12\theta\right)^2\ge1$ is trivial since then $\sigma^{(1/2)}_{K,N'}(\theta)=\infty$ by convention.

According to our curvature assumption, each point $x\in\mathsf{M}$ has a neighborhood $M(x)$ such that probability measures in $\mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$ which are supported in $M(x)$ can be joined by a geodesic in $\mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$ satisfying (2.1). By compactness of M_c, there exist $\lambda>0$, $n\in\mathbb{N}$, finitely many disjoint sets $L_1,L_2,\dots,L_n$ covering M_c, and closed sets $M_j\supseteq B_\lambda(L_j)$ for $j=1,\dots,n$, such that probability measures in $\mathcal{P}_\infty(M_j,\mathsf{d},\mathsf{m})$ can be joined by geodesics in $\mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$ satisfying (2.1). Choose $\kappa\in\mathbb{N}$ such that

$$2^{-\kappa}\mathsf{diam}(M_c,\mathsf{d},\mathsf{m})\le\lambda.$$

Our next claim is:

Claim 5.3. *Property* $\mathsf{C}(\kappa)$ *is satisfied.*

In order to prove this claim, we consider a geodesic Γ in $\mathcal{P}_\infty(\mathsf{M},\mathsf{d},\mathsf{m})$ satisfying $\Gamma(0),\Gamma(1) \in \mathcal{P}_\infty(M_b,\mathsf{d},\mathsf{m})$ and numbers $s,t \in I_\kappa$ with $t-s = 2^{-\kappa}$. Let $\hat{\mathsf{q}}$ be a coupling of $\Gamma(l2^{-\kappa})$ for $l = 0,\dots,2^\kappa$ on $\mathsf{M}^{2^\kappa+1}$ such that for $\hat{\mathsf{q}}$-almost every $(x_l)_{l=0,\dots,2^\kappa} \in \mathsf{M}^{2^\kappa+1}$ the points x_s, x_t lie on some geodesic connecting x_0 and x_1 with

$$\mathsf{d}(x_s,x_t) = |t-s|\mathsf{d}(x_0,x_1) \le 2^{-\kappa}\mathsf{diam}(M_c,\mathsf{d},\mathsf{m}) \le \lambda. \tag{5.1}$$

Define probability measures $\Gamma_j(s)$ and $\Gamma_j(t)$ for $j = 1,\dots,n$ by

$$\Gamma_j(s)(A) := \frac{1}{\alpha_j}\Gamma(s)(A\cap L_j) = \frac{1}{\alpha_j}\hat{\mathsf{q}}\underbrace{(\mathsf{M}\times\cdots\times(\underset{\substack{\uparrow \\ (2^\kappa s+1)\text{-th factor}}}{A}\cap L_j)\times\mathsf{M}\times\cdots\times\mathsf{M})}_{(2^\kappa+1)\text{ factors}}$$

and

$$\Gamma_j(t)(A) := \frac{1}{\alpha_j}\hat{\mathsf{q}}(\mathsf{M}\times\cdots\times L_j\times \underset{\substack{\uparrow \\ (2^\kappa t+1)\text{-th factor}}}{A}\times\cdots\times\mathsf{M})$$

provided that $\alpha_j := \Gamma_s(L_j) \neq 0$. Otherwise, define $\Gamma_j(s)$ and $\Gamma_j(t)$ arbitrarily. Then $\mathsf{supp}(\Gamma_j(s)) \subseteq \overline{L_j}$ which combined with inequality (5.1) implies

$$\mathsf{supp}(\Gamma_j(s)) \cup \mathsf{supp}(\Gamma_j(t)) \subseteq \overline{B_\lambda(L_j)} \subseteq M_j.$$

