Functional Analysis

WS 2015/2016
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Problem Sheet 9.

Due 15.1.2016.

Problem 1. (Ranges of operators) (5+5 Points)

a) Give an example of $T \in \mathcal{L}(l_2(\mathbb{R}), l_2(\mathbb{R}))$ with $\mathcal{R}(T) \neq l_2(\mathbb{R})$ but $\overline{\mathcal{R}(T)} = l_2(\mathbb{R})$.
Hint: For the latter property it suffices to show that $\mathcal{R}(T)$ contains all sequences with only finitely many nonzero elements.

b) Let X be a Banach space, H be a Hilbert space and $K \in \mathcal{L}(X, H)$.Show that K is a compact operator (i.e. that $\overline{K(B(0,1))}$ is compact in H) if and only if there is a sequence $k \mapsto K_k$ of operators $K_k \in \mathcal{L}(X, H)$ with finite dimensional range such that $K_k \to K$ in $\mathcal{L}(X, H)$.
Hint: For $k \in \mathbb{N}$ choose a cover $\overline{K(B(0,1))} \subset \cup_{i=1}^{N(k)} B(y_i, \frac{1}{k})$ and set $H_k = span\{y_1, \ldots, y_{N_k}\}$. Denote by $P_k : H \to H_k$ the orthogonal projection, and set $K_k = P_k \circ K$.

Problem 2. (Limit of linear operators) (5+5 Points)

Let Y be a Banach space. Let X be a normed space and $D \subset X$ a dense subset.

a) Let $T \in \mathcal{L}(D, Y)$. Prove that there is a unique continuous extension $\tilde{T}$ onto X of T with $\tilde{T} \in \mathcal{L}(X, Y)$.

b) Let $k \mapsto T_k$ be a bounded sequence in $\mathcal{L}(X, Y)$. Assume that

$$\lim_{k\to\infty} T_k x$$

exists for all $x \in D$. Prove that

$$Tx := \lim_{k\to\infty} T_k x$$

exists for all $x \in X$ and $T \in \mathcal{L}(X, Y)$.

Problem 3. (The exponential map on operators) (2+2+2+2+2 Points)

Let X be a Banach space. We say that two operators $A, B \in \mathcal{L}(X)$ commute if $AB = BA$.

a) Find X and $A, B, C \in \mathcal{L}(X)$ such that A, B and B, C commute but A, C do not.

b) Let $A \in \mathcal{L}(X)$. Define $e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$. Show that A, e^A commute.

c) Show that if $A, B \in \mathcal{L}(X)$ commute, then $e^{A+B} = e^A e^B$.

d) Show that in general, $e^{A+B} \neq e^A e^B$.

e) Show that $\frac{d}{dt} e^{tA} = A e^{tA}$.

Problem 4. (Corollaries of Lax-Milgram) (5+5 Points)

Let X be a Hilbert space. Prove Corollaries 5.4 and 5.5 below:

a) Let $a : X \times X \to \mathbb{K}$ be a continuous coercive sesquilinear form. Show that for every $T \in X'$ there exists a unique $x_0 \in X$ such that $a(y, x_0) = T(y)$ for all $y \in X$. Moreover the map $T \mapsto x_0$ is conjugately linear and $\|x_0\|_X \leq c\,\|T\|_{X'}$, with c depending on a.

b) Let $A \in \mathcal{L}(X)$. If there exists $c_0 > 0$ with $\operatorname{Re}(Ax, x) \geq c_0\,\|x\|_X^2$, then A is invertible and $\|A^{-1}\|_{\mathcal{L}(X)} \leq \frac{1}{c_0}$.