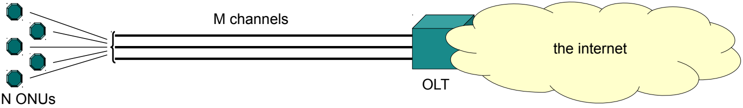
On the stability of a dynamical system arising in telecommunication networks

Maite Wilke Berenguer

Technische Universität Berlin
(RTG 1845)

Young Women in Probability 2014
Bonn, May 2014

„the last/first mile“

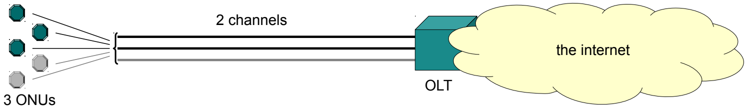


Aurzada; Scheutzow; Reisslein; and Maier (2010):

Towards a Fundamental Understanding of the Stability and Delay of Offline WDM EPONs.
IEEE/OSA Journal of Optical Communications and Networking, **2**, 51–66

Setting

$N = 3$, $M = 2$



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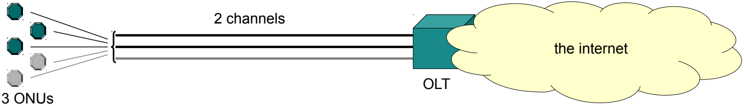

- Poisson arrivals with parameter $\lambda > 0$ [bits / s]
- channel capacity $C = 1$ [bits / s]

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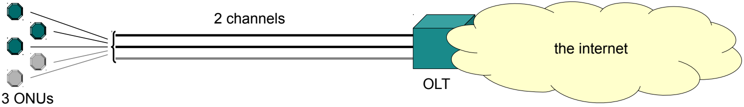

- Poisson arrivals with parameter $\lambda > 0$ [bits / s]
- channel capacity $C = 1$ [bits / s]

normalised traffic load $\rho := \frac{\lambda}{C} = \lambda$

Protocol

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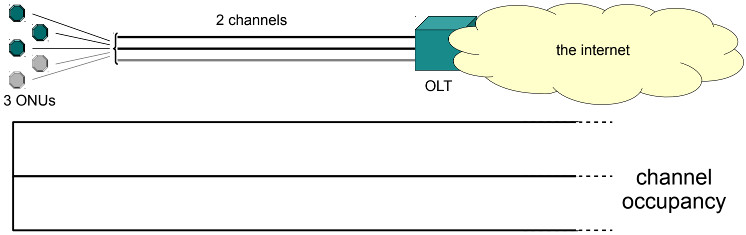

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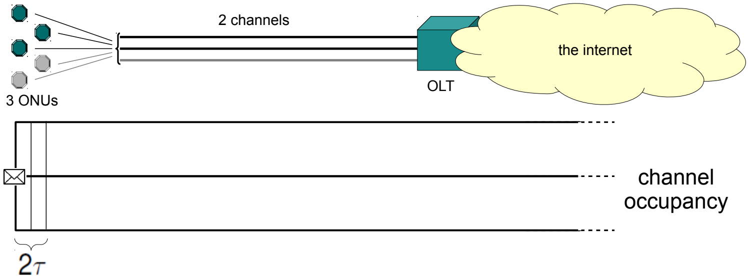

- ONUs send report (✉) to OLT

Protocol

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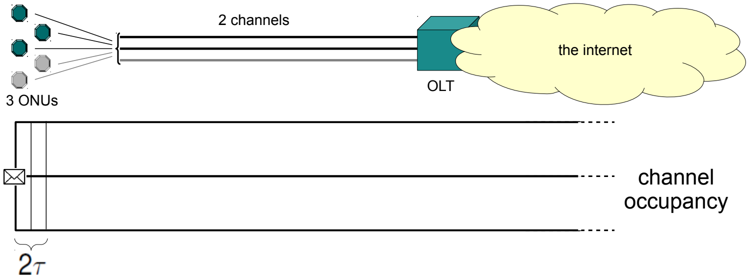

- ONUs send report (✉) to OLT
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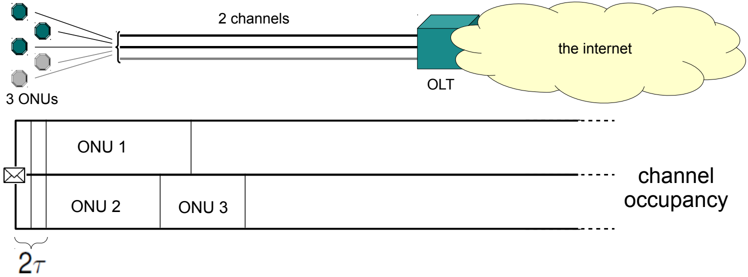