Therefore, for each $j \in \{1,\dots,n\}$, the assumption "$(\mathsf{M},\mathsf{d},\mathsf{m})$ satisfies $\mathsf{CD}^*(K,N)$ locally" can be applied to the probability measures $\Gamma_j(s)$ and $\Gamma_j(t) \in \mathcal{P}_\infty(M_j,\mathsf{d},\mathsf{m})$. It yields the existence of a midpoint $\eta_j(s,t)$ of $\Gamma_j(s)$ and $\Gamma_j(t)$ with the property that

$$\mathsf{S}_{N'}(\eta_j(s,t)|\mathsf{m}) \le \sigma_{K,N'}^{(1/2)}(\theta_{s,t})\mathsf{S}_{N'}(\Gamma_j(s)|\mathsf{m}) + \sigma_{K,N'}^{(1/2)}(\theta_{s,t})\mathsf{S}_{N'}(\Gamma_j(t)|\mathsf{m}) \tag{5.2}$$

for all $N' \ge N$ where

$$\theta_{s,t} := \inf_{[x]\in\mathcal{G}_\kappa^\Gamma} \mathsf{d}(x_s,x_t).$$

Define

$$\eta(s,t) := \sum_{j=1}^n \alpha_j \eta_j(s,t).$$

Then, $\eta(s,t)$ is a midpoint of $\Gamma(s) = \sum_{j=1}^n \alpha_j\Gamma_j(s)$ and $\Gamma(t) = \sum_{j=1}^n \alpha_j\Gamma_j(t)$. Moreover, since the $\Gamma_j(s)$ are mutually singular for $j = 1,\dots,n$ and since M is non-branching, also the $\eta_j(s,t)$ are mutually singular for $j = 1,\dots,n$ due to Lemma 2.6. Therefore, for all $N' \ge N$,

$$\mathsf{S}_{N'}(\eta(s,t)|\mathsf{m}) = \sum_{j=1}^n \alpha_j^{1-1/N'}\mathsf{S}_{N'}(\eta_j(s,t)|\mathsf{m}) \tag{5.3}$$

and

$$\mathsf{S}_{N'}(\Gamma(s)|\mathsf{m}) = \sum_{j=1}^n \alpha_j^{1-1/N'}\mathsf{S}_{N'}(\Gamma_j(s)|\mathsf{m}), \tag{5.4}$$

whereas

$$\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m}) \ge \sum_{j=1}^n \alpha_j^{1-1/N'}\mathsf{S}_{N'}(\Gamma_j(t)|\mathsf{m}), \tag{5.5}$$

since the $\Gamma_j(t)$ are not necessarily mutually singular for $j = 1, \ldots, n$. Summing up (5.2) for $j = 1, \ldots, n$ and using (5.3)–(5.5) yields

$$\mathsf{S}_{N'}(\eta(s,t)|\mathsf{m}) \leq \sigma^{(1/2)}_{K,N'}(\theta_{s,t})\mathsf{S}_{N'}(\Gamma(s)|\mathsf{m}) + \sigma^{(1/2)}_{K,N'}(\theta_{s,t})\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m})$$

for all $N' \geq N$. This proves property $\mathsf{C}(\kappa)$.

In order to finish the proof let two probability measures $\nu_0, \nu_1 \in \mathcal{P}_\infty(M_b, \mathsf{d}, \mathsf{m})$ be given. By assumption there exists a geodesic Γ in $\mathcal{P}_\infty(\mathsf{M}, \mathsf{d}, \mathsf{m})$ connecting them. According to our second claim, property $\mathsf{C}(\kappa)$ is satisfied and according to our first claim, this implies $\mathsf{C}(\mathsf{k})$ for all $k = \kappa - 1, \kappa - 2, \ldots, 0$. Property $\mathsf{C}(0)$ finally states that there exists a midpoint $\eta \in \mathcal{P}_\infty(\mathsf{M}, \mathsf{d}, \mathsf{m})$ of $\Gamma(0) = \nu_0$ and $\Gamma(1) = \nu_1$ with

$$\mathsf{S}_{N'}(\eta|\mathsf{m}) \leq \sigma^{(1/2)}_{K,N'}(\theta)\mathsf{S}_{N'}(\Gamma(0)|\mathsf{m}) + \sigma^{(1/2)}_{K,N'}(\theta)\mathsf{S}_{N'}(\Gamma(1)|\mathsf{m}),$$

for all $N' \geq N$ where

$$\theta := \inf_{x_0 \in \mathcal{S}_0, x_1 \in \mathcal{S}_1} \mathsf{d}(x_0, x_1).$$

