

Summary of the course

Functional analysis and partial differential equations

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This is only a summary of the main results and arguments discussed in class and *not* a complete set of lecture notes. These notes can thus not replace the careful study of the literature. Among others the two books [Alt, Brezis] are recommended.

These notes are based on the books mentioned above, lecture notes by Stefan Müller from fall 2012–2013 and further sources which are not always mentioned specifically. These notes are only for the use of the students in the class 'Functional analysis and partial differential equations' at the University of Bonn, Fall term 2027–2027.

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References

- [Alt] H.W. Alt, Lineare Funktionalanalysis, 5. Auflage, Springer, 2006.
- [Brezis] H. Brezis, Functional analysis, Sobolev spaces and partial differential equations, Springer, 2011.

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1 Structures

We consider the following increasingly richer structures on a set or a vector space.

- Open sets (topological spaces)
- Distance (metric spaces)
- Length in a vector space (normed spaces, Banach spaces)
- Length and angle/ scalar product in a vector space (Pre-Hilbert spaces, Hilbert spaces)

In these notes **all vector spaces will be vector spaces of the fields**

$$\mathbb{K} = \mathbb{R} \quad \text{or} \quad \mathbb{K} = \mathbb{C}.$$

1.1 Topological spaces

Let X be a set. Then 2^X denotes the set of all subsets of X (including the empty set).

Definition 1.1. A topological space $(X, \mathcal{T})$ is pair consisting of a set X and a subset $\mathcal{T}$ of 2^X with the following properties.

- (i) $\emptyset \in \mathcal{T}, X \in \mathcal{T}$.
- (ii) If $U \in \mathcal{T}, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$.
- (iii) If Λ is an arbitrary set and $U_\lambda \in \mathcal{T}$ for all $\lambda \in \Lambda$ then $\bigcup_{\lambda \in \Lambda} U_\lambda \in \mathcal{T}$.

Notions The set $\mathcal{T}$ is called a topology on X . If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X then $\mathcal{T}_1$ is called *finer* (or *stronger*) than $\mathcal{T}_2$ if $\mathcal{T}_1 \supset \mathcal{T}_2$. In this case $\mathcal{T}_2$ is called *coarser* (or *weaker*) than $\mathcal{T}_1$.

A set $A \subset X$ is called *open* if $A \in \mathcal{T}$. It is called *closed* if the complement $A^c := X \setminus A$ is open. By definition of a topological space a finite intersection and an arbitrary union of open sets is open. It follows from the formula $(\bigcap_{\lambda \in \Lambda} A_\lambda)^c = \bigcup_{\lambda \in \Lambda} A_\lambda^c$ that a finite union and an arbitrary intersection of closed sets is closed.

For $x \in X$ a set $U \subset X$ is an *open neighbourhood* of x if $U \in \mathcal{T}$ and $x \in U$.

Definition 1.2 (Continuous maps). Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let f be a map from X to Y . Then f is continuous if

$$V \in \mathcal{T}_Y \implies f^{-1}(V) \in \mathcal{T}_X,$$

i.e. if the preimage of every open set is open.

The map f is continuous at a point $x \in X$ if for every $V \in \mathcal{T}_Y$ with $f(x) \in V$ there exists $U \in \mathcal{T}_X$ such that $x \in U$ and $f(U) \subset V$.

The map f is called a *homeomorphism* if f is bijective and f and f^{-1} are continuous.

Notation $f^{-1}(V) := \{x \in X : f(x) \in V\}$. Note the properties

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B), \quad f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B), \quad f^{-1}(Y \setminus A) = X \setminus f^{-1}(A).$$

Definition 1.3 (Hausdorff space). *A topological space $(X, \mathcal{T})$ is called Hausdorff if for different points there exist disjoint open neighbourhoods, i.e.,*

$$x \neq y \implies \exists U_x, U_y \in \mathcal{T} \text{ such that } x \in U_x, y \in U_y \text{ and } U_x \cap U_y = \emptyset.$$

1.2 Metric spaces

Definition 1.4. *A pair (X, d) is called a metric space if $d : X \times X \rightarrow [0, \infty)$ has the following properties.*

- (i) (definiteness) $d(x, y) = 0 \Leftrightarrow x = y$,
- (ii) (symmetry) $d(y, x) = d(x, y)$ for all $x, y \in X$,
- (iii) (triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

A map $d : X \times X \rightarrow [0, \infty)$ with the above properties is called a metric on X . If $A \subset X$ and if d_A denotes the restriction of d to $A \times A$ then (A, d_A) is again a metric space.

Examples.

- (i) Let $X = \mathbb{R}^n$, $d_\infty(x, y) = \max_{i=1, \dots, n} |x_i - y_i|$.
- (ii) $X = \mathbb{R}^n$, $1 \leq p < \infty$, $d_p(x, y) = (\sum_{i=1}^n |x_i - y_i|^p)^{1/p}$. For $p = 2$ we obtain the standard Euclidean metric on $\mathbb{R}^n$ and we usually write $|x - y| := d_2(x, y)$.
- (iii) Let $1 \leq p < \infty$, $l_p := \{(x_k)_{k \in \mathbb{N}}, x_k \in \mathbb{R} : \sum_{k=1}^\infty |x_k|^p < \infty\}$, $d_p(x, y) := (\sum_{k=1}^\infty |x_k - y_k|^p)^{1/p}$. Then (l_p, d_p) is a metric space.
- (iv) Let l_∞ be the space of bounded sequences and $d_\infty(x, y) = \sup_{k \in \mathbb{N}} |x_k - y_k|$. Then (l_∞, d_∞) is a metric space.
- (v) (Pull-back metric) Let X be set, let (Y, d_Y) be a metric space and let $f : X \rightarrow Y$ be injective. Then $d_X(x_1, x_2) := d_Y(f(x_1), f(x_2))$ is a metric on X .
- (vi) Let $X = \mathbb{R} \cup \{-\infty\} \cup \{\infty\} = [-\infty, \infty]$, define $f : X \rightarrow [-1, 1]$ by

$$f(x) = \begin{cases} -1 & \text{if } x = -\infty \\ \frac{x}{1+|x|} & \text{if } x \in \mathbb{R} \\ 1 & \text{if } x = \infty \end{cases}$$

and set $d_X(x_1, x_2) = |f(x_1) - f(x_2)|$. By (v) the pair (X, d_X) is a metric space.