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- ONUs send all packets reported while new packets arrive

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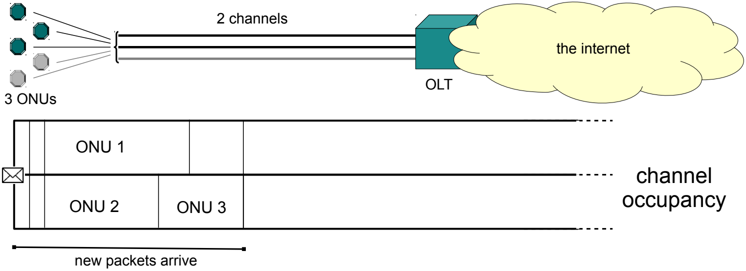

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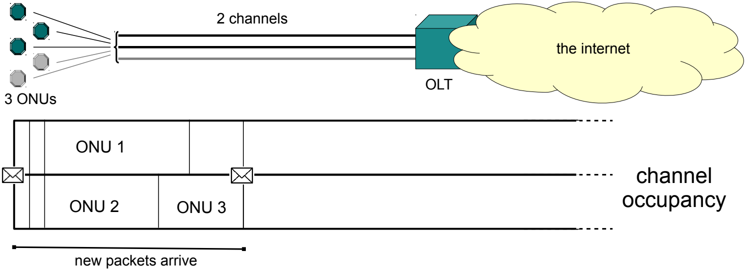

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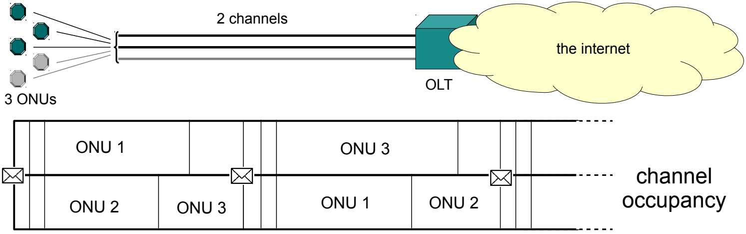

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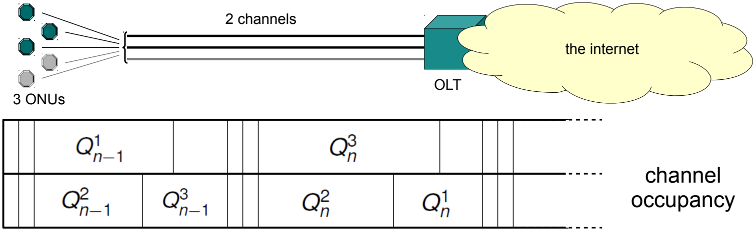
Protocol

synchronized reporting



- ONUs send report (⌧) to OLT
- OLT sends grant to ONUs (largest-processing-time-first schedule)
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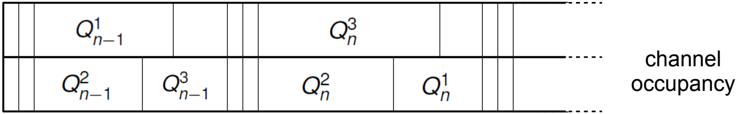
- ONUs send new report to OLT

Q^i_n : duration of the *n*th transmission of *i*th ONU

Immediate reporting



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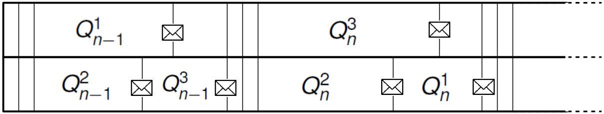
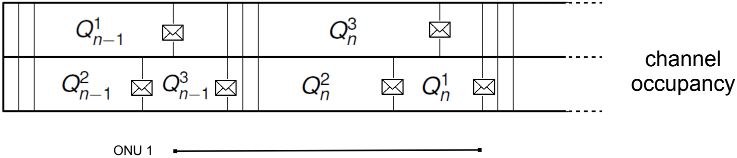


channel occupancy

Immediate reporting



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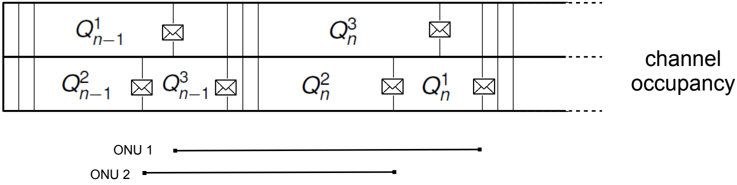