This proves Theorem 5.1. □

Corollary 5.4 ($\mathsf{CD}^*_{\mathsf{loc}}(K-, N) \Leftrightarrow \mathsf{CD}^*(K, N)$)**.** *Fix two numbers $K, N \in \mathbb{R}$. A non-branching metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ fulfills the reduced curvature-dimension condition $\mathsf{CD}^*(K', N)$ locally for all $K' < K$ if and only if it satisfies the condition $\mathsf{CD}^*(K, N)$ globally.*

Proof. Given any $K' < K$, the condition $\mathsf{CD}^*(K', N)$ is deduced from $\mathsf{CD}^*_{\mathsf{loc}}(K', N)$ according to the above localization theorem. Due to the stability of the reduced curvature-dimension condition stated in Theorem 3.1, $\mathsf{CD}^*(K', N)$ for all $K' < K$ implies $\mathsf{CD}^*(K, N)$. □

Proposition 5.5 ($\mathsf{CD}^*_{\mathsf{loc}}(K-, N) \Leftrightarrow \mathsf{CD}_{\mathsf{loc}}(K-, N)$)**.** *Fix two numbers $K, N \in \mathbb{R}$. A metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ fulfills the reduced curvature-dimension condition $\mathsf{CD}^*(K', N)$ locally for all $K' < K$ if and only if it satisfies the original condition $\mathsf{CD}(K', N)$ locally for all $K' < K$.*

Proof. As remarked in the past, we content ourselves with the case $K > 0$. Again, the general one can be deduced from analogous calculations. The implication "$\mathsf{CD}^*_{\mathsf{loc}}(K-, N) \Leftarrow \mathsf{CD}_{\mathsf{loc}}(K-, N)$" follows from analogous arguments leading to part (i) of Proposition 2.5.

The implication "$\mathsf{CD}^*_{\mathsf{loc}}(K-, N) \Rightarrow \mathsf{CD}_{\mathsf{loc}}(K-, N)$" is based on the fact that the coefficients $\tau^{(t)}_{K,N}(\theta)$ and $\sigma^{(t)}_{K,N}(\theta)$ are "almost identical" for $\theta \ll 1$: In order to be precise, we consider $0 < K' < \tilde{K} < K$ and $\theta \ll 1$ and compare the relevant coefficients $\tau^{(t)}_{K',N}(\theta)$ and $\sigma^{(t)}_{\tilde{K},N}(\theta)$:

$$\begin{aligned}
\left[\tau^{(t)}_{K',N}(\theta)\right]^N &= t \left[\frac{\sin\left(t\theta\sqrt{\frac{K'}{N-1}}\right)}{\sin\left(\theta\sqrt{\frac{K'}{N-1}}\right)} \right]^{N-1} \\
&= t^N \left[\frac{1 - \frac{1}{6}t^2\theta^2\frac{K'}{N-1} + O(\theta^4)}{1 - \frac{1}{6}\theta^2\frac{K'}{N-1} + O(\theta^4)} \right]^{N-1} \\
&= t^N \left[1 + \tfrac{1}{6}(1-t^2)\theta^2\tfrac{K'}{N-1} + O(\theta^4) \right]^{N-1} \\
&= t^N \left[1 + \tfrac{1}{6}(1-t^2)\theta^2 K' + O(\theta^4) \right].
\end{aligned}$$

And accordingly,

$$
\begin{aligned}
\left[\sigma^{(t)}_{\tilde{K},N}(\theta)\right]^N &= \left[\frac{\sin\left(t\theta\sqrt{\frac{\tilde{K}}{N}}\right)}{\sin\left(\theta\sqrt{\frac{\tilde{K}}{N}}\right)}\right]^N \\
&= t^N\left[\frac{1-\frac{1}{6}t^2\theta^2\frac{\tilde{K}}{N}+O(\theta^4)}{1-\frac{1}{6}\theta^2\frac{\tilde{K}}{N}+O(\theta^4)}\right]^N \\
&= t^N\left[1+\tfrac{1}{6}(1-t^2)\theta^2\tfrac{\tilde{K}}{N}+O(\theta^4)\right]^N \\
&= t^N\left[1+\tfrac{1}{6}(1-t^2)\theta^2\tilde{K}+O(\theta^4)\right].
\end{aligned}
$$

