Option Pricing

Chapter 2 –Brownian motion and stochastic calculus–

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Variance of logreturns

Binomial model

- logreturn between i and $i+1$: $L_i = \log(S_{t_{i+1}}) - \log(S_{t_i})$
- logreturn between i and $i+2$:

$$\log(S_{t_{i+2}}) - \log(S_{t_i}) = L_i + L_{i+1}$$

- logreturns L_i and L_{i+1} are independent $\Longrightarrow$

$$\mathrm{var}[\log(S_{t_{i+2}}) - \log(S_{t_i})] = \mathrm{var}[L_i + L_{i+1}] = \mathrm{var}(L_i) + \mathrm{var}(L_{i+1})$$

Variance of the logreturns is proportional to the number of steps:

$$\mathrm{var}[\log(S_{t_{i+k}}) - \log(S_{t_i})] \sim k$$

Data analysis

Can we empirically confirm that the variance is prop. to time?

Data: closing prices of Siemens shares 3 Jan 2005 - 5 April 2011
(Source: `http://de.finance.yahoo.com/`)

	st. deviation
daily logreturns	
weekly logreturns	
monthly logreturns	

Variance of logreturns

The Black-Scholes (BS) model is the continuous time limit of the binomial model.
Similar assumptions:

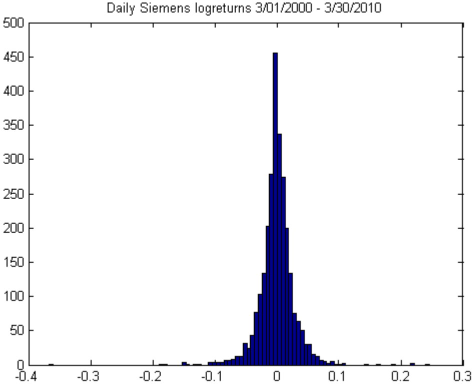
- logreturn variance is proportional to time
- logreturns on disjoint time intervals are independent

In addition:

In the BS model logreturns are assumed to be normally distributed

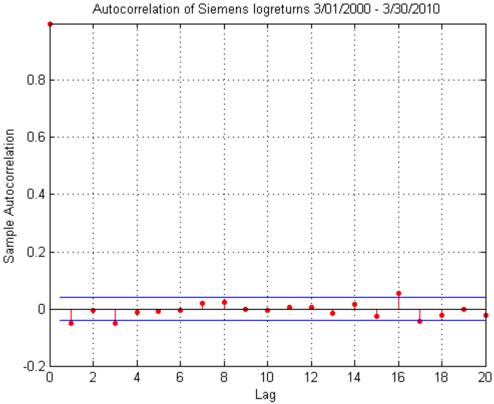
Data analysis

Histogram of Siemens logreturns 3/01/2000 - 3/30/2010
(Source: `http://de.finance.yahoo.com/`)



Data analysis cont'd

Autocorrelation of Siemens logreturns 3/01/2000 - 3/30/2010



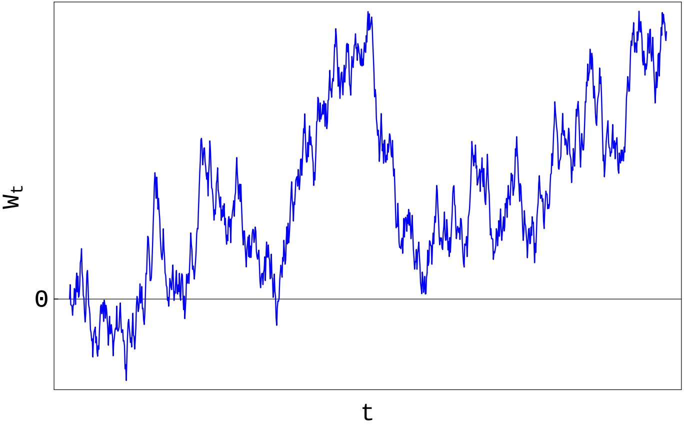
Brownian motion

- A stochastic process $(W_t)_{t\geq 0}$ with continuous paths is called a Brownian motion, if