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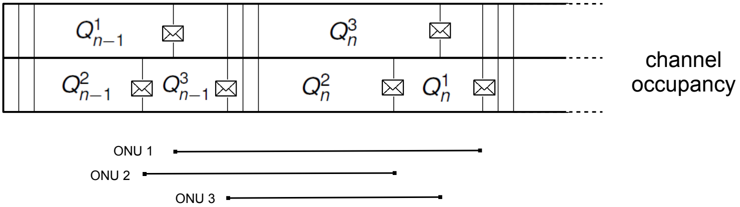

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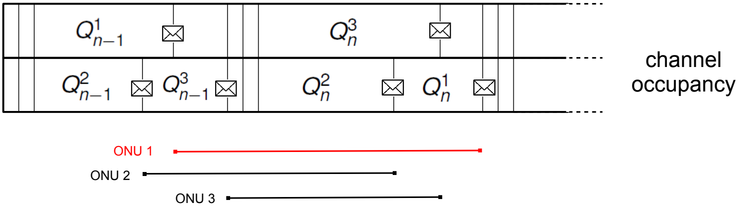

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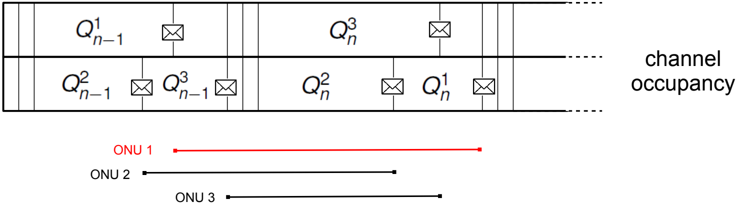
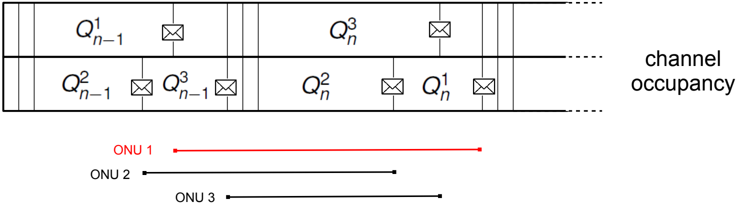


$$-Q^1_{n-1} + Q^2_{n-1} + Q^3_{n-1} + 2\tau + Q^1_n + Q^2_n$$

Immediate reporting



$$-Q^1_{n-1} + Q^2_{n-1} + Q^3_{n-1} + 2\tau + Q^1_n + Q^2_n$$

$$\Rightarrow Q^1_{n+1} \sim \text{Poi}(\rho(-Q^1_{n-1} + Q^2_{n-1} + Q^3_{n-1} + 2\tau + Q^1_n + Q^2_n))$$

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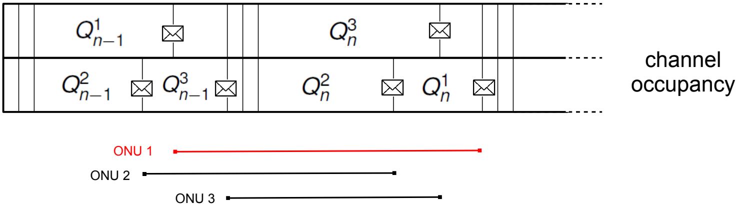

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and thus (for random $A_{n+1}, B_{n+1} \in \{-1, 0, 1\}^{3\times 3}$)

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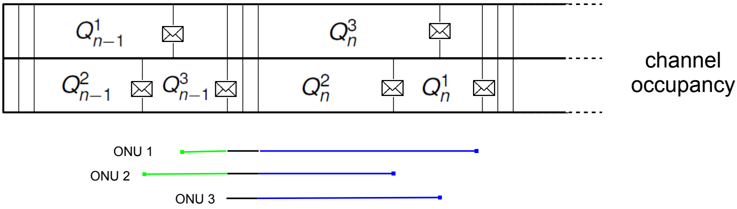

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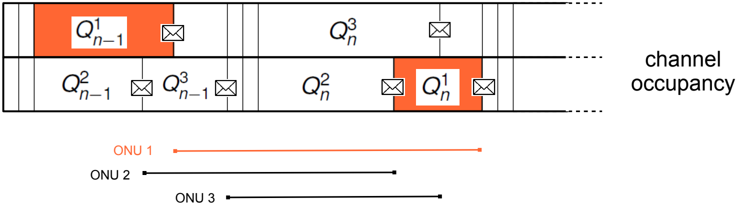