Now we choose $\theta^* > 0$ in such a way that

$$
\tau^{(t)}_{K',N}(\theta) \leq \sigma^{(t)}_{\tilde{K},N}(\theta)
$$

for all $0 \leq \theta \leq \theta^*$ and all $t \in [0,1]$. According to our curvature assumption, each point $x \in \mathsf{M}$ has a neighborhood $M(x) \subseteq \mathsf{M}$ such that every two probability measures $\nu_0, \nu_1 \in \mathcal{P}_\infty(M(x), \mathsf{d}, \mathsf{m})$ can be joined by a geodesic in $\mathcal{P}_\infty(\mathsf{M}, \mathsf{d}, \mathsf{m})$ satisfying (2.1). In order to prove that $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ satisfies $\mathsf{CD}(K', N)$ locally, we set for $x \in \mathsf{M}$,

$$
M'(x) := M(x) \cap B_{\theta^*}(x)
$$

and consider $\nu_0, \nu_1 \in \mathcal{P}_\infty(M'(x), \mathsf{d}, \mathsf{m})$. As indicated above, due to $\mathsf{CD}^*_{\mathsf{loc}}(\tilde{K}, N)$ there exist an optimal coupling q of $\nu_0 = \rho_0\mathsf{m}$ and $\nu_1 = \rho_1\mathsf{m}$ and a geodesic $\Gamma : [0,1] \to \mathcal{P}_\infty(\mathsf{M}, \mathsf{d}, \mathsf{m})$ connecting ν_0 and ν_1 such that

$$
\begin{aligned}
&\mathsf{S}_{N'}(\Gamma(t)|\mathsf{m}) \leq \\
&\leq -\int_{\mathsf{M}\times\mathsf{M}}\left[\sigma^{(1-t)}_{\tilde{K},N'}(\underbrace{\mathsf{d}(x_0,x_1)}_{\leq\theta^*})\rho_0^{-1/N'}(x_0) + \sigma^{(t)}_{\tilde{K},N'}(\underbrace{\mathsf{d}(x_0,x_1)}_{\leq\theta^*})\rho_1^{-1/N'}(x_1)\right] d\mathsf{q}(x_0,x_1) \\
&\leq -\int_{\mathsf{M}\times\mathsf{M}}\left[\tau^{(1-t)}_{K',N'}(\mathsf{d}(x_0,x_1))\rho_0^{-1/N'}(x_0) + \tau^{(t)}_{K',N'}(\mathsf{d}(x_0,x_1))\rho_1^{-1/N'}(x_1)\right] d\mathsf{q}(x_0,x_1)
\end{aligned}
$$

for all $t \in [0,1]$ and all $N' \geq N$. □

Remark 5.6. *The proofs of Theorem 4.1 and Theorem 5.1, respectively, do not extend to the original curvature-dimension condition* $\mathsf{CD}(K,N)$*. An immediate obstacle is that no analogous statements of rather technical tools like Lemma 2.1 and Proposition 2.10 are known due to the more complicated nature of the coefficients* $\tau^{(t)}_{K,N}(\cdot)$*. It is still an open question whether* $\mathsf{CD}(K,N)$ *satisfies the tensorization or the local-to-global property.*

6 Geometric and Functional Analytic Consequences

6.1 Geometric Results

The weak versions of the geometric statements derived from $\mathsf{CD}(K,N)$ in [Stu06b] follow by using analogous arguments replacing the coefficients $\tau^{(t)}_{K,N}(\cdot)$ by $\sigma^{(t)}_{K,N}(\cdot)$.

Note that we do not use the assumption of non-branching metric measure spaces in this whole section and that Corollary 6.3 and Theorem 6.5 follow immediately from the strong versions in [Stu06b] in combination with Proposition 2.5(ii).