  - ▷ the initial value is given by $W_0 = 0$,
  - ▷ for any $0 = t_0 \leq t_1 \leq \cdots \leq t_n$, the increments $W_{t_1} - W_{t_0}, \cdots, W_{t_n} - W_{t_{n-1}}$ are independent,
  - ▷ for any $0 \leq s < t$ the increment $(W_t - W_s)$ is normally distributed with

$$W_t - W_s \sim \mathcal{N}\Big(0, t - s\Big).$$

BM trajectories



Properties of Brownian motion

- Variance of increments is proportional to time period

$$\mathrm{var}(W_t - W_s) = t - s$$

This is also called $\sqrt{dt}$-**effect**:

$$\text{st. deviation of } (W_t - W_s) = \sqrt{t - s}$$

Properties of Brownian motion

- Non-vanishing quadratic variation

Let $\Pi : 0 = t_0 < t_1 < \cdots < t_n = t$ be a finite partition of $[0, t]$. We call

$$|\Pi| := \max_{1 \leq i \leq n} |t_i - t_{i-1}|$$

the *mesh* of Π.

A stochastic process $X : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ is of finite quadratic variation if there exists an increasing process $[X, X]$ such that

$$P - \lim_{|\Pi| \to 0} \sum_{\Pi} |X_{t_i} - X_{t_{i-1}}|^2 = [X, X]$$

- $P - \lim_n X_n = X$ means: X_n converges to X in probability
- $[X, X]$ is called quadratic variation of X

Properties of Brownian motion

Observe that a Brownian motion W satisfies

$$E \sum_{\Pi} |W_{t_i} - W_{t_{i-1}}|^2 = t.$$

Indeed, W is of finite quadratic variation:

Theorem

$\lim_{|\Pi| \to 0} \sum_{\Pi} |W_{t_i} - W_{t_{i-1}}|^2 = t$, *in probability.*

Proof with CLT: Let $\Pi : 0 = t_0 < t_1 < \cdots < t_n = t$ with $t_i = \frac{i}{n} t$. By the CLT, for large n the random variable

$$N := \frac{\sum_{i=1}^{n} \left(|W_{t_i} - W_{t_{i-1}}|^2 - \frac{t}{n} \right)}{\sqrt{n}\, \mathrm{std}(|W_{t_i} - W_{t_{i-1}}|^2)}$$

is approximately $\mathcal{N}(0,1)$-distributed.

Proof cont'd

$\Longrightarrow$

$$
\begin{aligned}
\sum_{i=1}^{n} \left(|W_{t_i} - W_{t_{i-1}}|^2\right) &= t + \sqrt{n}\,\mathrm{std}(|W_{t_i} - W_{t_{i-1}}|^2) N \\
&= t + \sqrt{n}\sqrt{2}\frac{t}{n} N \\
&= t + \sqrt{2}\frac{t}{\sqrt{n}} N,
\end{aligned}
$$

where in the second equality we used that a $\mathcal{N}(0, \sigma^2)$-distributed r.v. N satisfies: $\quad \mathrm{var}(N^2) = 2\sigma^4$.
Thus, for n large we have

$$
\sum_{i=1}^{n} |W_{t_i} - W_{t_{i-1}}|^2 \approx t.
$$

Proof cont'd

Verification by simulation: Simulate 500 paths, and let s_n denote the empirical standard deviation of $Q_n = \sum_{i=1}^n |W_{t_i} - W_{t_{i-1}}|^2$. Then

n	s_n	s_n/s_{n-1}
1000		
4000		
16000		
64000		

Matlab source code:

```
npaths = 500;
msteps = 1000;
t = 1;

dt = t/msteps;
x = sqrt(dt)*randn(npaths,msteps);
qv = sum(x.^2,2);

estd=std(qv);
hist(qv,50);
title(strcat('sum of quadratic increments - st. dev:',...
num2str(estd),' \newline number of time steps = ', ...
num2str(msteps),' \newline number of simulations = ',...
num2str(npaths)));
```

Properties of Brownian motion

- The trajectories of a Brownian motion

$$\mathbb{R}_+ \ni t \mapsto W_t(\omega)$$

are nowhere differentiable, for almost all $\omega \in \Omega$.