Semimetric If $d : X \times X \rightarrow [0, \infty)$ satisfies only (ii), (iii) and $d(x, x) = 0$ for all $x \in X$, then d is called a semimetric. If d is a semimetric one can define an equivalence relation by $x \sim y$ if and only if $d(x, y) = 0$. The quotient space $\tilde{X} := X / \sim$ consists of equivalence classes $[x] := \{z \in X : x \sim z\} = \{z \in X : d(z, x) = 0\}$. It follows from the triangle inequality that the expression $\tilde{d}([x], [y]) := d(x, y)$ is well-defined. Moreover one easily sees that $\tilde{d}$ is a metric (not just a semimetric) on $\tilde{X}$. In this way one can improve every semimetric to a metric by passing to equivalence classes. This idea was e.g. used in the definition of the Lebesgue spaces $L^p(X, \mu)$ which consist of equivalence classes of functions that only differ on sets of measure zero.

Diameter, distance of sets, balls

Let (X, d) be a metric space, let $A_1, A_2, A \subset X$, let $x \in X$ and let $r > 0$. We define the diameter of a set

$$\text{diam}(A) := \sup\{d(x, y) : x, y \in A\},$$

the distance of two sets

$$\text{dist}(A_1, A_2) := \inf\{d(x, y) : x \in A_1, y \in A_2\},$$

the distance of a point from a set

$$\text{dist}(x, A) := \inf\{d(x, y) : y \in A\},$$

the r -neighbourhood of a set

$$B_r(A) := \{y \in X : \text{dist}(y, A) < r\}$$

and the ball of radius r around x

$$B(x, r) := \{y \in X : d(x, y) < r\}.$$

Another notation is $B_r(x)$.

Definition 1.5 (Interior and closure). *Let (X, d) be a metric space and $A \subset X$. The interior A° is defined by*

$$A^\circ := \{x \in X : B_\varepsilon(x) \subset A \text{ for an } \varepsilon > 0\} \subset A, \quad (1.1)$$

the closure $\bar{A}$ by

$$\bar{A} := \{x \in X : B_\varepsilon(x) \cap A \neq \emptyset \text{ for all } \varepsilon > 0\} \supset A \quad (1.2)$$

and the boundary ∂A by

$$\partial A = \bar{A} \setminus A^\circ. \quad (1.3)$$

A subset $A \subset X$ is called dense if $\bar{A} = X$ and X is called separable if there exists a countable dense subset.

Example. If $M = \{u \in C^0([0, 1]) \mid u > 0\}$, then $M^\circ = M$ and $\bar{M} = \{u \in C^0([0, 1]) \mid u \geq 0\}$.

Proposition 1.6 (Topology induced by a metric). *Let (X, d) be a metric space and let $\mathcal{T}_d$ consist of all the sets with the following property:*

$$\forall x \in U \exists \varepsilon > 0 \quad B_\varepsilon(x) \subset U.$$

Then $\mathcal{T}_d$ is a topology on X and $(X, \mathcal{T}_d)$ is a Hausdorff space.

Proof. Easy exercise. □

In the following we will always consider the topology $\mathcal{T}_d$ on (X, d) (unless stated otherwise) and we will often write only $\mathcal{T}$ instead of $\mathcal{T}_d$. One easily sees that $B(x, r)$ is open and we call this set the open ball of radius r around x . We define the closed ball of radius r by

$$\bar{B}(x, r) = \{y \in X : d(x, y) \leq r\}$$

If X is a normed space (see next subsection) then $\bar{B}(x, r)$ agrees with the closure of $B(x, r)$ of the open ball and $\partial B(x, r) = \{y \in X : d(x, y) = r\}$. In general the closure $\bar{B}(x, r)$ can be a strict subset of $B(x, r)$. Consider, e.g., the set $X = \{0, 1\}$ with the discrete metric $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$. Then $B(0, 1) = \{0\}$ is open and closed, but $\bar{B}(0, 1) = \{0, 1\}$.

Definition 1.7 (Convergence in a metric space). *Let (X, d) be a metric space and $(x_k)_{k \in \mathbb{N}} \subset X$. The sequence (x_k) is called a Cauchy-sequence, if*

$$d(x_k, x_l) \rightarrow 0 \quad \text{as } k, l \rightarrow \infty.$$

The point x is called the limit of (x_k) (we write $x_k \rightarrow x$) if $d(x_k, x) \rightarrow 0$ as $k \rightarrow \infty$.

The space (X, d) is called complete, if every Cauchy-sequence converges in X .

Examples.

- (i) Let $X = \mathbb{Q}$, equipped with the standard metric on $\mathbb{R}$. Then X is not complete.
- (ii) Let $1 \leq p \leq \infty$. Then the spaces (l_p, d_p) of sequences introduced after Definition 1.4 are complete.
- (iii) The space (l_1, d_∞) is not complete
- (iv) If (X, d) is a complete metric space and A is a dense subsets of X with $A \neq X$ then the metric space (A, d_A) , where d_A denotes the restriction of d to $A \times A$ is not complete.

[12.10.2026, Lecture 1]
[15.10.2026, Lecture 2]

A key feature in a metric space is that open sets (or equivalently closed sets) are completely characterized in terms of convergence of *sequences*. More precisely we have

Proposition 1.8. *Let (X, d) be a metric space and $A \subset X$. Then the following two statements are equivalent*

(i) A is closed, i.e. $X \setminus A \in \mathcal{T}_d$.

(ii) A is sequentially closed, i.e. for every sequence $(x_k) \subset A$ which converges to a point $x^* \in X$ we have $x^* \in A$.

Moreover for an arbitrary set $A \subset X$ the closure $\overline{A}$ agrees with the sequential closure, i.e.,

$$\overline{A} = \{x^* \in X : \exists \text{ sequence } (x_k) \subset A \text{ such that } \lim_{k \rightarrow \infty} x_k = x^*\}.$$

Proof. Exercise. □

Proposition 1.9. Let (X, d) be a complete metric and let $A \subset X$ be closed and let d_A be the restriction of d to $A \times A$. Then the space (A, d_A) is complete.