Proposition 6.1 (Generalized Brunn-Minkowski inequality)**.** *Assume that* $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *satisfies the condition* $\mathsf{CD}^*(K, N)$ *for two real parameters* K, N *with* $N \geq 1$. *Then for all measurable sets* $A_0, A_1 \subseteq \mathsf{M}$ *with* $\mathsf{m}(A_0), \mathsf{m}(A_1) > 0$ *and all* $t \in [0, 1]$,

$$\mathsf{m}(A_t) \geq \sigma_{K,N}^{(1-t)}(\Theta) \cdot \mathsf{m}(A_0)^{1/N} + \sigma_{K,N}^{(t)}(\Theta) \cdot \mathsf{m}(A_1)^{1/N} \tag{6.1}$$

where A_t *denotes the set of points which divide geodesics starting in* A_0 *and ending in* A_1 *with ratio* $t : (1-t)$ *and where* Θ *denotes the minimal/maximal length of such geodesics*

$$\Theta := \begin{cases} \inf\limits_{x_0 \in A_0, x_1 \in A_1} \mathsf{d}(x_0, x_1), & K \geq 0 \\ \sup\limits_{x_0 \in A_0, x_1 \in A_1} \mathsf{d}(x_0, x_1), & K < 0. \end{cases}$$

The Brunn-Minkowski inequality implies further geometric consequences, for example the Bishop-Gromov volume growth estimate and the Bonnet-Myers theorem.

For a fixed point $x_0 \in \mathsf{supp}(\mathsf{m})$ we study the growth of the volume of closed balls centered at x_0 and the growth of the volume of the corresponding spheres

$$v(r) := \mathsf{m}\left(\overline{B_r(x_0)}\right) \quad \text{and} \quad s(r) := \limsup_{\delta \to 0} \tfrac{1}{\delta} \mathsf{m}\left(\overline{B_{r+\delta}(x_0)} \setminus B_r(x_0)\right),$$

respectively.

Theorem 6.2 (Generalized Bishop-Gromov volume growth inequality)**.** *Assume that the metric measure space* $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *satisfies the condition* $\mathsf{CD}^*(K, N)$ *for some* $K, N \in \mathbb{R}$. *Then each bounded closed set* $M_{b,c} \subseteq \mathsf{supp}(\mathsf{m})$ *is compact and has finite volume. To be more precise, if* $K > 0$ *then for each fixed* $x_0 \in \mathsf{supp}(\mathsf{m})$ *and all* $0 < r < R \leq \pi\sqrt{N/K}$,

$$\frac{s(r)}{s(R)} \geq \left(\frac{\sin(r\sqrt{K/N})}{\sin(R\sqrt{K/N})}\right)^N \quad \textit{and} \quad \frac{v(r)}{v(R)} \geq \frac{\int_0^r \sin\left(t\sqrt{K/N}\right)^N dt}{\int_0^R \sin\left(t\sqrt{K/N}\right)^N dt}. \tag{6.2}$$

In the case $K < 0$, *analogous inequalities hold true (where the right-hand sides of (6.2) are replaced by analogous expressions according to the definition of the coefficients* $\sigma_{K,N}^{(t)}(\cdot)$ *for negative* K*).*

Corollary 6.3 (Generalized Bonnet-Myers theorem)**.** *Fix two real parameters* $K > 0$ *and* $N \geq 1$. *Each metric measure space* $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *satisfying the condition* $\mathsf{CD}^*(K, N)$ *has compact support and its diameter* L *has an upper bound*

$$L \leq \pi\sqrt{\frac{N}{K}}.$$

Note that in the sharp version of this estimate the factor N is replaced by $N - 1$.

6.2 Lichnerowicz Estimate

In this subsection we follow the presentation of Lott and Villani in [LV07].

Definition 6.4. *Given* $f \in \mathsf{Lip}(\mathsf{M})$, *we define* $|\nabla^- f|$ *by*

$$|\nabla^- f|(x) := \limsup_{y \to x} \frac{[f(y) - f(x)]_-}{\mathsf{d}(x, y)}$$

where for $a \in \mathbb{R}$, $a_- := \max(-a, 0)$.

