The joint distribution of excursion and hitting times of the Brownian motion with Application to Parisian Option Pricing

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Motivation

Definition

The MinParisianHit Option is triggered either when the age of the excursion above L reaches time d or a barrier $B > L$ is hit by the underlying price process S. The MaxParisianHit Option is triggered when both the barrier B is hit and the excursion age exceeds duration d above L.

Let the stock price process $(S_t)_{t\geq 0} = \left(S_0 e^{(r-\frac{\sigma^2}{2})t+\sigma W_t}\right)_{t\geq 0}$ follow a geometric Brownian motion and $\mathbb{Q}$ denote the equivalent martingale measure and define

$$
\begin{aligned}
g_{L,t}^S &= \sup\{s \leq t : S_s = L\} \\
d_{L,t}^S &= \inf\{s \geq t : S_s = L\} \\
\tau_d^+(S) &= \inf\{t \geq 0 | 1_{S_t > L}(t - g_{L,t}^S) \geq d\} \\
H_B(S) &= \inf\{t \geq 0 | S_t = B\}
\end{aligned}
$$

Motivation

The fair price of a MinParisianHit Up-and-In Call option with payoff $(S_T - K)^+$ can be written in terms of hitting times in the following way using Girsanov theorem

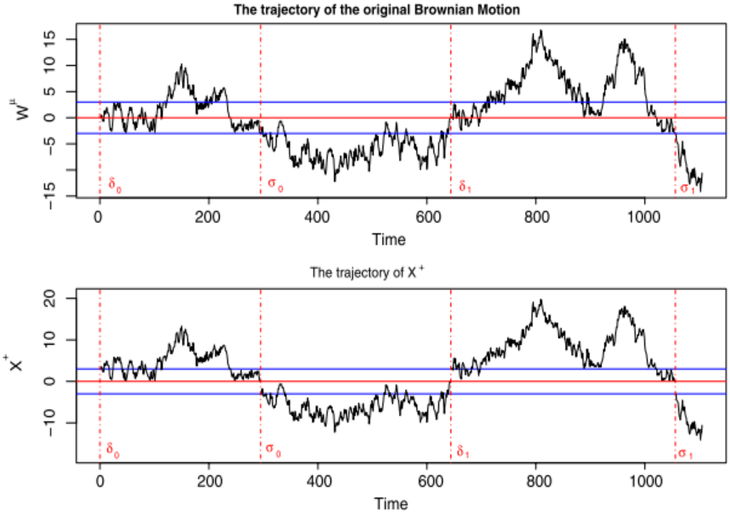
$$
\begin{aligned}
minPHC_i^u(x, T, K, L, d, r) &= e^{-rT}\mathbb{E}_{\mathbb{Q}}\left((S_T - K)^+ 1_{\min\{\tau_d^+(S), H_B(S)\}\leq T}\right) \\
&= e^{-(r+\frac{1}{2}m^2)T}\mathbb{E}_{\mathbb{P}}\left((S_0 e^{\sigma Z_T} - K)^+ e^{mZ_T} 1_{\min\{\tau_d^+(Z), H_b(Z)\}\leq T}\right) \\
&= e^{-(r+\frac{1}{2}m^2)T}\int_{\frac{1}{\sigma}\ln\frac{K}{S_0}}^{\infty}(S_0 e^{\sigma z} - K)e^{mz}\boxed{\mathbb{P}\left(Z_T \in dz, \min\{\tau_d^+(Z), H_b(Z)\} \leq T\right)}
\end{aligned}
$$

where $(Z_t)_{t\geq 0} = (W_t + mt)_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion and

$$m = \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right) \qquad l = \frac{1}{\sigma}\ln\frac{L}{S_0} \qquad b = \frac{1}{\sigma}\ln\frac{B}{S_0}$$

$$\tau_d^+(Z) = \inf\{t \geq 0 | 1_{Z_t > l}(t - g_{l,t}) \geq d\} \qquad g_{l,t}^Z = \sup\{u \leq t | Z_u = l\} \qquad H_b(Z) = \inf\{t \geq 0 | Z_t = b\}$$