Proof. Let $(x_k) \subset A$ be a Cauchy sequence in A . Then x is also a Cauchy sequence in X and hence has a limit x_* . Since A is closed $x_* \in A$. Thus (A, d_A) is complete. □

In metric spaces continuity can also be characterized in terms of sequences.

Proposition 1.10 (Continuity in metric spaces). Let (X, d_X) and (Y, d_Y) be metric spaces and $f: X \rightarrow Y$. Then the following three statements are equivalent.

(i) f is continuous as a map from $(X, \mathcal{T}_{d_X})$ to $(Y, \mathcal{T}_{d_Y})$.

(ii) ($\varepsilon - \delta$ definition of continuity)

$$\forall x \in X \forall \varepsilon > 0 \exists \delta > 0 \quad d_X(z, x) < \delta \implies d_Y(f(z), f(x)) < \varepsilon.$$

(iii) (sequential continuity) For every $x \in X$ and every sequence $(x_k) \subset X$ with $x_k \rightarrow x$, we have $f(x_k) \rightarrow f(x)$.

Proof. Exercise, see Analysis 1. □

Definition 1.11. Let d_1 and d_2 be metrics on X . We say that d_1 is stronger than d_2 if the topology induced by d_1 is stronger than the one induced by d_2 . We say that d_1 and d_2 are equivalent if they induce the same topology (equivalently, if d_1 is stronger than d_2 and d_2 is stronger than d_1).

Proposition 1.12. Let d_1 and d_2 be metrics on X . Then the following statements are equivalent.

(i) The metric d_1 is stronger than d_2 .

(ii) The identity map $x \mapsto x$ is continuous as a map from $(X, \mathcal{T}_{d_1})$ to $(X, \mathcal{T}_{d_2})$.

(iii) Every sequence which converges in d_1 converges also in d_2 .

(iv) $\forall x \in X \forall \varepsilon > 0 \exists \delta > 0 \quad d_1(x, y) < \delta \implies d_2(x, y) < \varepsilon$

Proof. (i) $\iff$ (ii): this follows directly from the definition of continuity and Definition 1.11.

(ii) $\iff$ (iii): this follows since continuity and sequential continuity are equivalent.

(ii) $\iff$ (iv): this follows from the $\varepsilon - \delta$ characterization of continuity. □

Example. Let l_1 denote the spaces of summable sequences. The metric $d_1(x, y) := \sum_{k=0}^{\infty} |x_k - y_k|$ is stronger than the metric $d_{\infty}(x, y) = \sup_{k \in \mathbb{N}} |x_k - y_k|$ (since $d_{\infty} \leq d_1$), but the two metrics are not equivalent. To see this consider the sequence $(x^{(j)})$ where $x_k^{(j)} = 1/j$ for $k \leq j$ and $x_k^{(j)} = 0$ for $k > j$. Then $d_{\infty}(x^{(j)}, 0) \rightarrow 0$ as $j \rightarrow \infty$, but $d_1(x^{(j)}, 0) = 1$.

An important result states that every non-complete space can be seen as a dense subset in a complete metric space (up to isometry).

Definition 1.13 (Isometry). *Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \rightarrow Y$ is called an isometry if*

$$d(f(x), f(y)) = d(x, y).$$

Note that an isometry is automatically injective. Thus if f is an isometry then f is a bijective isometry from X to $f(X)$. The spaces X and $f(X)$ (equipped with the restriction of the metric d_Y) are indistinguishable as metric spaces. In particular a sequence $(x_k) \subset X$ is convergent in X if and only if $f \circ x$ is convergent in $f(X)$ and x is a Cauchy sequence in X if and only if $f \circ x$ is a Cauchy sequence in $f(X)$.

Theorem 1.14 (Completion). *Let (X, d) be a metric space. Then there exists a complete metric space $(\tilde{X}, \tilde{d})$ and an isometry $J : X \rightarrow \tilde{X}$ such that $J(X)$ is dense in $\tilde{X}$.*

Proof. Step 1: Construction of $\tilde{X}$ and $\tilde{d}$.

Let $X^{\mathbb{N}}$ denote the space of all sequences in X . Let $\hat{X} \subset X^{\mathbb{N}}$ denote the subspace of all Cauchy sequences. On $\hat{X}$ we introduce the following equivalence relation. We say that $x \sim y$ if $\lim_{j \rightarrow \infty} d(x_j, y_j) = 0$. Let $\tilde{X}$ denote the corresponding space of equivalence classes of Cauchy sequences. For two equivalence classes $[x], [y] \in \tilde{X}$ we define

$$\tilde{d}([x], [y]) := \lim_{j \rightarrow \infty} d(x_j, y_j).$$

Since x, y are Cauchy sequences, the triangle inequality implies that the limit on the right hand side exists. Similarly one shows that $\tilde{d}$ is well-defined, i.e. the right hand side only depends on the equivalence classes of the Cauchy sequences x and y . The properties of a metric follow easily.

Step 2: Completeness of $(\tilde{X}, \tilde{d})$.

Let $[x^{(k)}]$ be a Cauchy sequence in $\tilde{X}$. Since $x^{(k)}$ is a Cauchy sequence in X there exists $j_k \in \mathbb{N}$ such that

$$d(x_i^{(k)}, x_j^{(k)}) \leq \frac{1}{k} \quad \text{if } i, j \geq j_k.$$

Therefore

$$\begin{aligned} d(x_{j_k}^{(k)}, x_{j_l}^{(l)}) &\leq d(x_{j_k}^{(k)}, x_j^{(k)}) + d(x_j^{(k)}, x_j^{(l)}) + d(x_j^{(l)}, x_{j_l}^{(l)}) \\ &\leq \frac{1}{k} + d(x_j^{(k)}, x_j^{(l)}) + \frac{1}{l} \quad \text{if } j \geq j_k \text{ and } j \geq j_l \\ &\rightarrow \frac{1}{k} + \tilde{d}([x^{(k)}], [x^{(l)}]) + \frac{1}{l} \quad \text{as } j \rightarrow \infty. \end{aligned}$$