Theorem 6.5 (Lichnerowicz estimate, Poincaré inequality). *We assume that* $(\mathsf{M},\mathsf{d},\mathsf{m})$ *satisfies* $\mathsf{CD}^*(K,N)$ *for two real parameters* $K>0$ *and* $N\geq 1$. *Then for every* $f\in\mathsf{Lip}(\mathsf{M})$ *fulfilling* $\int_{\mathsf{M}} f d\mathsf{m}=0$ *the following inequality holds true,*

$$\int_{\mathsf{M}} f^2 d\mathsf{m} \leq \frac{1}{K}\int_{\mathsf{M}} |\nabla^- f|^2 d\mathsf{m}. \tag{6.3}$$

Remark 6.6. *In 'regular' cases,* $\mathcal{E}(f,f):=\int_{\mathsf{M}}|\nabla^- f|^2\, d\mathsf{m}$ *is a quadratic form which – by polarization – then defines uniquely a bilinear form* $\mathcal{E}(f,g)$ *and a self-adjoint operator* L *('generalized Laplacian') through the identity* $\mathcal{E}(f,g)=-\int_{\mathsf{M}} f\cdot Lg\, d\mathsf{m}$.

The inequality (6.3) means that L *admits a spectral gap* λ_1 *of size at least* K,

$$\lambda_1 \geq K.$$

In the sharp version, corresponding to the case where $(\mathsf{M},\mathsf{d},\mathsf{m})$ *satisfies* $\mathsf{CD}(K,N)$, *the spectral gap is bounded from below by* $K\frac{N}{N-1}$.

7 Universal Coverings of Metric Measure Spaces

7.1 Coverings and Liftings

Let us recall some basic definitions and properties of coverings of metric (or more generally, topological) spaces. For further details we refer to [BBI01].

Definition 7.1 (Covering). *(i) Let* E *and* X *be topological spaces and* $p:E\to X$ *a continuous map. An open set* $V\subseteq X$ *is said to be evenly covered by* p *if and only if its inverse image* $p^{-1}(V)$ *is a disjoint union of sets* $U_i\subseteq E$ *such that the restriction of* p *to* U_i *is a homeomorphism from* U_i *to* V *for each* i *in a suitable indexset* I. *The map* p *is a covering map (or simply covering) if and only if every point* $x\in X$ *has an evenly covered neighborhood. In this case, the space* X *is called the base of the covering and* E *the covering space.*

(ii) A covering map $p:E\to X$ *is called a universal covering if and only if* E *is simply connected. In this case,* E *is called universal covering space for* X.

The existence of a universal covering is guaranteed under some weak topological assumptions. More precisely:

Theorem 7.2. *If a topological space* X *is connected, locally pathwise connected and semi-locally simply connected, then there exists a universal covering* $p:E\to X$.

For the exact meaning of the assumptions we again refer to [BBI01].

Example 7.3. *(i) The map* $p:\mathbb{R}\to\mathsf{S}^1$ *given by* $p(x)=(\cos(x),\sin(x))$ *is a covering map.*

(ii) The universal covering of the torus by the plane $P:\mathbb{R}^2\to\mathsf{T}^2:=\mathsf{S}^1\times\mathsf{S}^1$ *is given by* $P(x,y):=(p(x),p(y))$ *where* $p(x)=(\cos(x),\sin(x))$ *is defined as in (i).*

We consider a covering $p:E\to X$. For $x\in X$ the set $p^{-1}(x)$ is called the *fiber* over x. This is a discrete subspace of E and every $x\in X$ has a neighborhood V such that $p^{-1}(V)$ is homeomorphic to $p^{-1}(x)\times V$. The disjoint subsets of $p^{-1}(V)$ mapped homeomorphically onto V are called the *sheets* of $p^{-1}(V)$. If V is connected, the sheets of $p^{-1}(V)$ coincide with the connected components of $p^{-1}(V)$. If E and X are connected, the cardinality of $p^{-1}(x)$ does not depend on $x\in X$ and is called the *number of sheets*. This number may be infinity.

Every covering is a local homeomorphism which implies that E and X have the same local topological properties.

Remark 7.4. *Consider length spaces (E, d_E) and (X, d_X) and a covering map $p : E \to X$ which is additionally a local isometry. If X is complete, then so is E.*

We list two essential lifting statements in topology referring to [BS] for further details and the proofs.