Motivation



(Dassios and Wu. *Perturbed Brownian motion and its application to Parisian option pricing.* (2))

Martingale Problem

Let $W^{\epsilon,\mu} = \mu t + W_t^{\epsilon}$ be the perturbed drifted Brownian motion around 0. We define a three state semi-Markov process for $b > 0$

$$X_t = \begin{cases} 1^\star & \text{if } W_t^{\epsilon,\mu} > b \\ 1 & \text{if } 0 < W_t^{\epsilon,\mu} < b \\ 2 & \text{if } W_t^{\epsilon,\mu} < 0 \end{cases}$$

Let $U_t = t - \bar{g}_t^{\epsilon}$ denote the time elapsed in current state 2 or state 1 and $1^\star$ combined (state $1^\star$ is an absorbing state). (X_t, U_t) becomes a Markov process. Hence, X_t is a three state semi-Markov process.
We consider a bounded function $f_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1^*, 1, 2$, defined as $f_{X_t}(U_t, t)$. The generator $\mathcal{A}$ is an operator making

$$f_{X_t}(U_t, t) - \int_0^t \mathcal{A} f_{X_s}(U_s, s) ds$$

a martingale. Hence, solving $\mathcal{A} f = 0$ provides us with martingales of the form $f_{X_t}(U_t, t)$.

Martingale Problem

We have

$$\mathcal{A}f_1(u,t) = \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial u} + \lambda_{12}(u)\left(f_2(0,t) - f_1(u,t)\right) + \lambda_{11^\star}(u)\left(Ae^{-\beta t}h(u) - f_1(u,t)\right)$$
$$\mathcal{A}f_2(u,t) = \frac{\partial f_2}{\partial t} + \frac{\partial f_2}{\partial u} + \lambda_{21}(u)\left(f_1(0,t) - f_2(u,t)\right)$$

Since we are not interested in what happens after the absorbing state 1^* has been reached, we do not define $\mathcal{A}f_{1^*}$, the generator starting from state 1^*. We define the times for $d, b > 0$

$$H_b^\epsilon = \inf\{t \geq 0 | W_t^{\epsilon,\mu} = b\}$$
$$\bar{g}_t^\epsilon = \sup\{s \leq t | W_s^{\epsilon,\mu} = 0\}$$
$$\tau_d^{\epsilon,+} = \inf\{t > 0 | 1_{W_t^{\epsilon,\mu} > 0}(t - \bar{g}_t) > d\}$$

Martingale Problem

Definition

Given a function $F(\beta)$, the inverse Laplace transform of F, denoted by $\mathcal{L}^{-1}\{F(\beta)\}$, is the function f whose Laplace transform is F, i.e.

$$f(t) = \mathcal{L}_{\beta}^{-1}\{F(\beta)\}|_t \quad \Longleftrightarrow \quad \mathcal{L}_t\{f(t)\}(\beta) := \int_0^{\infty} e^{-\beta t} f(t) dt = F(\beta)$$

Note that we consider the inverse Laplace transform with respect to the transformation variable β at the evaluation point t.

Martingale Problem

Lemma

For the ϵ-perturbed process we find the Laplace transform

$$A\mathbb{E}_\epsilon\left(e^{-\beta H_b^\epsilon} h(u_{H_b^\epsilon}) 1_{H_b^\epsilon < \tau_d^{\epsilon,+}}\right) + B\mathbb{E}_\epsilon\left(e^{-\beta \tau_d^{\epsilon,+}} 1_{\tau_d^{\epsilon,+} < H_b^\epsilon}\right) = \frac{Be^{-\beta d}\bar{P}_1(d) + A\int_0^d e^{-\beta w} h(w) p_{11^\star}(w)\,dw}{1 - \tilde{P}_{21}(\beta)\hat{P}_{12}(\beta)} \quad (1)$$

