

To be handed in through eCampus on 28.10.26.

**Exercise 1** (4 points)

Show that

- (i)  $l_p(\mathbb{R})$  is separable for  $1 \leq p < \infty$ ,
- (ii)  $l_\infty(\mathbb{R})$  is not separable.

**Exercise 2** (4 points)

Let  $(X, d)$  be a metric space ( $X \neq \emptyset$ ),  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{K} = \mathbb{C}$  and consider the Banach space  $(\mathcal{B}(X), \|\cdot\|_\infty)$  of bounded functions on  $X$ , i.e.

$$\mathcal{B}(X) := \{f: X \rightarrow \mathbb{K} \mid \|f\|_\infty < \infty\} \quad \text{with} \quad \|f\|_\infty := \sup_{x \in X} |f(x)|.$$

Fix  $x_0 \in X$  and consider  $L: (X, d) \rightarrow (\mathcal{B}(X), \|\cdot\|_\infty)$  where for  $x \in X$  we define

$$L(x)(y) := d(x, y) - d(x_0, y) \quad \text{for all } y \in X.$$

- (i) Show that  $L$  is well-defined, i.e.  $\|L(x)\|_\infty < \infty$  for all  $x \in X$ .
- (ii) Show that  $L$  is an isometry, i.e.  $\|L(x) - L(y)\|_\infty = d(x, y)$  for all  $x, y \in X$ .

**Exercise 3** (4 points)

Let  $\Omega \subset \mathbb{R}^n$  be open and bounded,  $0 < \alpha < \beta < 1$ . Prove the following statements:

- (i)  $C^{0,\beta}(\Omega) \subset C^{0,\alpha}(\Omega)$  and there exists a constant  $C = C(\Omega, \alpha, \beta)$  such that  $\|f\|_{0,\alpha} \leq C \|f\|_{0,\beta}$  for all  $f \in C^{0,\beta}(\Omega)$ .
- (ii) The inclusion  $C^{0,\beta}(\Omega) \subset C^{0,\alpha}(\Omega)$  is strict.
- (iii) There exists  $f \in C^0(\Omega)$  such that  $f \notin C^{0,\alpha}(\Omega)$  for all  $0 < \alpha < 1$ .

**Exercise 4** (4 points)

We consider the space  $\mathcal{B}(\mathbb{N}, \mathbb{R})$  of bounded real sequences (we use the convention that  $0 \notin \mathbb{N}$ ). We define

$$\|a\|_\infty = \sup_{n \in \mathbb{N}} |a_n|, \quad \|a\|_{\infty,-1} = \sup_{n \in \mathbb{N}} \frac{|a_n|}{n}$$

to get the following normed spaces

$$E_0 = (\mathcal{B}(\mathbb{N}, \mathbb{R}), \|\cdot\|_\infty), \quad E_{-1} = (\mathcal{B}(\mathbb{N}, \mathbb{R}), \|\cdot\|_{\infty,-1}),$$

and consider the following maps:

$$(M(a))_j = j^{-1} \sum_{k=1}^j a_k \quad (\text{the average of the entries})$$

$$(F(a))_j = a_j a_{2j} \quad (\text{product of entries})$$

- (i) Prove that  $M$  is not continuous from  $E_{-1}$  to  $E_0$ .
- (ii) Prove that  $F$  is continuous from  $E_0$  to  $E_0$ .