Definition 7.5. *Let $\alpha, \beta : [0,1] \to X$ be two curves in X with the same end points meaning that $\alpha(0) = \beta(0) = x_0 \in X$ and $\alpha(1) = \beta(1) = x_1 \in X$. We say that α and β are homotopic relative to $\{0,1\}$ if and only if there exists a continuous map $H : [0,1] \times [0,1] \to X$ satisfying $H(t,0) = \alpha(t)$, $H(t,1) = \beta(t)$ as well as $H(0,t) = x_0$ and $H(1,t) = x_1$ for all $t \in [0,1]$. We call H a homotopy from α to β relative to $\{0,1\}$.*

Theorem 7.6 (Path lifting theorem)**.** *Let $p : E \to X$ be a covering and let $\gamma : [0,1] \to X$ be a curve in X. We assume that $e_0 \in E$ satisfies $p(e_0) = \gamma(0)$. Then there exists a unique curve $\alpha : [0,1] \to E$ such that $\alpha(0) = e_0$ and $p \circ \alpha = \gamma$.*

Theorem 7.7 (Homotopy lifting theorem)**.** *Let $p : E \to X$ be a covering, let $\gamma_0, \gamma_1 : [0,1] \to X$ be two curves in X with starting point $x_0 \in X$ and terminal point $x_1 \in X$, and let $\alpha_0, \alpha_1 : [0,1] \to E$ be the lifted curves such that $\alpha_0(0) = \alpha_1(0)$. Then every homotopy $H : [0,1] \times [0,1] \to X$ from γ_0 to γ_1 relative to $\{0,1\}$ can be lifted in a unique way to a homotopy $H' : [0,1] \times [0,1] \to E$ from α_0 to α_1 relative to $\{0,1\}$ satisfying $H'(0,0) = \alpha_0(0) = \alpha_1(0)$.*

We consider a universal covering $p : E \to X$ and distinguished points $x_0 \in X$ as well as $e_0 \in p^{-1}(x_0) \subseteq E$. The above lifting theorems enable us to define a function

$$\Phi : \pi_1(X, x_0) \to p^{-1}(x_0)$$

such that for $[\gamma] \in \pi_1(X, x_0)$, $\Phi([\gamma])$ is the (unique) terminal point of the lift of γ to E starting at e_0. Then Φ has the following property:

Theorem 7.8 (Cardinality of fibers)**.** *The function Φ is a one-to-one correspondence of the fundamental group $\pi_1(X, x_0)$ and the fiber $p^{-1}(x_0)$.*

7.2 Lifted Metric Measure Spaces

We consider now a non-branching metric measure space $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ satisfying the reduced curvature-dimension condition $\mathsf{CD}^*(K, N)$ locally for two real parameters $K > 0$ and $N \geq 1$ and a distinguished point $x_0 \in \mathsf{M}$. Moreover, we assume that (M, d) is a semi-locally simply connected length space. Then, according to Theorem 7.2, there exists a universal covering $p : \hat{\mathsf{M}} \to \mathsf{M}$. The covering space $\hat{\mathsf{M}}$ inherits the length structure of the base M in the following way: We say that a curve $\hat{\gamma}$ in $\hat{\mathsf{M}}$ is "admissible" if and only if its composition with p is a continuous curve in M. The length $\mathsf{Length}(\hat{\gamma})$ of an admissible curve in $\hat{\mathsf{M}}$ is set to the length of $p \circ \hat{\gamma}$ with respect to the length structure in M. For two points $x, y \in \hat{\mathsf{M}}$ we define the associated distance $\hat{\mathsf{d}}(x,y)$ between them to be the infimum of lengths of admissible curves in $\hat{\mathsf{M}}$ connecting these points:

$$\hat{\mathsf{d}}(x,y) := \inf\{\mathsf{Length}(\hat{\gamma}) | \hat{\gamma} : [0,1] \to \hat{\mathsf{M}} \text{ admissible}, \hat{\gamma}(0) = x, \hat{\gamma}(1) = y\}. \tag{7.1}$$

Endowed with this metric, $p : (\hat{\mathsf{M}}, \hat{\mathsf{d}}) \to (\mathsf{M}, \mathsf{d})$ is a local isometry.