Let $(y_l) \subset X$ be the diagonal sequence defined by $y_l := x_{j_l}^{(l)}$. Then the previous estimate shows that $d(y_k, y_l) \leq \frac{1}{k} + \tilde{d}([x^{(k)}], [x^{(l)}]) + \frac{1}{l}$. Thus y is a Cauchy sequence. Moreover

$$\tilde{d}([x^{(l)}], [y]) = \lim_{k \rightarrow \infty} d(x_k^{(l)}, y_k) = \lim_{k \rightarrow \infty} d(x_k^{(l)}, x_{j_k}^{(k)}) \quad (1.4)$$

For $k \geq j_l$ we have

$$d(x_k^{(l)}, x_{j_k}^{(k)}) \leq d(x_k^{(l)}, x_{j_l}^{(l)}) + d(x_{j_l}^{(l)}, x_{j_k}^{(k)}) \leq \frac{1}{l} + d(y_l, y_k).$$

Taking first the limit $k \rightarrow \infty$ and then the limit $l \rightarrow \infty$ and using the fact that y is a Cauchy sequence we deduce from (1.4) that $\tilde{d}([x^{(l)}], [y]) \rightarrow 0$ as $l \rightarrow \infty$. Thus the Cauchy sequence $[x^{(l)}]$ converges to y . This finishes the proof that $(\tilde{X}, \tilde{d})$ is complete.

Step 3: Embedding of X .

The map J is defined by mapping a point $a \in X$ to the (equivalence class of) the constant sequence which takes the value a . Thus $\tilde{d}(J(a), J(b)) = d(a, b)$. To show that $J(X)$ is dense in $\tilde{X}$ let $[x] \in \tilde{X}$. Recall that $J(x_k)$ is (the equivalence class of) the constant sequence with value x_k . Then

$$\tilde{d}([x], J(x_k)) = \lim_{l \rightarrow \infty} d(x_l, x_k).$$

Since x is a Cauchy sequence we deduce that $\lim_{k \rightarrow \infty} \tilde{d}([x], J(x_k)) = 0$. Thus $J(X)$ is dense in $\tilde{X}$ (recall that by Proposition 1.8 the closure agrees with the sequential closure). $\square$

1.3 Normed spaces

Let X be a $\mathbb{K}$ vector space.

Definition 1.15 (Normed space). *A pair $(X, \|\cdot\|)$ is called a normed space if X is a $\mathbb{K}$ vector space and $\|\cdot\| : X \rightarrow [0, \infty)$ is a map with the following properties*

- (i) (Definiteness) $\|x\| = 0 \implies x = 0$.
- (ii) (Homogeneity) $\|\alpha x\| = |\alpha| \|x\| \quad \forall \alpha \in \mathbb{K}, x \in X$.
- (iii) (Triangle inequality) $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in X$.

The map $\|\cdot\|$ is called a norm. Note also that the homogeneity condition (ii) implies that $\|0\| = 0$. Thus condition (i) can also be written as $\|x\| = 0 \iff x = 0$.

A map $\|\cdot\| : X \rightarrow [0, \infty)$ which satisfies the second and third condition is called a seminorm. One can pass from a seminorm to a norm by considering the quotient space $\tilde{X} = X / \sim$, where $x \sim y$ if $\|x - y\| = 0$ with the norm $\|[x]\|_{\sim} := \|x\|$.

If $(X, \|\cdot\|)$ is a normed space then $d(x, y) = \|x - y\|$ is a metric on X . The notions of convergence, continuity and completeness on a normed space are defined using this metric.

It follows from the triangle inequality that

$$|\|x\| - \|y\|| \leq \|x - y\|,$$

that is the norm is Lipschitz-continuous with Lipschitz constant 1.

Definition 1.16 (Banach space). *A normed space is called a Banach space if it is complete under the above metric d .*

Example. For $1 \leq p \leq \infty$ the sequence spaces l_p are Banach spaces with norms $\|x\|_p := (\sum_{k=0}^{\infty} |x_k|^p)^{1/p}$ (for $p < \infty$) and $\|x\|_{\infty} := \sup_{k \in \mathbb{N}} |x_k|$, respectively.

Definition 1.17. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on the $\mathbb{K}$ vector space X . We say that $\|\cdot\|_1$ is stronger than $\|\cdot\|_2$ if the corresponding metric d_1 is stronger than d_2 . We say that the two norms are equivalent if the corresponding metrics are equivalent.

Proposition 1.18. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on the $\mathbb{K}$ vector space X . The norm $\|\cdot\|_1$ is stronger than $\|\cdot\|_2$ if and only if there exists a constant $C > 0$ such that

$$\|x\|_2 \leq C\|x\|_1 \quad \forall x \in X.$$

The two norms are equivalent if and only if there exists constant $c > 0$ and $C > 0$ such that

$$c\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1 \quad \forall x \in X.$$

Proof. It suffices to prove the first statement. To do so apply the characterization (iv) in Proposition 1.12 to the point $y = 0$ with $\varepsilon = 1$. Then the assertion follows with $C = \frac{1}{\delta}$ in view of the homogeneity of the norm. $\square$

[15.10.2026, Lecture 2]
[19.10.2026, Lecture 3]

1.4 Hilbert spaces

For $\alpha \in \mathbb{C}$ we denote by $\bar{\alpha}$ the complex conjugate.

Definition 1.19. Let X be a $\mathbb{K}$ vector space. A map $(x_1, x_2) \mapsto (x_1, x_2)_X$ from $X \times X$ to $\mathbb{K}$ is called a sesquilinear form if for all $x, y, x_1, x_2, y_1, y_2 \in X$ and all $\alpha \in \mathbb{K}$ we have

- (i) $(\alpha x, y)_X = \alpha(x, y)_X \quad (x, \alpha y)_X = \bar{\alpha}(x, y)_X,$
- (ii) $(x_1 + x_2, y)_X = (x_1, y)_X + (x_2, y)_X, \quad (x, y_1 + y_2)_X = (x, y_1)_X + (x, y_2)_X.$

A sesquilinear form is called symmetric if for all $x, y \in X$

$$(y, x)_X = \overline{(x, y)_X}.$$

A symmetric sesquilinear form is called positive semidefinite if

$$\forall x \in X \quad (x, x)_X \geq 0$$

A symmetric sesquilinear form is called positive definite if

$$\forall x \in X \quad (x, x)_X \geq 0 \quad \text{and} \quad (x, x)_X = 0 \Leftrightarrow x = 0.$$

Remark. If $\mathbb{K} = \mathbb{R}$ a sesquilinear form is bilinear.