Proof: Assuming f having the form $f_i(u,t) = e^{-\beta t} g_i(u)$ and solving $\mathcal{A}f \equiv 0$ with the constraints $g_1(d) = B$ and $g_2(\infty) = 0$ and $g_{1^\star}(u) = Ah(u)$ we can solve for g_1 which provides us with martingales of the form $M_t := f_{X_t}(U_t, t) = e^{-\beta t} g_{X_t}(U_t)$. Let $T = \min\{H_b^\epsilon, \tau_d^{\epsilon,+}\}$, then Optional Sampling suggests

$$\begin{aligned}
g_1(0) = \mathbb{E}(M_0) &= \mathbb{E}(M_{T\wedge t}) = \mathbb{E}(M_T 1_{T<t}) + \mathbb{E}(M_t 1_{T>t}) \\
&= \mathbb{E}(M_{H_b^\epsilon} 1_{H_b^\epsilon < \tau_d^{\epsilon,+}} 1_{H_b^\epsilon < t}) + \mathbb{E}(M_{\tau_d^{\epsilon,+}} 1_{\tau_d^{\epsilon,+} < H_b^\epsilon} 1_{\tau_d^{\epsilon,+} < t}) + \mathbb{E}(M_t 1_{T>t}) \\
&\xrightarrow[t\to\infty]{} \mathbb{E}\left(e^{-\beta H_b^\epsilon} g_{1^\star}(U_{H_b^\epsilon}) 1_{H_b^\epsilon < \tau_d^{\epsilon,+}}\right) + \mathbb{E}\left(e^{-\beta \tau_d^{\epsilon,+}} g_1(U_{\tau_d^{\epsilon,+}}) 1_{\tau_d^{\epsilon,+} < H_b^\epsilon}\right) \\
&= A\mathbb{E}\left(e^{-\beta H_b^\epsilon} h(u_{H_b^\epsilon}) 1_{H_b^\epsilon < \tau_d^{\epsilon,+}}\right) + B\mathbb{E}\left(e^{-\beta \tau_d^{\epsilon,+}} 1_{\tau_d^{\epsilon,+} < H_b^\epsilon}\right)
\end{aligned}$$

Single Laplace transform of Hitting and Excursion time

Proposition

The Laplace transform of the hitting and excursion time for drifted Brownian motion $\mu t + W_t$ is given by

$$A\mathbb{E}\left(e^{-\beta H_b} h(u_{H_b}) 1_{H_b<\tau_d^+}\right) + B\mathbb{E}\left(e^{-\beta \tau_d^+} 1_{\tau_d^+<H_b}\right) =$$

$$\Bigg\{2Be^{-\beta d}\sum_{k=0}^{\infty}\left[g(k,0) - e^{\mu b} g(k+\frac{1}{2},0) + \mu\left[f(k,0) - f(k+1,0)\right]\right] +$$

$$+ A\int_0^d e^{-\beta w} h(w)\sqrt{\frac{2}{\pi w^3}} e^{\mu b - \frac{\mu^2 w}{2}} \sum_{k=0}^{\infty}\left(\frac{(2k+1)^2 b^2}{w} - 1\right) e^{-\frac{(2k+1)^2 b^2}{2w}} dw\Bigg\} \times$$

$$\times \left\{2\sum_{k=0}^{\infty}\left[\sqrt{2\beta+\mu^2} f(k,\beta) + g(k,\beta)\right]\right\}^{-1}$$

where

$$f(k,\beta) = e^{-\sqrt{2\beta+\mu^2}2kb}\mathcal{N}\left(\sqrt{(2\beta+\mu^2)d} - \frac{2kb}{\sqrt{d}}\right) - e^{\sqrt{2\beta+\mu^2}2kb}\mathcal{N}\left(-\sqrt{(2\beta+\mu^2)d} - \frac{2kb}{\sqrt{d}}\right)$$

$$g(k,\beta) = \sqrt{\frac{2}{\pi d}} e^{-\frac{(2\beta+\mu^2)d}{2} - \frac{2(kb)^2}{d}}$$

