Functional Analysis

WS 2015/2016
Prof. Dr. M. Disertori
P. Gladbach; R. Schubert

Problem Sheet 8.

Due 8.1.2016.

Problem 1. (Continuity of the projection) (10 Points)

Let X be a Hilbert space, $K \subseteq X$ nonempty, closed and convex. Show that the orthogonal projection $P : X \to A$ is continuous.
Hint: Show that $\operatorname{Re}(x - y, P(x) - P(y)) \geq \|P(x) - P(y)\|^2$.

Problem 2. (Compactness in Hölder spaces) (10 Points)

Let $0 < \beta < \alpha \leq 1$. Show that the unit ball of $C^{0,\alpha}([0,1])$ is compact in $C^{0,\beta}([0,1])$.
Hint: Use the Arzela-Ascoli theorem.

Problem 3. (Boundedness and precompactness) (7+3 Points)

Consider the following sets:

i) $E_1 = \{f : (0,1) \to \mathbb{R} : f(x) = x^{-\alpha}, 0 \leq \alpha < 1\}$,

ii) $E_2 = \{f : (0,1) \to \mathbb{R} : f(x) = x^{-\alpha}, -\infty < \alpha \leq 1 - \delta\}$ (with fixed $\delta > 0$),

iii) $E_3 = \{f : (0,1) \to \mathbb{R} : f(x) = \sin(\omega x), \omega \in \mathbb{R}\}$.

iv) $E_4 = \{f \in C^2([0,1]) : \|f\|_\infty (1 + \|f''\|_\infty) \leq 1\}$.

a) Decide whether E_1, E_2, E_3 as subsets of $L^1((0,1))$ are bounded and whether they are precompact.

b) Decide whether E_4 as a subset of $C([0,1])$ is precompact.

Problem 4. (Precompactness criterium in L^2) (10 Points)

Suppose $A \subset L^2(\mathbb{R}^n)$. For $f \in A$ denote by $\hat{f}$ its Fourier transform. Prove that A is precompact if and only if the following three statements are true.

i) $\sup_{f \in A} \|f\|_{L^2}$

ii) $\limsup_{R\to\infty} \sup_{f\in A} \int_{\mathbb{R}^n \setminus B_R(0)} |f(x)|^2 \,\mathrm{d}x = 0$

iii) $\limsup_{R\to\infty} \sup_{f\in A} \int_{\mathbb{R}^n \setminus B_R(0)} \left|\hat{f}(k)\right|^2 \mathrm{d}k = 0$

Hint: Observe that there is a decomposition $f = f_1 + f_2$ *with* $\operatorname{supp} \hat{f}_1 \subset B_R(0)$ *and* $\left\|\hat{f}_2\right\|_{L^2} < \varepsilon$.