One can show that the quadratic variation of a differentiable function $f : [0, t] \to \mathbb{R}$ is zero. Since the quadratic variation of the Brownian motion is non-zero, the paths can not be differentiable.

Brownian motion is a martingale

Theorem

Brownian motion is a martingale.

Proof. later

Conditional expectation

Let $X : \Omega \to \mathbb{R}$ be an integrable random variable on a probability space $(\Omega, \mathcal{F}, P)$, and let $\mathcal{G}$ be a σ-field such that $\mathcal{G} \subset \mathcal{F}$.

Lemma

There exists a unique r.v. Y such that

- Y *is* $\mathcal{G}$*-measurable*
- $E[ZX] = E[ZY]$ *for any* $\mathcal{G}$*-measurable bounded r.v.* Z*;*

Definition: The random variable Y is referred to as the conditional expectation of X relative to $\mathcal{G}$, and usually denoted by $E[X|\mathcal{G}]$.

An alternative definition

Suppose that $X : \Omega \to \mathbb{R}$ is a *square-integrable* random variable, i.e. $E(X^2) < \infty$. In this case the conditional expectation $E[X|\mathcal{G}]$ is the unique $\mathcal{G}$-measurable r.v. that minimizes the mean square error:

$$\boxed{\min_{Z \text{ is } \mathcal{G}-\text{measurable}} E[(X-Z)^2]}$$

I.e. $E[X|\mathcal{G}]$ is the $\mathcal{G}$-measurable r.v. closest to X in a least squares sense ($E[X|\mathcal{G}]$ is the L^2 projection of X onto the set of $\mathcal{G}$-measurable r.v.'s).

Properties of conditional expectations

- Linearity: for any α, $\beta \in \mathbb{R}$

$$\boxed{E[\alpha X + \beta Y|\mathcal{G}] = \alpha E[X|\mathcal{G}] + \beta E[Y|\mathcal{G}]}$$

- Monotonicity: If $X \geq Y$ then $E[X|\mathcal{G}] \geq E[Y|\mathcal{G}]$, P-a.s.
- $E[E[X|\mathcal{G}]] = E[X]$
- $E[X|\mathcal{G}] = X$ if X is $\mathcal{G}$-measurable
- Tower property: if $\mathcal{H} \subset \mathcal{G}$, then

$$E[E[X|\mathcal{G}]|\mathcal{H}] = E[X|\mathcal{H}]$$

(the "coarser information" prevails)

- $E[X|\mathcal{G}] = E[X]$ if X is independent of $\mathcal{G}$

Filtrations

- A family of σ-fields $(\mathcal{F}_t)$, indexed by $\mathbb{R}_+$ (think of time), is called a filtration if

$$\mathcal{F}_s \subset \mathcal{F}_t \text{ whenever } s < t.$$

- A stochastic process $X : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ is said to be adapted to the filtration $(\mathcal{F}_t)$ if for any $t \geq 0$ we have that

$$X_t \text{ is } \mathcal{F}_t - \text{measurable}.$$

In the remainder we will only work with the filtration generated by a Brownian motion W. This is, essentially, the smallest filtration $(\mathcal{F}_t)$ such that for all $t \geq 0$

$$W_t \text{ is } \mathcal{F}_t - \text{measurable}.$$

Mathematical definition: $\mathcal{F}_t = \sigma(W_s : s \leq t) \vee P$-null sets in $\mathcal{F}$

Martingales

Definition: A stochastic process $M : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ is called a martingale wrt a filtration $(\mathcal{F}_t)$ if

- M is adapted to $(\mathcal{F}_t)$,
- M_t is integrable for all $t \geq 0$,
- $E[M_t|\mathcal{F}_s] = M_s$, for all $0 \leq s \leq t$.

