Functional Analysis

WS 2015/2016
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Problem Sheet 1.

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Problem 1. (Base of a topology) (4+3+3 Points)

Let X be a set. Consider a family of subsets $\mathcal{B} \subseteq 2^X$ such that

(i) If $U, V \in \mathcal{B}$ then for every $x \in U \cap V$ there is some $W \in \mathcal{B}$ such that $x \in W$ and $W \subseteq U \cap V$.

(ii) $\bigcup_{U \in \mathcal{B}} U = X$.

Define $\mathcal{T}_{\mathcal{B}} = \left\{ \bigcup_{U \in \mathcal{C}} U \,:\, \mathcal{C} \subseteq \mathcal{B} \right\}$. Let $\mathcal{T}$ be a topology over X. Any $\mathcal{B} \subseteq 2^X$ satisfying (i) and (ii) with $\mathcal{T}_{\mathcal{B}} = \mathcal{T}$ is called a base of $\mathcal{T}$.

a) Show that $\mathcal{T}_{\mathcal{B}}$ is indeed the coarsest topology containing $\mathcal{B}$.

b) Find at least two bases of the standard topology on $\mathbb{R}^n$.

c) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. Find at least two bases of the product topology on $X \times Y$.

Problem 2. (Cocountable topology) (3+1+4+2 Points)

We define

$$\mathcal{T} := \{ U \subset \mathbb{R} \,:\, U = \emptyset \text{ or } \mathbb{R} \setminus U \text{ is countable} \}.$$

Here we call a set A countable if A is either finite or if there is a bijective map $j : \mathbb{N} \to A$.

a) Prove that $\mathcal{T}$ is a topology on $\mathbb{R}$.

b) Show that $(\mathbb{R}, \mathcal{T})$ is not a Hausdorff space.

c) Prove that a sequence $x : \mathbb{N} \to \mathbb{R}$ converges to $x^* \in \mathbb{R}$ with respect to $\mathcal{T}$ if and only if there is $k_0 \in \mathbb{N}$ such that $x_k = x^*$ for all $k \geq k_0$. Hence limits are unique even though $(\mathbb{R}, \mathcal{T})$ is not Hausdorff.

d) Show that there exists $A \subset \mathbb{R}$ which is (with respect to $\mathcal{T}$) sequentially closed but not closed.

Problem 3. (Generating new metrics) (5+2+3 Points)

a) Consider a concave nondecreasing function $\psi \in C^2([0,\infty))$ with $\psi(0) = 0$ and $\psi(x) > 0$ for $x > 0$. Show that for any metric d the function $\psi \circ d$ is a metric, too.

   Hint: Show first that ψ' is a nonincreasing function.

b) Define $d : \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by

$$d(x,y) = \sum_{k=1}^{\infty} 2^{-k} \frac{|x_k - y_k|}{1 + |x_k - y_k|}.$$

   (i) Prove that d is a metric on the space of sequences $\mathbb{R}^{\mathbb{N}}$.

   (ii) Let $(x^j)_{j\in\mathbb{N}}$, $x^j = (x^j_k)_{k\in\mathbb{N}}$, be a sequence in $\mathbb{R}^{\mathbb{N}}$. Prove that $d(x^j, 0) \to 0$ if and only if $\lim_{j\to\infty} x^j_k = 0$ for all $k \in \mathbb{N}$.

Problem 4. (Pointwise convergence) (4+4+2 Points)

Let $X = \mathbb{R}^{\mathbb{R}} = \{f : \mathbb{R} \to \mathbb{R}\}$. For $f \in X$, $x \in \mathbb{R}$, and $\delta > 0$ let $B_{f,x,\delta} = \{g \in X : |g(x) - f(x)| < \delta\}$.

a) Find an explicit base (see Problem 1) for the coarsest topology $\mathcal{T}$ over X containing all the $B_{f,x,\delta}$.

b) Show that a sequence $(f_j)_{j\in\mathbb{N}}$ converges to f in $\mathcal{T}$ if and only if it converges pointwise.

c) Show that $\mathcal{T}$ is Hausdorff.