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Partial differential equations and modelling

Problem Sheet 1

Due Friday, April 18, 2025

Problem 1 (2+2+4 points). *Rigid motions*

(i) Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^n$ and suppose that $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $|\varphi(x) - \varphi(y)| = |x - y|$ for all $x, y \in \mathbb{R}^n$ and $\varphi(0) = 0$. Show that φ is linear.
(ii) Let $U \subset \mathbb{R}^n$ be open and let $\varphi : U \to \mathbb{R}^n$ be a C^1 map such that $|\varphi(x) - \varphi(y)| = |x - y|$ for all $x, y \in \mathbb{R}^n$. Show that $D\varphi$ is constant.
(iii) Let $U \subset \mathbb{R}^n$ be open and connected. Let $\varphi : U \to \mathbb{R}^n$ be a C^1 map such that $D\varphi(x) \in SO(n)$ for all $x \in U$. Show that for every $x_0 \in U$ there exists a $r > 0$ such that $|\varphi(x) - \varphi(y)| = |x - y|$ for all $x, y \in B_r(x_0)$. Show further that $\varphi(x) = Ax + b$ for some $A \in SO(n)$.

Hints: (i) Show first that $\varphi((1 - \lambda)x + \lambda y) = (1 - \lambda)\varphi(x) + \lambda\varphi(y)$, for $\lambda \in (0, 1)$. To do so, show that the closed balls $\overline{B}_r(x')$ and $\overline{B}_s(y')$ have a unique intersection point if $r, s > 0$ and $r + s = |x' - y'|$.
(ii) Show first that $(D\varphi(y))^T D\varphi(x) = Id$ for all x, y by differentiating the identity $|\varphi(x) - \varphi(y)|^2 = |x - y|^2$ with respect to x_i and y_j. Alternatively, reduce the problem to (i).
(iii) You may use the inverse function theorem for the bound $|\varphi(x) - \varphi(y)| \geq |x - y|$.

Problem 2 (8 points). *Linearized rigid motion*

A linearized rigid motion on a body $\mathcal{B} \subset \mathbb{R}^n$ is a vectorfield u on $\mathcal{B}$ (i.e., a map $u : \mathcal{B} \to \mathbb{R}^n$) such that ∇u is constant and skew-symmetric.

Let u be a smooth vectorfield on $\mathcal{B}$ and assume that $\mathcal{B}$ is open and connected. Show that the following statements are equivalent:

(a) u is a linearized rigid motion
(b) for all $p, q \in \mathcal{B}$ one has $(p - q) \cdot [u(p) - u(q)] = 0$.
(c) $\nabla u(p)$ is skew-symmetric for each $p \in \mathcal{B}$

Hints: Consider also the following seemingly weaker statement:
(b') For all $p \in \mathcal{B}$ there exists an $r > 0$ such that $(p - q) \cdot [u(p) - u(q)] = 0$ for all $q \in B_r(p)$.
(a) $\Longrightarrow$ (c) and (b)$\Longrightarrow$ (b') are trivial and (a) $\Longrightarrow$ (b) is easy.
For (b') $\Longrightarrow$ (a) you can argue similarly to Problem 1 (ii).
For (b') $\Longleftrightarrow$ (c) you can consider a function of the form $f(t) = (p - q) \cdot u(h(t))$.
Alternatively, for (c) $\Longrightarrow$ (a) you can start from the identity $\partial_j u_i + \partial_i u_j = 0$, differentiate with respect to x_k and use the three versions of the resulting equation obtained by cyclically permuting i, j, k.