Now, let ξ be the family of all sets $\hat{E} \subseteq \hat{\mathsf{M}}$ such that the restriction of p onto $\hat{E}$ is a local isometry from $\hat{E}$ to a measurable set $E := p(\hat{E})$ in M. This family ξ is stable under intersections, and the smallest σ-algebra $\sigma(\xi)$ containing ξ is equal to the Borel-σ-algebra $\mathcal{B}(\hat{\mathsf{M}})$ according to the local compactness of $(\hat{\mathsf{M}}, \hat{\mathsf{d}})$. We define a function $\hat{\mathsf{m}} : \xi \to [0, \infty[$ by $\hat{\mathsf{m}}(\hat{E}) = \mathsf{m}(p(\hat{E})) = \mathsf{m}(E)$ and extend it in a unique way to a measure $\hat{\mathsf{m}}$ on $(\hat{\mathsf{M}}, \mathcal{B}(\hat{\mathsf{M}}))$.

Definition 7.9. *(i) We call the metric $\hat{\mathsf{d}}$ on $\hat{\mathsf{M}}$ defined in (7.1) the lift of the metric d on M.*

(ii) The measure $\hat{\mathsf{m}}$ *on* $(\hat{\mathsf{M}}, \mathcal{B}(\hat{\mathsf{M}}))$ *constructed as described above is called the lift of* m.

(iii) We call the metric measure space $(\hat{\mathsf{M}}, \hat{\mathsf{d}}, \hat{\mathsf{m}})$ *the lift of* $(\mathsf{M}, \mathsf{d}, \mathsf{m})$.

Theorem 7.10 (Lift)**.** *Assume that* $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *is a non-branching metric measure space satisfying* $\mathsf{CD}^*_{\mathsf{loc}}(K, N)$ *for two real parameters* $K > 0$ *and* $N \geq 1$ *and that* (M, d) *is a semi-locally simply connected length space. Let* $\hat{\mathsf{M}}$ *be a universal covering space for* M *and let* $(\hat{\mathsf{M}}, \hat{\mathsf{d}}, \hat{\mathsf{m}})$ *be the lift of* $(\mathsf{M}, \mathsf{d}, \mathsf{m})$. *Then,*

(i) $(\hat{\mathsf{M}}, \hat{\mathsf{d}}, \hat{\mathsf{m}})$ *has compact support and its diameter has an upper bound*

$$\mathsf{diam}(\hat{\mathsf{M}}, \hat{\mathsf{d}}, \hat{\mathsf{m}}) \leq \pi \sqrt{\frac{N}{K}}.$$

(ii) The fundamental group $\pi_1(\mathsf{M}, x_0)$ *of* $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ *is finite.*

Proof. (i) Due to the construction of the lift, the local properties of $(\mathsf{M}, \mathsf{d}, \mathsf{m})$ are transferred to $(\hat{\mathsf{M}}, \hat{\mathsf{d}}, \hat{\mathsf{m}})$. That means, $(\hat{\mathsf{M}}, \hat{\mathsf{d}}, \hat{\mathsf{m}})$ is a non-branching metric measure space $(\hat{\mathsf{M}}, \hat{\mathsf{d}}, \hat{\mathsf{m}})$ satisfying $\mathsf{CD}^*(K, N)$ locally. Theorem 5.1 implies that $(\hat{\mathsf{M}}, \hat{\mathsf{d}}, \hat{\mathsf{m}})$ satisfies $\mathsf{CD}^*(K, N)$ globally and therefore, the diameter estimate of Bonnet-Myers – Corollary 6.3 – can be applied.

(ii) If the fundamental group $\pi_1(\mathsf{M}, x_0)$ were infinite then the support of $\hat{\mathsf{m}}$ could not be compact according to Theorem 7.8.

□

Remark 7.11. *Note that there exists a universal cover for any Gromov-Hausdorff limit of a sequence of complete Riemannian manifolds with a uniform lower bound on the Ricci curvature [SW04]. The limit space may have infinite topological type [Men00].*

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