BM is a martingale

Theorem

A Brownian motion W is a martingale wrt the filtration generated by W.

Proof. Let $0 \leq s \leq t$. Then

$$\begin{aligned} E[W_t|\mathcal{F}_s] &= E[(W_t - W_s) + W_s|\mathcal{F}_s] \\ &= E[W_t - W_s|\mathcal{F}_s] + E[W_s|\mathcal{F}_s] \\ &= E[W_t - W_s] + W_s \\ &= W_s. \end{aligned}$$

Stochastic calculus

Aim: We want to give meaning to integrals of the form

$$\int_0^T \Delta(t) dW_t, \tag{1}$$

where Δ is adapted to the filtration generated by W.

For differentiable functions $g : \mathbb{R}_+ \to \mathbb{R}$ one usually defines

$$\int_0^T \Delta(t) dg_t := \int_0^T \Delta(t) g'(t) dt.$$

Since the paths of W are not differentiable we have to choose an other way to define (1).

Further reading: S. Shreve: Stochastic Calculus for Finance II, Chapter 4.

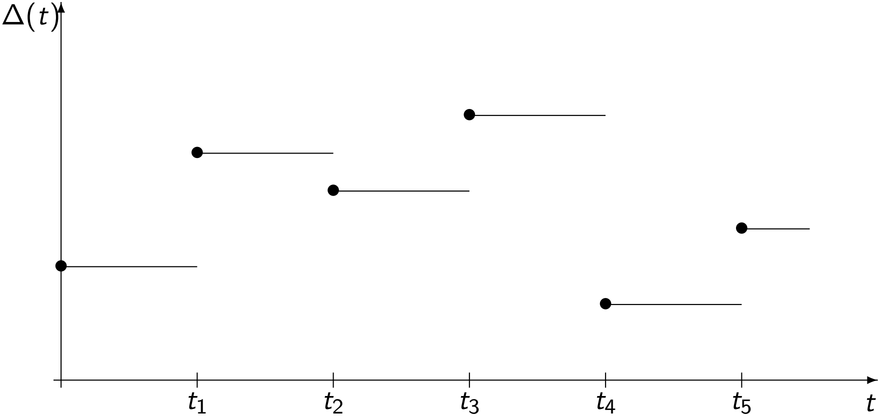
Buy-and-hold strategies

Let $\Pi : 0 = t_0 < t_1 < \cdots < t_n = T$ be a finite partition of $[0, T]$.

Definition: A buy-and-hold strategy is an adapted process Δ that is constant on each subinterval $[t_j, t_{j+1}[$ of a partition Π.

- Interpretation: $\Delta(t) = \#$ shares of an asset in a portfolio. The portfolio is only adjusted at trading times $t_0, t_1, \dots, t_n$.
- Adaptedness implies that the strategy is non-anticipating: $\Delta(t)$ depends only on the information available at time t.
- In mathematics buy-and-hold strategies are called *simple integrands*.

A trajectory of a simple integrand process



Simple Ito integrals

What are the gains from a buy-and-hold strategy Δ?

- Suppose the price of an asset evolves according to a Brownian motion W.
- Let $t \in [0, T]$ and k such that $t_k \leq t \leq t_{k+1}$. Then the gain up to t is given by

$$I_t = \Delta(t_k)(W_t - W_{t_k}) + \sum_{j=0}^{k-1} \Delta(t_j)(W_{t_{j+1}} - W_{t_j}). \tag{2}$$

The process I_t is called *simple Ito integral process* of Δ wrt W.
Notation: $I_t = \int_0^t \Delta(s) dW_s$.

