Functional Analysis

WS 2015/2016
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Problem Sheet 3.

Due 20.11.2015.

Problem 1. (Separability) (10 Points)

Prove: The space $B(X, \mathbb{R})$ is separable if and only if X is a finite set.
This proves in particular, that the space of bounded sequences $l_\infty = B(\mathbb{N}, \mathbb{R})$ is not separable.
Hint: To every set $A \subseteq \mathbb{N}$ assign a function $f_A \in B(X, \mathbb{R})$ such that $\|f_A - f_{A'}\| = 1$ whenever $A \neq A'$.

Problem 2. (Hölder continuous functions) (4+4+4+4+4+5∗ Points)

Reminder: For $u : [a, b] \to \mathbb{R}$, $\alpha \in (0, 1]$, define $[u]_\alpha = \sup_{x \neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$. Define $\|u\|_\alpha = \|u\|_{C^0} + [u]_\alpha$ and $C^{0,\alpha}([a, b]) = \{u : [a, b] \to \mathbb{R} \, : \, \|u\|_\alpha < \infty\}$.

a) Show that $(C^{0,\alpha}([a, b]), \|\cdot\|_\alpha)$ is a Banach space.

b) Show that $\|u\|_\beta \leq \|u\|_\alpha$ for $u : [0, 1] \to \mathbb{R}$ whenever $\beta \in (0, \alpha]$.

c) Let $\alpha \in (0, 1)$. Find $u \in C^{0.\alpha}([0, 1])$ with $[u]_\beta = \infty$ for all $\beta > \alpha$.

d) Find $u \in C^0([0, 1])$ with $[u]_\alpha = \infty$ for all $\alpha > 0$.

e) Let $u \in C^{0,\alpha}([0, 1])$ with $\alpha \in (0, 1]$. Define for $\lambda \in (0, \infty)$ the function $u_\lambda : [0, \lambda] \to \mathbb{R}$ as $u_\lambda(x) = u(x/\lambda)$. Calculate $[u_\lambda]_\alpha$ in terms of $[u]_\alpha$.

f∗) Show that $C^{0,\alpha}([0, 1])$ is not separable.

Hint: Proceed similarly as in Problem 1.

Problem 3. (The Hausdorff distance) (3+2+2+3+5∗ Points)

For $A \subseteq \mathbb{R}^n$ nonempty, $r > 0$, define $B(A, r) = \{x \in \mathbb{R}^n \, : \, \inf_{y \in A} \|x - y\| < r\}$.

a) Show that $B(A, r)$ is open, and that $B(B(A, r), s) = B(A, r + s)$.

b) Show that if $A_1 \subseteq A_2$, then $B(A_1, r) \subseteq B(A_2, r)$.

Now let $K_1, K_2 \subseteq \mathbb{R}^n$ be two nonempty compact sets. Define their Hausdorff distance

$$d_H(K_1, K_2) = \inf\{r > 0 \, : \, K_1 \subseteq B(K_2, r) \text{ and } K_2 \subseteq B(K_1, r)\}.$$

c) Let $x, y \in \mathbb{R}^n$, $r, s \geq 0$. Calculate $d_H(\overline{B(x, r)}, \overline{B(y, s)})$.

d) Show that d_H is a metric on $\mathcal{K} = \{K \subseteq \mathbb{R}^n \, : \, K \text{ is nonempty and compact}\}$.

e∗) Show that $(\mathcal{K}, d_H)$ is complete.