One often writes only (x_1, x_2) instead of $(x_1, x_2)_X$. Another common notation is $\langle x_1, x_2 \rangle$.

Definition 1.20. A positive definite symmetric sesquilinear form on a $\mathbb{K}$ vector space is called a scalar product. If $(\cdot, \cdot)_X$ is a scalar product then the pair $(X, (\cdot, \cdot)_X)$ is called a pre-Hilbert space.

Lemma 1.21. Let X be a $\mathbb{K}$ vector space and let $(\cdot, \cdot)$ be a positive semidefinite symmetric sesquilinear form and set

$$\|x\| := \sqrt{(x, x)}.$$

Then

(i) (Homogeneity)

$$\|\alpha x\| = |\alpha| \|x\| \quad \forall \alpha \in \mathbb{K}, x \in X. \quad (1.5)$$

(ii) (Cauchy-Schwarz inequality)

$$|(x, y)| \leq \|x\| \|y\| \quad \forall x, y \in X. \quad (1.6)$$

(iii) (Triangle inequality)

$$\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in X.$$

(iv) (Parallelogram identity)

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \forall x, y \in X.$$

In particular $\|\cdot\|$ is a seminorm on X . If, in addition, $(\cdot, \cdot)_X$ is positive definite then $\|\cdot\|$ is a norm.

Proof. Simple exercise (see also Analysis I+II) □

Definition 1.22. A Hilbert space is a pre-Hilbert space which is complete under the norm induced by the scalar product.

2 Function spaces

2.1 Spaces of bounded, continuous and differentiable functions

2.1.1 Bounded functions

Definition 2.1. Let X be a set, let $(Y, \|\cdot\|)$ be a normed space. Then the space of bounded functions $B(X; Y)$ is defined as

$$B(X; Y) := \{f : X \rightarrow Y : \|f\|_\infty := \sup\{\|f(x)\| : x \in X\} < \infty\}.$$

Proposition 2.2. Let Y be a Banach space. Then $(B(X; Y), \|\cdot\|_\infty)$ is a Banach space.

Remark. An important special case is $Y = \mathbb{R}$ or $Y = \mathbb{C}$. In this case we usually write $B(X)$ instead of $B(X; \mathbb{K})$.

Proof. It is easy to see that $\|\cdot\|_\infty$ is a norm so we only need to verify completeness. Let $(f_k) \subset B(X; Y)$ be a Cauchy sequence, i.e.,

$$\forall \delta > 0 \exists k_0(\delta) \forall j, k \geq k_0 \sup_{x \in X} \|f_j(x) - f_k(x)\| < \delta. \quad (2.1)$$

Taking $\delta = 1$ we deduce in particular that $\|f_j\|_\infty \leq M := \|f_{k_0(1)}\|_\infty + 1$ for all $j \geq k_0(1)$. Moreover for each $x \in X$ the sequence $(f_j(x))$ is a Cauchy sequence. Since Y is complete this sequence has a limit which we call $f(x)$. We have $\|f(x)\| \leq M + 1$ and thus the map $f : X \rightarrow Y$ belongs to $B(X; Y)$.

We finally show that $\lim_{j \rightarrow \infty} \|f_j - f\|_\infty = 0$. We have $\lim_{k \rightarrow \infty} \|f_k(x) - f(x)\| = 0$ and it thus follows from (2.1) that

$$\forall j \geq k_0(\delta) \quad \|f_j(x) - f(x)\| \leq \delta.$$

Since k_0 depends only on δ and not on x we get

$$\forall j \geq k_0(\delta) \quad \|f_j - f\|_\infty = \sup_{x \in X} \|f_j(x) - f(x)\| \leq \delta.$$

Thus $\limsup_{j \rightarrow \infty} \|f_j - f\|_\infty \leq \delta$. Since $\delta > 0$ was arbitrary this finishes the proof. $\square$

2.1.2 Continuous functions

Definition 2.3. Let $U \subset \mathbb{R}^n$ and Y be a normed space. The space of continuous functions from U to Y is defined as

$$C(U; Y) := \{f : U \rightarrow Y : f \text{ continuous}\}.$$

Another common notation is $C^0(U; Y)$.

The space of bounded and continuous functions on U is defined as

$$C_b(U; Y) := C(U; Y) \cap B(U; Y).$$

Remark. If $Y = \mathbb{R}$ or $Y = \mathbb{C}$ we usually write $C(U)$ and $C_b(U)$ respectively.

Theorem 2.4. Let Y be a Banach space. The space $C_b(U; Y)$ equipped with the norm

$$\|f\|_\infty := \sup\{\|f(x)\| : x \in U\}.$$

is a Banach space.

Proof. The space $C_b(U; Y)$ is a subspace of $B(U; Y)$. In view of Proposition 1.9 it suffices to show that $C_b(U; Y)$ is a closed subset. Assume that $f_k \in C_b(U; Y)$, $f \in B(U; Y)$ and $\lim_{k \rightarrow \infty} \|f_k - f\|_\infty = 0$. We need to show that f is continuous. For $x, y \in U$ we have

$$\|f(x) - f(y)\| \leq \|f_k(x) - f_k(y)\| + 2\|f - f_k\|_\infty.$$

For given $\varepsilon > 0$ we choose k so large such that $2\|f_k - f\|_\infty < \varepsilon/2$ and then $\delta > 0$ such that $\|f_k(x) - f_k(y)\| < \varepsilon/2$ for $|x - y| < \delta$. $\square$

Theorem 2.5. *Let $S \subset \mathbb{R}^n$ be compact. Then $C(S)$ is separable.*

Proof. Step 1: Uniform continuity.

Let $f \in C(S)$. It is known from Analysis I that f is uniformly continuous on S , that is for every $\varepsilon > 0$ there exists a $\delta_f > 0$ such that

$$|x - y| < \delta_f \implies |f(x) - f(y)| < \frac{\varepsilon}{2}. \quad (2.2)$$

Step 2: Partition of unity.

Let $\delta > 0$. Consider the lattice $L = (\frac{\delta}{4\sqrt{n}}\mathbb{Z})^n$. Then $\mathbb{R}^n = \bigcup_{x \in L} B(x, \delta/2)$. Since S is compact there exist finitely many balls $B(x_i, \delta/2)$ with $x_i \in L$ and $S \subset \bigcup_{i=1}^k B(x_i, \delta/2)$.

