Functional Analysis

WS 2015/2016
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Problem Sheet 12.

Due **Wednesday, 3.2.2016**.

Problem 1. (Small sets in $C([0,1])$) (5∗+5∗ Points)

a∗) For $n \in \mathbb{N}$ define the set $M_n \subset C([0,1])$ as

$$M_n = \{f \in C([0,1]) \,:\, \exists x^* \in [0, 1-1/n] \text{ s.t. } |f(x^*+h) - f(x^*)| \leq nh \text{ for all } h \in [0, 1-x^*]\} .$$

Show that M_n is closed and has empty interior in $C([0,1])$.

b∗) Define the set $M \subset C([0,1])$ as

$$M = \left\{f \in C([0,1]) \,:\, \exists x^* \in [0,1) \text{ s.t. the right derivative } f'_+(x^*) \text{ exists}\right\} .$$

Show that M has empty interior in $C([0,1])$.

Problem 2. (Weak convergence in $W^{1,p}$) (4×5∗ Points)

a∗) Let X, Y be Banach spaces. Show that a sequence (x_k, y_k) converges weakly in $(X \times Y)$ to (x, y) if and only if $x_k \rightharpoonup x$ and $y_k \rightharpoonup y$.

b∗) Let X be a Banach space, $Z \subset X$ a closed supspace of X(i.e. also a Banach space). Show that a sequence $z_k \in Z$ converges weakly in X to x if and only if $x \in Z$ and z_k converges weakly to x in Z.

c∗) Let X, Y be Banach spaces, $T \in \mathcal{L}(X, Y)$ invertible. Show that $x_k \rightharpoonup x$ in x if and only if $Tx_k \rightharpoonup Tx$ in Y.

d∗) Let $U \subseteq \mathbb{R}^n$ be open and $1 < p < \infty$. Show that $f_k \rightharpoonup f$ in $W^{1,p}(U)$ if and only if $f_k \rightharpoonup f$ in $L^p(U)$ and $\partial_j f_k \rightharpoonup \partial_j f$ in $L^p(U)$ for $j = 1, \ldots, n$.