Properties of simple integrals

Lemma

The simple integral process (2) is a martingale.

Lemma (Ito isometry)

Simple integrals satisfy

$$E[I_t^2] = E\int_0^t \Delta^2(u)du.$$

Lemma

The quadratic variation $[I, I]$ of the simple integral satisfies

$$[I, I]_t = \int_0^t \Delta^2(u)du.$$

General integrands

Next: Ito integral for continuously varying trading strategies.

Definition: By a *general integrand* we mean any process $\Delta(t)$ such that

- Δ is adapted,
- Δ is square-integrable, i.e. $E \int_0^T \Delta^2(t) dt < \infty$,
- there exists a sequence of simple integrands Δ_n such that

$$\lim_n E \int_0^T (\Delta_n(t) - \Delta(t))^2 dt = 0. \tag{3}$$

Remark: Assumption (3) is not very restrictive.

Ito integrals as L^2 limits

Let Δ be a general integrand, and (Δ_n) an approximating sequence of simple integrands.

- $\int_0^t \Delta_n(s) dW_s$ already defined for all n,
- Ito isometry implies

$$\begin{aligned}\lim_{n,m} E\left(\int_0^T (\Delta_n(t) - \Delta_m(t)) dW_t\right)^2 &= \lim_{n,m} E \int_0^T (\Delta_n(t) - \Delta_m(t))^2 dt \\ &= 0.\end{aligned}$$

Consequently, $\int_0^T \Delta_n(t) dW_t$ is a Cauchy sequence in $L^2(\Omega)$, and possesses a limit denoted by

$$\int_0^T \Delta(t) dW_t. \tag{4}$$

Definition: (4) is called Ito integral of Δ wrt W.

Properties of Ito integrals

$I_t = \int_0^t \Delta(s) dW_s$ has the following properties.

- **Continuity:** there exists a version of the Ito integral process I_t such that the paths $t \mapsto I_t(\omega)$ are continuous for all ω.
- **Adaptivity:** I_t is $\mathcal{F}_t$-measurable,
- **Linearity:** if $J_t = \int_0^t \Gamma(s) dW_s$ is another integral, then

$$I_t + J_t = \int_0^t (\Delta(s) + \Gamma(s)) dW_s,$$

- **Martingale property:** I_t is a martingale,
- **Ito isometry:** $E(I_t^2) = E \int_0^t \Delta^2(s) ds$,
- **Quadratic variation:** $[I, I]_t = \int_0^t \Delta^2(s) ds$.

Ito formula

Theorem

Let $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be once continuously differentiable in t, and twice continuously differentiable in x. Then

$$\begin{aligned} f(T, W_T) &= f(0, W_0) + \int_0^T f_t(t, W_t)dt + \int_0^T f_x(t, W_t)dW_t \\ &\quad + \frac{1}{2}\int_0^T f_{xx}(t, W_t)dt \end{aligned}$$

Proof of the Ito formula

Proof for $f(x) = \frac{1}{2}x^2$.

$$\begin{aligned} f(y) - f(x) &= f'(x)(y-x) + \frac{1}{2} f''(x)(y-x)^2 \\ &= x(y-x) + \frac{1}{2}(y-x)^2. \end{aligned}$$

Let $\Pi : 0 = t_0 < t_1 < \cdots < t_n = T$ be a partition. Then

$$\begin{aligned} f(W_T) - f(W_0) &= \sum_{j=0}^{n-1} (f(W_{t_{j+1}}) - f(W_{t_j})) \\ &= \sum_{j=0}^{n-1} W_{t_j}(W_{t_{j+1}} - W_{t_j}) + \frac{1}{2} \sum_{j=0}^{n-1} (W_{t_{j+1}} - W_{t_j})^2 \end{aligned}$$

Note that $\sum_{j=0}^{n-1} W_{t_j}(W_{t_{j+1}} - W_{t_j})$ is a simple integral. Thus, letting $|\Pi| \to 0$, we get

$$f(W_T) - f(W_0) = \int_0^T W_t dW_t + \frac{1}{2} T.$$

General case: use Taylor's formula.