We will show that there exist finitely many continuous functions $\eta_i: S \rightarrow [0, 1]$ such that

$$\sum_i \eta_i(x) = 1 \quad \forall x \in S, \quad \eta_i = 0 \text{ on } \mathbb{R}^n \setminus B(x_i, \delta)$$

(such a family is called a partition of unity).

Let $\psi: \mathbb{R}^n \rightarrow [0, 1]$ be a continuous function such that $\psi \geq \frac{1}{2}$ on $B(0, \frac{1}{2})$ and $\psi = 0$ on $\mathbb{R}^n \setminus B(0, 1)$. Set $\psi_i(x) = \psi(\frac{x - x_i}{\delta})$. Then $\psi_i \geq 1/2$ on $B(x_i, \delta/2)$ and thus $\sum_{i=1}^k \psi_i(x) \geq 1/2$. Moreover $\psi_i = 0$ on $\mathbb{R}^n \setminus B(x_i, \delta)$. Hence the functions

$$\eta_i(x) := \frac{\psi_i(x)}{\sum_{j=1}^k \psi_j(x)}$$

have the desired properties.

Step 3: Approximation by a finite-dimensional space.

Let $f \in C(S)$, let $\varepsilon > 0$ and let δ_f be as in Step 1. Assume that $0 < \delta < \delta_f$ and let $B(x_i, \delta/2)$ and η_i be as in Step 2. Set

$$g(x) = \sum_{i=1}^k f(x_i) \eta_i(x).$$

We claim that $\|f - g\|_\infty \leq \varepsilon$. To see this recall that $\sum_i \eta_i(x) = 1$ and thus

$$f(x) - g(x) = \sum_{i=1}^k [f(x) - f(x_i)] \eta_i(x).$$

If $\eta_i(x) \neq 0$ then $|x - x_i| < \delta$ and thus $|f(x) - f(x_i)| < \varepsilon$. Thus

$$|f(x) - g(x)| < \sum_{i=1}^k \frac{\varepsilon}{2} \eta_i(x) \leq \varepsilon$$

Step 4: Approximation by a countable set.

Let $\varepsilon, \delta_f, \delta$ and η_i be as in Step 3, let $D \subset \mathbb{K}$ be countable and dense and set

$$A_\delta := \{g: S \rightarrow \mathbb{K} : g(x) = \sum_{i=1}^k d_i \eta_i(x), d_i \in D\}$$

Then A_δ is countable. There exist $d_i \in D$ such that $|f(x_i) - d_i| < \varepsilon$. Together with Step 3 it follows that there exists $g \in A_\delta$ with $\|f - g\|_\infty < 2\varepsilon$.

Step 5: Conclusion.

Finally set $A = \bigcup_{k=1}^\infty A_{\frac{1}{k}}$. Then A is countable. If $\varepsilon > 0$, $f \in C(S)$ and $k > 1/\delta_f$ then by Step 4 there exist $g \in A_k \subset A$ such that $\|f - g\|_\infty < 2\varepsilon$. Hence A is dense in $C(S)$. $\square$

Remark 2.6. In contrast to Theorem 2.5 one can show that $B(S)$ is not separable unless S is finite.

If $U \subset \mathbb{R}^n$ is open, then continuous maps $f: U \rightarrow \mathbb{K}$ need not be bounded. One can still define a metric on such continuous functions by exhausting U with compact sets.

Theorem 2.7. Let $U \subset \mathbb{R}^n$ be open and let Y be a normed space. Let $K_i \subset U$ be an increasing sequence of compact sets with $\bigcup_{i=1}^{\infty} K_i = U$ and assume that for every $x \in U$ there exists $r > 0$ and i such that $B(x, r) \subset K_i$. For $f \in C(U; Y)$ define

$$[f]_i := \sup_{x \in K_i} \|f(x)\|, \quad d(f, g) := \sum_{i=1}^{\infty} 2^{-i} \frac{[f - g]_i}{1 + [f - g]_i}. \quad (2.3)$$

Then $[\cdot]_i$ is a seminorm on $C(U; Y)$ and d is a metric. If Y is a Banach space then $(C(U; Y), d)$ is complete. If Y is separable then $(C(U; Y), d)$ is separable.

Remark.

- (i) The procedure to use countably many seminorms (or semimetrics) to obtain a metric as in (2.3) is used in a number of other situations. The metric d is sometimes called the Frechet metric generated by the seminorms $[\cdot]_i$.
- (ii) There exists no norm on $C(U; Y)$ such that the induced metric is equivalent to d .
- (iii) For each open set $U \subset \mathbb{R}^n$ there exists such a sequence K_i . One may take $K_i = \{x \in U : \text{dist}(x, \mathbb{R}^n \setminus U) \geq \frac{1}{i}, |x| \leq i\}$.
- (iv) Different choices of the sequence K_i lead to equivalent metrics d (compare Proposition 2.8 below).

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[22.10.2026, Vorlesung 4]

Proof. It is easy to see that $[\cdot]_i$ is a seminorm (note that $[f]_i < \infty$ since K_i is compact). Thus $d_i(f, g) = [f - g]_i$ is a semimetric. From this it easily follows that then $(f, g) \mapsto \frac{d_i(f, g)}{1 + d_i(f, g)}$ and d are also semimetrics (exercise). To see that d is actually a metric assume that $d(f, g) = 0$. Then $d_i(f, g) = 0$ for all i and hence $f = g$ on K_i . Since $U = \bigcup_{i=1}^{\infty} K_i$ we get $f = g$.

To prove completeness let (f_j) be a Cauchy sequence with respect to d . This implies that (f_j) is a Cauchy sequence for each d_i . It follows from the completeness of $C(K_i; Y)$ that there exist functions $g_i \in C(K_i; Y)$ such that $\lim_{j \rightarrow \infty} \sup_{x \in K_i} \|f_j(x) - g_i(x)\| = 0$. Since $K_i \subset K_{i+1}$ one easily sees that for $k < i$ we have $g_k = g_i|_{K_k}$. Thus we can define $g: X \rightarrow Y$ by $g(x) = g_i(x)$ if $x \in K_i$. By construction we have $\lim_{j \rightarrow \infty} d_i(f_j, g) = 0$ for each i . This implies that $d(f_j, g) \rightarrow 0$ (see Proposition 2.8). Finally we need to show that g is continuous. This follows from the fact that $g|_{K_i}$ is continuous and that for each $x \in U$ there exists a ball $B(x, r)$ which is contained in one K_i .

