

Advanced Topics in Analysis and the Calculus of Variations (V5B5)

Mathematical problems in nonlinear elasticity

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Monday 8-10 ct, Thursday 10-12 ct
Room N 0.007 (Endenicher Allee 60/ Neubau)

First Lecture Thursday, October 15, 10.15h

Content The fundamental problem of nonlinear elasticity is to study the existence and properties of minimizers or stationary points of the elastic energy functional

$$I(u) = \int_{\Omega} W(Du(x)) \, dx.$$

Here $\Omega \subset \mathbb{R}^n$ is an open and bounded set (the so-called reference configuration of the elastic body) and $u : \Omega \rightarrow \mathbb{R}^n$ is a deformation. The stored energy density function W is a map from the space $\mathbb{R}^{n \times n}$ of $n \times n$ matrices to $\mathbb{R} \cup \{\infty\}$. This map has some physically natural properties which make the mathematical analysis quite subtle. In particular W is frame-indifferent, i.e. $W(RF) = W(F)$ for all $R \in SO(n)$, and $W(F) = \infty$ if $\det F \leq 0$. These seemingly harmless properties in particular imply in particular that W cannot be a convex function. This rules out the use of 'soft' functional analytic methods.

Indeed, the study of the functional I led to interesting relations with a number of other areas of mathematics, including geometric function theory, the theory of quasiconformal mappings, compensated compactness and the study of rigidity and flexibility phenomena.

The course will begin with the fundamental theorem of Ball [2] on the existence of minimizers. We will then discuss a number of solved and unsolved problems [3], including global invertibility of minimizers, regularity,

validity of the Euler-Lagrange equations for minimizers and the existence and analysis of highly oscillatory solutions in case W describes a materials which undergoes a solid-solid phase transformation.

If time allows, the course will also give an introduction to the behaviour of thin elastic objects which show a strikingly different behaviour than bulk elastic materials [1, 4]. They can easily undergo large deformations, form a fascinating variety of complex patterns and often provide optimal structures. For this reason they play a fundamental role in engineering as well as in biology. From a mathematical point of view thin elastic structure are also very interesting in view of their universal behaviour which is dictated by geometry rather than the specific properties of a concrete material.

Prerequisites Sobolev spaces, some knowledge in partial differential equations and functional analysis.

References

- [1] B. Audoly and Y. Pomeau, Elasticity and geometry - from hair curls to the non-linear response of shells, Oxford Univ. Press, 2010.
- [2] J.M. Ball, Convexity conditions and existence theorems in nonlinear elasticity, Archive for Rational Mechanics and Analysis, **63** (1977), 337–403.
- [3] J.M. Ball, Some open problems in elasticity, Geometry, mechanics, and dynamics, pp. 3–59, Springer, 2002
- [4] S. Müller, Mathematical problems in thin elastic sheets: scaling limits, packing, crumpling and singularities, in: Vector-valued partial differential equations and applications, pp. 125–193, Lecture Notes in Math. **2179**, Springer, Cham, 2017.