Ito processes

Definition: An Ito process X_t is a stochastic process of the form

$$X_t = x + \int_0^t \Delta(s)dW_s + \int_0^t \Theta(s)ds, \qquad (5)$$

where $x \in \mathbb{R}$, and Δ, Θ are adapted processes.

Properties of Ito processes:

- **Uniqueness:** Suppose that X_t satisfies (5), and additionally $X_t = y + \int_0^t \Gamma(s)dW_s + \int_0^t \beta(s)ds$. Then $x = y$, and (a.s.)

$$\Gamma = \Delta, \qquad \text{and} \qquad \beta = \Theta.$$

- **Quadratic variation:** $[X, X]_t = \int_0^t \Delta^2(s)ds$.

Ito formula for Ito processes

Definition: Let X be as in (5). The integral of a process Γ wrt X is defined as

$$\int_0^t \Gamma(s) dX_s = \int_0^t \Gamma(s)\Delta(s) dW_s + \int_0^t \Gamma(s)\Theta(s) ds.$$

Theorem

Let $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be once continuously differentiable in t, and twice continuously differentiable in x. Then

$$\begin{aligned} f(T, X_T) &= f(0, X_0) + \int_0^T f_t(t, X_t) dt + \int_0^T f_x(t, X_t) dX_t \\ &\quad + \frac{1}{2} \int_0^T f_{xx}(t, X_t) d[X, X]_t \end{aligned}$$

Example

Consider the Ito process

$$X_t = \int_0^t \sigma_s dW_s + \int_0^t \left(\alpha_s - \frac{\sigma_s^2}{2}\right) ds.$$

Differential notation (with no precise mathematical meaning):

$$dX_t = \sigma_t dW_t + \left(\alpha_t - \frac{\sigma_t^2}{2}\right) dt.$$

Suppose that the price of an asset is given by

$$S_t = S_0 e^{X_t}.$$

The Ito formula implies, with $f(x) = S_0 e^x$,

$$\begin{aligned} dS_t &= df(X_t) \\ &= \dots \end{aligned}$$

Example cont'd

$$
\begin{aligned}
dS_t &= df(X_t) \\
&= S_0 e^{X_t} dX_t + \frac{1}{2} S_0 e^{X_t} d[X,X]_t \\
&= S_t dX_t + \frac{1}{2} S_t d[X,X]_t \\
&= S_t \sigma_t dW_t + S_t \left(\alpha_t - \frac{\sigma_t^2}{2} \right) dt + \frac{1}{2} \sigma_t^2 S_t d_t \\
&= S_t \sigma_t dW_t + S_t \alpha_t dt.
\end{aligned}
$$

Observation: S solves the stochastic differential equation

$$
dS_t = S_t \sigma_t dW_t + S_t \alpha_t dt.
$$

Let $\mu : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be measurable functions. An equation of the form

$$dX_t = \mu(t, X_t)dt + \beta(t, X_t)dW_t, \quad X_0 = x, \tag{6}$$

is called *stochastic differential equation* (SDE) with *initial condition* $X_0 = x \in \mathbb{R}$.

A solution of the SDE (6) is an Ito process such that

$$X_t = X_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \beta(s, X_s)dW_s.$$

Stochastic differential equations

Theorem

Suppose that there exists a constant $K \geq 0$ such that for all $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}$

$$\begin{aligned} |\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| &\leq K|x-y|, \\ |\mu(t,x)| + |\sigma(t,x)| &\leq K(1+|x|). \end{aligned}$$

Then, for any $T > 0$, there exists a unique solution X of (6). The uniqueness of the solution means that if X and Y are two solutions, then P-a.s, for all $t \in [0,T]$, $X_t = Y_t$.