Double Laplace transform of Hitting and Excursion time

case $H_b < \tau_d^+$

Proposition

The double Laplace transform of Hitting and Excursion time of a drifted Brownian motion where $H_b < \tau_d^+$ is

$$\mathbb{E}_0\left(e^{-\beta H_b - \gamma\tau_d^+} 1_{H_b < \tau_d^+}\right) =$$

$$\int_0^d e^{-\beta w}\left[e^{-\gamma d}\left(1 - e^{-2\mu b}\mathcal{N}\left(\frac{\mu(d-w)-b}{\sqrt{d-w}}\right) - \mathcal{N}\left(\frac{-\mu(d-w)-b}{\sqrt{d-w}}\right)\right) +\right.$$

$$+\mathbb{E}_0^{\mu}(e^{-\gamma\hat{\tau}_d})\left(e^{-(\sqrt{2\gamma+\mu^2}+\mu)b}\mathcal{N}\left(\sqrt{(2\gamma+\mu^2)(d-w)} - \frac{b}{\sqrt{d-w}}\right) +\right.$$

$$\left.\left. + e^{\sqrt{2\gamma+\mu^2}-\mu)b}\mathcal{N}\left(-\sqrt{(2\gamma+\mu^2)(d-w)} - \frac{b}{\sqrt{d-w}}\right)\right)\right]\times$$

$$\times\sqrt{\frac{2}{\pi w^3}}e^{\mu b - \frac{\mu^2 w}{2}}\sum_{k=0}^{\infty}\left(\frac{(2k+1)^2 b^2}{w} - 1\right)e^{-\frac{(2k+1)^2 b^2}{2w}}dw\times$$

$$\times\left\{2\sum_{k=0}^{\infty}\left[\sqrt{2\beta+\mu^2}f(k,\beta) + g(k,\beta)\right]\right\}^{-1}$$

Proof:

Lemma

$$\mathbb{E}\left(e^{-\beta H_b} h(u_{H_b}) 1_{H_b<\tau_d^+}\right) = \frac{\int_0^d e^{-\beta w} h(w) \sqrt{\frac{2}{\pi w^3}} e^{\mu b - \frac{\mu^2 w}{2}} \sum_{k=0}^{\infty} \left(\frac{(2k+1)^2 b^2}{w} - 1\right) e^{-\frac{(2k+1)^2 b^2}{2w}} dw}{2\sum_{k=0}^{\infty}\left[\sqrt{2\beta+\mu^2} f(k,\beta) + g(k,\beta)\right]}$$

Define $h(u_{H_b}) := \mathbb{E}\left(e^{-\gamma\tau_d^+} | H_b\right)$ and the l.h.s becomes

$$\mathbb{E}\left(e^{-\beta H_b} h(u_{H_b}) 1_{H_b<\tau_d^+}\right) = \mathbb{E}\left(e^{-\beta H_b}\mathbb{E}\left(e^{-\gamma\tau_d^+}|H_b\right) 1_{H_b<\tau_d^+}\right) = \mathbb{E}\left(e^{-\beta H_b} e^{-\gamma\tau_d^+} 1_{H_b<\tau_d^+}\right)$$

On the other hand, we have

$$\begin{aligned} h(u_{H_b}) &= \mathbb{E}_0\left(e^{-\gamma(H_b+d-u_{H_b})} 1_{\tilde{H}_0>d-u_{H_b}} | H_b\right) + \mathbb{E}_0\left(e^{-\gamma(H_b+\tilde{H}_0+\hat{\tau}_d)} 1_{\tilde{H}_0<d-u_{H_b}} | H_b\right) \\ &= e^{-\gamma H_b}\left[e^{-\gamma(d-u_{H_b})}\mathbb{P}_b(\tilde{H}_0>d-u_{H_b}) + \mathbb{E}_b\left(e^{-\gamma\tilde{H}_0} 1_{\tilde{H}_0<d-u_{H_b}}\right)\mathbb{E}_0(e^{-\gamma\hat{\tau}_d})\right] \end{aligned}$$

case $\tau_d^+ < H_b$

Proposition

The double Laplace transform of Hitting and Excursion time of a drifted Brownian motion where $\tau_d^+ < H_b$ is