To prove separability let $\varepsilon > 0$ and let i_0 be such that $2^{-i_0+2} < \varepsilon$. Then for any set $d(f, g) \leq d_{i_0}(f, g) + \varepsilon$ and the result follows easily from the proof of Theorem 2.5. $\square$

Proposition 2.8. *Let $U \subset \mathbb{R}^n$ be open and let $(C(U; Y), d)$ be as in Theorem 2.7. Let $(f_j) \subset C(U; Y)$ be a sequence. Then the following statements are equivalent:*

- (i) $\lim_{j \rightarrow \infty} d(f_j, g) = 0$;
- (ii) $\forall i \quad \lim_{j \rightarrow \infty} d_i(f_j, g) = 0$;
- (iii) *for all compact sets $K \subset U$ $\lim_{j \rightarrow \infty} \sup_{x \in K} \|f_j(x) - g(x)\| = 0$.*

Remark. The convergence in (iii) is often called locally uniform convergence.

Proof. (iii) $\implies$ (ii) and (i) $\implies$ (ii): obvious.

(ii) $\implies$ (i): Let $\varepsilon > 0$. Let $2^{-i_0+2} < \varepsilon$. Then $d(f_j, g) \leq \sum_{i=1}^{i_0} \frac{d_i(f_j, g)}{1+d_i(f_j, g)} + \varepsilon/2$. By assumption there exist j_0 such that for $j \geq j_0$ we have $d_i(f_j, g) < \frac{\varepsilon}{2^{i_0}}$ for all $i \leq i_0$. Thus $d(f_j, g) < \varepsilon$ for $j \geq j_0$. This proves (i).

(ii) $\implies$ (iii): For each $x \in K$ there exists an $r > 0$ and an i such that $B(x, r) \subset K_i$. Since K is compact, finitely many of these balls $B(x_j, r_j) \subset K_j$ cover K . Since the sets K_i are increasing in i there exists i_0 such that $K \subset K_{i_0}$. By assumption $d_{i_0}(f_j, g) \rightarrow 0$. This implies (iii). $\square$

2.1.3 Differentiable functions

Let $U \subset \mathbb{R}^n$ be open, $f: U \rightarrow \mathbb{R}^m$ and $x \in U$. The i -th partial derivative of f in x is defined as

$$\partial_i f(x) = \lim_{h \rightarrow 0} \frac{f(x + h e_i) - f(x)}{h}.$$

Inductively one defines higher order partial derivatives $\partial_{ij}^2 f(x)$ etc.

If $\alpha \in \mathbb{N}^n$ is a multiindex and $|\alpha| := \sum_{i=1}^n \alpha_i$ its order, we define the partial derivative of f of order α as

$$\partial^\alpha f = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} f.$$

Furthermore we define for $l \in \mathbb{N}$

$$D^l f := \left(\partial^\alpha f \right)_{|\alpha|=l} \quad \text{and} \quad |D^l f(x)| := \left(\sum_{|\alpha|=l} |\partial^\alpha f(x)|^2 \right)^{1/2}.$$

By $C^k(U; \mathbb{R}^m)$ we denote the space of k times differentiable functions $f: U \rightarrow \mathbb{R}^m$ whose derivatives are continuous, that is

$$C^k(U; \mathbb{R}^m) := \left\{ f: U \rightarrow \mathbb{R}^m : f \text{ k-times differentiable, } \partial^\alpha f \text{ cont. on } U \forall |\alpha| \leq k \right\}.$$

Definition 2.9. *Let $U \subset \mathbb{R}^n$ be open and bounded. We define*

$$C^k(\overline{U}; \mathbb{R}^m) := \{ f \in C^k(U; \mathbb{R}^m) : D^l f \text{ has a continuous extension to } \overline{U} \quad \forall l \leq k \}$$

and, for $0 \leq l \leq k$,

$$[f]_{C^l(\overline{U})} := \sup_{x \in \overline{U}} |D^l f(x)|, \quad \|f\|_{C^k} := \sum_{l=0}^k [f]_{C^l}.$$

Moreover we set $C^\infty(\overline{U}; \mathbb{R}^m) = \bigcap_{k=0}^\infty C^k(\overline{U}; \mathbb{R}^m)$ and

$$d(f, g) = \sum_{l=0}^\infty 2^{-l} \frac{[f]_{C^l}}{1 + [f]_{C^l}}.$$

Theorem 2.10. For $0 \leq l \leq k$ the expression $[\cdot]_{C^l}$ is a seminorm on $C^k(\overline{U}; \mathbb{R}^m)$, the expression $\|\cdot\|_{C^k}$ is a norm and $(C^k(\overline{U}; \mathbb{R}^m), \|\cdot\|_{C^k})$ is a separable Banach space. Moreover $(C^\infty(\overline{U}; \mathbb{R}^m), d)$ is a separable and complete metric space.

Remark. One can combine this result and Theorem 2.7 to introduce metrics on $C^k(U; \mathbb{R}^m)$ and $C^\infty(U; \mathbb{R}^m)$ so that these spaces become complete and separable metric space. Moreover $d(f_j, g) \rightarrow 0$ if and only if $D^l f_j$ converges uniformly to $D^l g$ on all compact subsets of U and all $l \leq k$ or all $l \in \mathbb{N}$, respectively.

Proof. Banach space: Exercise.

To establish separability one can proceed in a similar way as in the proof of Theorem 2.5 as long as one chooses in addition $f \in C^\infty(\mathbb{R}^n)$ and one uses the k -th order Taylor expansion in the definition of the approximating functions.

