Functional Analysis

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Problem Sheet 7.

Due 18.12.2015.

Problem 1. (Convex functions I) (3+3+4 Points)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex.

a) Show that f is locally bounded from above, i.e. that $\sup_{B_R} f < \infty$ for every $R > 0$.
Hint: Find a finite set $X_R \subset \mathbb{R}^n$ such that $B_R \subset \operatorname{conv} X_R$.

b) Show that f is locally bounded, i.e. that $\sup_{B_R} |f| < \infty$ for every $R > 0$.
Hint: Use that f is bounded from above in ∂B_R.

c) Show that f is locally Lipschitz, i.e. that $\sup_{x,y \in B_R, x \neq y} \frac{|f(x)-f(y)|}{|x-y|} < \infty$ for every $R > 0$.
Hint: Show that there are $z \in \partial B_{2R}$, $\lambda > 0$ such that $y - x = \lambda(z - x)$.

Problem 2. (Convex functions II) (5+5∗+5 Points)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex.

a) Show that $f = \sup\{g(x) \mid g : \mathbb{R}^n \to \mathbb{R} \text{ affine}, g \leq f\}$.
Hint: Show that the epigraph $\{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid y \geq f(x)\}$ is closed and convex.

b∗) Show that the **subdifferential** $\partial^- f(x) = \{a \in \mathbb{R}^n \mid f(y) \geq f(x) + a \cdot (y - x) \text{ for all } y \in \mathbb{R}^N\}$ is nonempty for all $x \in \mathbb{R}^n$.

c) Let $(\Omega, \mathcal{A}, \mu)$ be a probability space, i.e. a measure space with $\mu(\Omega) = 1$, $X : \Omega \to \mathbb{R}^n$ a Borel measurable map with $\int_\Omega |X| \, d\mu < \infty$. Show that

$$f\left(\int_\Omega X \, d\mu\right) \leq \int_\Omega f(X) \, d\mu.$$

This is **Jensen's inequality**.

Problem 3. (The Legendre transform) (5+5 Points)

Let $f : \mathbb{R}^n \to (-\infty, \infty]$ with $\inf f < \infty$. Define the **Legendre transform** $f^* : \mathbb{R}^n \to (-\infty, \infty]$ as

$$f^*(y) = \sup_{x \in \mathbb{R}^n} y \cdot x - f(x).$$

a) Let $p \in [1, \infty)$. Calculate the Legendre transform of $x \in \mathbb{R}^n \mapsto \frac{1}{p} |x|^p$.

b) Let $f : \mathbb{R}^n \to [0, \infty)$. Show that the double Legendre transform of f is $f^{**}(x) = \sup\{g(x) \mid g : \mathbb{R}^n \to \mathbb{R} \text{ convex}, g \leq f\}$.
Hint: Fix $y \in \mathbb{R}^n$ and consider the function $h_y(x) = \inf_{z \in \mathbb{R}^n} y \cdot (x - z) + f(z)$.

Problem 4. (Compactness properties) (3+3+5∗+2+5∗+2 Points)

Let (X,d) be a metric space and let $A \subset X$. Prove the following statements.

a) A is sequentially compact if A is compact.

b) If A is sequentially compact then (A, d) is complete and A is precompact.

c∗) If (A, d) is complete and A is precompact then A is compact.

Let X be complete. Prove the following statements (you may use a),b) and c) even if you did not prove them.)

d) A is precompact if and only if $\bar{A}$ is compact.

e∗) Let X be additionally a vector space with the metric induced by a norm. Then A is compact if and only if it is closed and for every $\varepsilon > 0$ there is a bounded subset B of a finite dimensional subspace of X such that $d_H(A, B) < \varepsilon$.

Remember that d_H is the Hausdorff-distance between sets. In this sense compact subsets of vector spaces are nearly finite dimensional.

Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be continuous. Prove that

f) If $A \subset X$ is compact then $f(A) \subset Y$ is compact.