$$\mathbb{E}(e^{-\beta\tau_d^+ - \gamma H_b} 1_{\tau_d^+ < H_b}) = \Bigg\{ e^{-\beta d} \Big[e^{-b(\sqrt{2\gamma+\mu^2}-\mu)} \mathcal{N}\left(\frac{b}{\sqrt{d}} - \sqrt{(2\gamma+\mu^2)d} \right) - - e^{b(\sqrt{2\gamma+\mu^2}-\mu)} \mathcal{N}\left(-\frac{b}{\sqrt{d}} - \sqrt{(2\gamma+\mu^2)d} \right) \Big] \sum_{k=0}^{\infty} \Big[g(k,0) - e^{\mu b} g(k+\frac{1}{2},0) + \mu \left[f(k,0) - f(k+1,0) \right] \Big] \Bigg\} \times \Bigg\{ \sum_{k=0}^{\infty} \Big[\sqrt{2(\beta+\gamma)+\mu^2} f(k,\beta+\gamma) + g(k,\beta+\gamma) \Big] \times \Big[1 - \mathcal{N}\left(\frac{\mu t - b}{\sqrt{d}} \right) - e^{2\mu b} \mathcal{N}\left(\frac{-\mu t - b}{\sqrt{d}} \right) \Big] \Bigg\}^{-1}$$

Proof:

Lemma

$$\mathbb{E}\left(e^{-\beta\tau_d^+}1_{\tau_d^+<H_b}\right) = \frac{e^{-\beta d}\sum_{k=0}^{\infty}\left[g(k,0) - e^{\mu b}g(k+\frac{1}{2},0) + \mu\left[f(k,0) - f(k+1,0)\right]\right]}{\sum_{k=0}^{\infty}\left[\sqrt{2\beta+\mu^2}f(k,\beta) + g(k,\beta)\right]}$$

We define a new generator starting at time τ_d^+. State $1^\star$ is a killed state.

$$\mathcal{A}f_1(u,t) = \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial u} + \lambda_{11^\star}(u)\left(e^{-\gamma t} - f_1(u,t)\right)$$

At time τ_d^+ we are in state 1 with $f_1(d,0) = g_1(d)$. Solving $\mathcal{A}f \equiv 0$ with constraint $g_1(\infty) = 0$ we derive $g_1(d)$. As a result we have found a martingale $M_t := f_{X_t}(U_t, t)$ with $M_0 = f_1(d,0) = g_1(d)$. Also, with T_b being the first hitting time of b of our process starting at τ_d^+ and hence $H_b = \tau_d^+ + T_b$,

$$M_{T_b} = f_{1^\star}(U_{T_b}, T_b) = e^{-\gamma T_b}$$

Hence, OST yields $g_1(d) = \mathbb{E}(e^{-\gamma T_b})$ and the double Laplace becomes

$$\mathbb{E}(e^{-\beta\tau_d^+}e^{-\gamma H_b}1_{\tau_d^+<H_b}) = \mathbb{E}(e^{-\beta\tau_d^+}1_{\tau_d^+<H_b}\mathbb{E}(e^{-\gamma H_b}|\tau_d^+))$$
$$= \mathbb{E}(e^{-\beta\tau_d^+}1_{\tau_d^+<H_b}\mathbb{E}(e^{-\gamma(\tau_d^+ + T_b)}|\tau_d^+)) = g_1(d)\mathbb{E}(e^{-(\beta+\gamma)\tau_d^+}1_{\tau_d^+<H_b})$$

Application to MinParisianHit options

$$minPHC_i^u = e^{-(r+\frac{1}{2}m^2)T} \int_{\frac{1}{\sigma}\ln\frac{K}{x}}^{\infty} (xe^{\sigma z} - K)e^{mz} \boxed{\mathbb{P}\left(Z_T \in dz, \min\{\tau_d^+(Z), H_b(Z)\} \leq T\right)}$$

