Functional Analysis

WS 2015/2016
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Problem Sheet 10.

Due 22.1.2016.

Problem 1. (Frechet Differentiability) (5+5 Points)

Consider $F(u) = \sin(u)$.

a) Prove that $F : C([0,1]) \to C([0,1])$ is Frechet differentiable and compute $DF(0)$.
Hint: Guess the derivative and then apply the mean-value theorem twice.

b) Prove that $F : L^2(0,1) \to L^2(0,1)$ is not Frechet differentiable.
Hint: Proof by contradiction. Get a candidate from a) and show that there is a sequence $f_k \to 0$ contradicting the Frechet Differentiabiltiy condition.

Problem 2. (Duality) (2+3+2+3+3∗+3∗ Points)

Let X, Y, Z be Banach spaces.

a) Show that $X \times Y$ has a norm making it Banach.

b) Show that $(X \times Y)'$ is isomorphic to $X' \times Y'$.

c) Show that $\mathcal{L}(\mathbb{K}, X)$ is isomorphic to X.

d) Show that the adjoint of an operator $T \in \mathcal{L}(X, Y)$ is in $\mathcal{L}(Y', X')$ and that $\|T^*\|_{\mathcal{L}(Y',X')} = \|T\|_{\mathcal{L}(X,Y)}$.
Hint: To show equality of the norms, use the Hahn-Banach theorem.

e∗) Show for $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, Z)$ that $(S \circ T)^* = T^* \circ S^*$.

f∗) Show that Z is isomorphic to a closed subspace of its bidual $(Z')'$.
*Hint: Use (d) for suitable X, Y. In general, Z is not equal to Z''. If it is, Z is called **reflexive**.*

Problem 3. (Dual space of c_0 and c) (6+4 Points)

Define

$$c_0 := \{x \in l^\infty(\mathbb{R}) : \lim_{n\to\infty} x_n = 0\},$$
$$c := \{x \in l^\infty(\mathbb{R}) : \lim_{n\to\infty} x_n \text{ exists in } \mathbb{R}\}.$$

Note that c_0 and c equipped with the l^∞ norm are Banach spaces.

a) Show that $l^1(\mathbb{R})$ is isometrically isomorphic to the dual space of c_0.

b) Give an Isomorphism between c_0 and c. This yields an isomorphism $l^1(\mathbb{R}) \to c'$.

Problem 4. (The Bilaplace equation) (10 Points)

Let $U \subseteq \mathbb{R}^n$ be bounded, open, with Lipschitz boundary.

a) Show that $\int_U (\Delta u)^2\,dx = \sum_{i,j=1}^n \int_U |\partial_i \partial_j u|^2\,dx$ for all $u \in C_c^\infty(U)$.

b) Let $f \in L^2(U)$. Consider the Bilaplace equation

$$\begin{cases} \Delta^2 u = f & \text{in } U \\ u = 0 & \text{on } \partial U \\ \partial_\nu u = 0 & \text{on } \partial U, \end{cases} \tag{1}$$

where $\Delta^2 u = \Delta(\Delta u)$ is the Bilaplace operator and $\nu \in S^{n-1}$ is the unit outer normal to U. Determine a weak formulation in $W_0^{2,2}(U)$ and show that this weak equation has a unique solution.

Hint: Use integration by parts to find a suitable bilinear form $a(u,v)$. Do not worry about boundary values. (They are realized by $u \in W_0^{2,2}(U)$)