The corresponding results for $C^\infty(\overline{U}; \mathbb{R}^m)$ then follow by using the relation between the Frechet metric d and the seminorms $[\cdot]_{C^l}$ as in the proof of Theorem 2.7. $\square$

Definition 2.11 (Functions with compact support). (i) Let $S \subset \mathbb{R}^n$, Y a Banach space, and $f: S \rightarrow Y$. The support of f is defined as

$$\text{supp } f := \overline{\{x \in S : f(x) \neq 0\}}.$$

(ii) The space of continuous functions with compact support is defined as

$$C_c(S; Y) := \{f \in C(S; Y) : \text{supp } f \subseteq S \text{ and } \text{supp } f \text{ compact}\}.$$

(iii) If $U \subset \mathbb{R}^n$ is open and $Y = \mathbb{R}^m$ we define

$$\begin{aligned} C_c^k(U; \mathbb{R}^m) &:= C^k(U; \mathbb{R}^m) \cap C_c(U; \mathbb{R}^m), \\ C_c^\infty(U; \mathbb{R}^m) &:= C^\infty(U; \mathbb{R}^m) \cap C_c(U; \mathbb{R}^m). \end{aligned}$$

Remarks.

- (i) For $m = 1$ we often write $C_c^k(U) := C_c^k(U, \mathbb{R})$. Some authors (e.g. H.W. Alt) write $C_0^k(U; \mathbb{R}^m)$ instead of $C_c^k(U; \mathbb{R}^m)$ etc.
- (ii) Define $f(x) = 1 - x$ if $x \in [0, 1)$ and $f(x) = 0$ if $x \in [1, \infty)$. Then $f \in C_c([0, \infty))$ and $f \in C_c([0, 1])$ but $f \notin C_c((0, 1))$ and $f \notin C_c([0, 1))$.
- (iii) The space $C_c(U; \mathbb{R}^m)$ is dense in $C(U; \mathbb{R}^m)$ (equipped with the Frechet metric defined above).

2.1.4 Hölder spaces

Interesting subspaces of the space of continuous functions arise when we consider functions with a given modulus of continuity $\rho: [0, \infty) \rightarrow [0, \infty)$, i.e., functions for which $\|f(x) - f(y)\| \leq C_f \rho(d(x, y))$ where $\lim_{t \downarrow 0} \rho(t) = 0$. The most important examples are Hölder continuous functions, which correspond to $\rho(t) = t^\alpha$. For simplicity we focus on functions on $A \subset \mathbb{R}^n$.

Definition 2.12. Let $\alpha \in (0, 1]$. Let $A \subset \mathbb{R}^n$. We say that $f: A \rightarrow \mathbb{R}^m$ is Hölder continuous with exponent α if f is continuous and

$$[f]_{\alpha, A} := \sup_{x, y \in A, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha} < \infty.$$

We define

$$C^{0, \alpha}(A; \mathbb{R}^m) := \{f \in C_b(A; \mathbb{R}^m) : [f]_{\alpha, A} < \infty\}, \quad \|f\|_{\alpha, A} := \sup_{x \in A} |f(x)| + [f]_{\alpha, A}.$$

If $U \subset \mathbb{R}^n$ is open and bounded then we define

$$C^{k, \alpha}(\bar{U}; \mathbb{R}^m) := \{f \in C^k(\bar{U}; \mathbb{R}^m) : [D^k f]_{\alpha, U} < \infty\}, \quad \|f\|_{k, \alpha, U} := \|f\|_{C^k(\bar{U}; \mathbb{R}^m)} + [D^k f]_{\alpha, U}$$

Remark.

- (i) Hölder continuous functions with exponent $\alpha = 1$ are called Lipschitz continuous and we write $\text{Lip}(f, A) := [f]_{1, A}$. Often the set A is dropped from the notation.
- (ii) If $U \subset \mathbb{R}^n$ is open, $f: U \rightarrow \mathbb{R}^m$ and $[f]_{\alpha, U} < \infty$ for some $\alpha > 1$ then f is differentiable at each point $x \in U$ and $Df(x) = 0$. If, in addition U is connected then f is constant in U . This is why we only consider Hölder exponents $\alpha \leq 1$.
- (iii) In analogy with Theorem 2.7 we can define a metric on the set $C_{loc}^{k, \alpha}(U) := \{f \in C^{k, \alpha}(K_i) : [D^k f]_{\alpha, K_i} < \infty \forall i\}$.

Proposition 2.13. The expression $[\cdot]_{\alpha, A}$ is a seminorm on $C^{0, \alpha}(A; \mathbb{R}^m)$ and the spaces $C^{0, \alpha}(A; \mathbb{R}^m)$ and $C^{k, \alpha}(\bar{U}; \mathbb{R}^m)$ are Banach spaces with the norms $\|\cdot\|_{\alpha, A}$ and $\|\cdot\|_{k, \alpha, U}$.

Remark. The space $C^{0, \alpha}(A; \mathbb{R}^m)$ is *not* separable (unless A is finite).

Proof. The assertions about the seminorms and norms are easy to prove. To verify completeness of $C^{0, \alpha}(A; \mathbb{R}^m)$ let (f_k) be a Cauchy sequence in that space. Then (f_k) is in particular a Cauchy sequence in $C_b(A; \mathbb{R}^m)$. As in Theorem 2.4 one proves that there exists $f \in C_b(A; \mathbb{R}^m)$ such that

$$\lim_{k \rightarrow \infty} \sup_{x \in A} \|f_k(x) - f(x)\| = 0. \quad (2.4)$$

Since (f_k) is a Cauchy sequence in $C^{0, \alpha}(A; \mathbb{R}^m)$ we also have

$$\forall \varepsilon > 0 \exists k_\varepsilon \forall j, k \geq k_\varepsilon \forall x, y \in A \quad |f_k(x) - f_j(x) - f_k(y) + f_j(y)| \leq \varepsilon |x - y|^\alpha. \quad (2.5)$$

Together with (2.4) this implies that

$$\forall \varepsilon > 0 \exists k_\varepsilon \forall j \geq k_\varepsilon \forall x, y \in A \quad |f(x) - f_j(x) - f(y) + f_j(y)| \leq \varepsilon |x - y|^\alpha. \quad (2.6)$$

Thus for all $\varepsilon > 0$ there exists k_ε such that $[f_k - f]_{\alpha, A} \leq \varepsilon$ if $k \geq k_\varepsilon$. Hence $[f_k - f]_{\alpha, A} \rightarrow 0$ and together with (2.4) this finishes the proof of completeness. Completeness of $C^{k, \alpha}(\bar{U}; \mathbb{R}^m)$ is shown similarly. $\square$