Proposition

The joint density of position at maturity and minimum of hitting and excursion time for standard Brownian motion is

$$\mathbb{P}(Z_T \in dz, \min\{\tau_d^+, H_b\} \leq T) = \int\limits_{t=0}^{T} \int\limits_{w=-\infty}^{b} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(z-w)^2}{2(T-t)}} \times$$

$$\times \left[\frac{\sum\limits_{k=-\infty}^{\infty} \frac{w+2kb}{d} e^{-\frac{(w+2kb)^2}{2d}}}{\sum\limits_{k=-\infty}^{\infty} \left(e^{-\frac{(2kb)^2}{2d}} - e^{-\frac{(2k+1)^2 b^2}{2d}} \right)} \mathcal{L}_\beta^{-1}\{H_1(\beta)\}|_t + \delta_{(w-b)} \mathcal{L}_\beta^{-1}\{H_2(\beta)\}|_t \right] dw\ dt$$

$$H_1(\beta) = \frac{e^{-\beta d} \sum\limits_{k=0}^{\infty} \left[g(k,0) - g(k+\frac{1}{2},0) \right]}{\sum\limits_{k=0}^{\infty} \left[\sqrt{2\beta} f(k,\beta) + g(k,\beta) \right]}, \quad H_2(\beta) = \frac{\sum\limits_{k=0}^{\infty} \sqrt{2\beta} f(k+\frac{1}{2},\beta) + g(k+\frac{1}{2},\beta)}{\sum\limits_{k=0}^{\infty} \left[\sqrt{2\beta} f(k,\beta) + g(k,\beta) \right]}$$

Proof: Let Z denote a standard Brownian motion and $\tau := \min\{\tau_d^+, H_b\}$.

$$\mathbb{P}(Z_T \in dz, \min\{\tau_d^+, H_b\} \leq T) = \int_{t=0}^{T} \int_{w=-\infty}^{b} \mathbb{P}(Z_T \in dz | \tau = t, Z_\tau \in dw) \mathbb{P}(\tau \in dt, Z_\tau \in dw)$$

$$= \int_{t=0}^{T} \int_{w=-\infty}^{b} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(z-w)^2}{2(T-t)}} dz \mathbb{P}(\tau \in dt, Z_\tau \in dw)$$

$$= \int_{t=0}^{T} \int_{w=-\infty}^{b} \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(z-w)^2}{2(T-t)}} dz \Big\{ \boxed{\mathbb{P}(\tau \in dt, Z_\tau \in dw | H_b < \tau_d^+) \mathbb{P}(H_b < \tau_d^+)} +$$

$$+ \boxed{\mathbb{P}(\tau \in dt, Z_\tau \in dw | \tau_d^+ < H_b) \mathbb{P}(\tau_d^+ < H_b)} \Big\}$$

case $H_b < \tau_d^+$:

$$\boxed{\mathbb{P}(\tau \in dt, Z_\tau \in dw | H_b < \tau_d^+)\mathbb{P}(H_b < \tau_d^+)}$$
$$= \mathbb{P}(Z_{H_b} \in dw | \tau = t, H_b < \tau_d^+)\mathbb{P}(\tau \in dt, H_b < \tau_d^+)$$
$$= \delta_{(w-b)} dw \mathcal{L}_\beta^{-1} \left\{ \mathbb{E}\left(e^{-\beta \min\{H_b, \tau_d^+\}} 1_{H_b < \tau_d^+} \right) \right\} \Big|_t dt$$

case $\tau_d^+ < H_b$:

$$\boxed{\mathbb{P}(\tau \in dt, Z_\tau \in dw | \tau_d^+ < H_b)\mathbb{P}(\tau_d^+ < H_b)}$$
$$= \mathbb{P}(Z_{\tau_d^+} \in dw | \tau = t, \tau_d^+ < H_b)\mathbb{P}(\tau \in dt, \tau_d^+ < H_b)$$
$$= \lim_{\epsilon \to 0} \mathbb{P}_\epsilon(Z_d \in dw | \inf_{0<s<d} Z_s > 0, \sup_{0<s<d} Z_s < b)\mathbb{P}(\tau \in dt, \tau_d^+ < H_b)$$
$$= \frac{\sum_{k=-\infty}^{\infty} \frac{w+2kb}{d} e^{-\frac{(w+2kb)^2}{2d}} dw}{\sum_{k=-\infty}^{\infty} \left(e^{-\frac{(2kb)^2}{2d}} - e^{-\frac{(2k+1)^2 b^2}{2d}} \right)} \mathcal{L}_\beta^{-1} \left\{ \mathbb{E}\left(e^{-\beta \tau_d^+} 1_{\tau_d^+ < H_b} \right) \right\} \Big|_t dt$$

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