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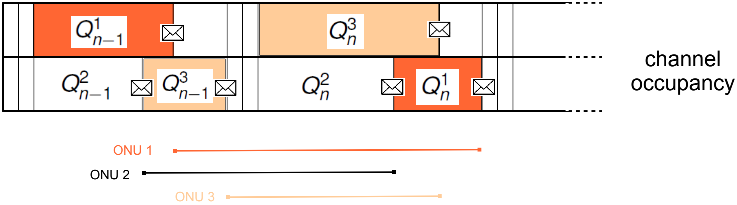

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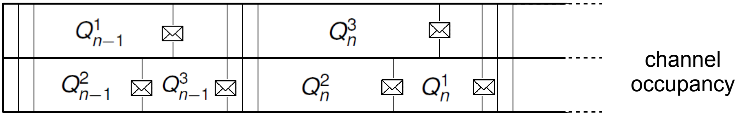

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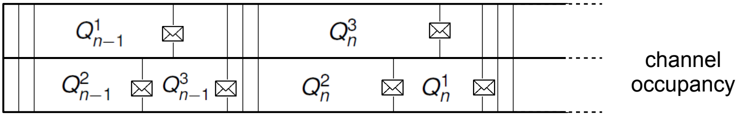

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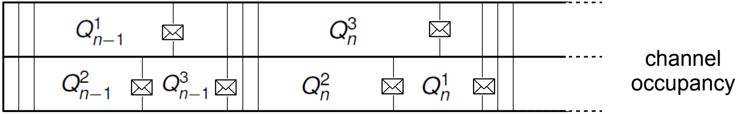
What is the critical value ρ_c?

Immediate reporting



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expect: $\rho_c = \frac{1}{2}$

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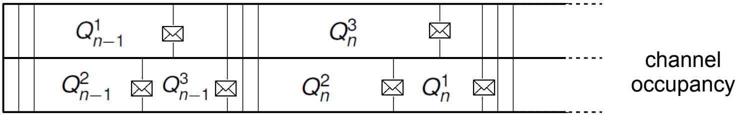

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What is the critical value ρ_c?

expect: $\rho_c = \frac{1}{2}$

simulations suggest: $\rho_c = \frac{1}{\sqrt{3}}$

The deterministic system

Bassir (2010)

Stabilitätsanalyse eines dynamischen Systems aus der Nachrichtentechnik: WDM EPONs, *Diplomarbeit (TU Berlin)*

$$\bar{Q}_{n+1} \sim \text{Poi}\left(\rho\left(A_{n+1}\bar{Q}_{n-1} + B_{n+1}\bar{Q}_n + 2\bar{\tau}\right)\right)$$

$$\bar{q}_{n+1} = \rho\left(a_{n+1}\bar{q}_{n-1} + b_{n+1}\bar{q}_n + 2\bar{\tau}\right)$$

The deterministic system

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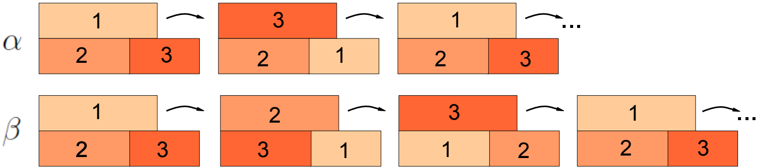

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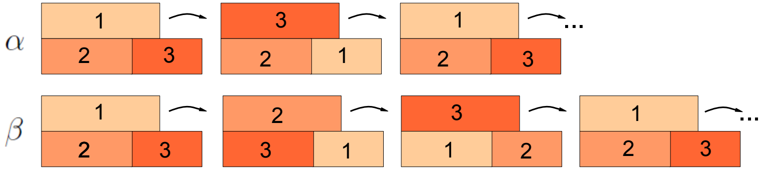

$$\rho_\alpha = \frac{1}{\sqrt{3}} \qquad \rho_\beta = \frac{1}{9}\left(1 - \frac{8}{\sqrt[3]{109+27\sqrt{17}}} + \sqrt[3]{109+27\sqrt{17}}\right)$$

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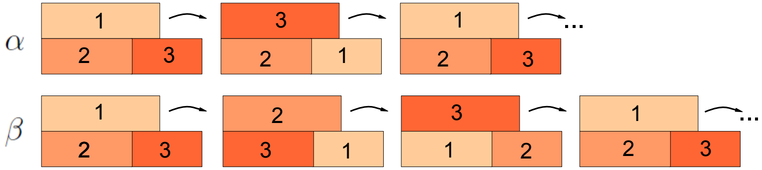

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$$\approx 0.57735 \qquad \approx 0.635024$$

simulations: an explanation

Code: courtesy of Frank Aurzada.

1. $Q_n^1 + Q_n^2 + Q_n^3$
2. $\begin{pmatrix} Q_n^1 \\ Q_n^2 \\ Q_n^3 \end{pmatrix}$

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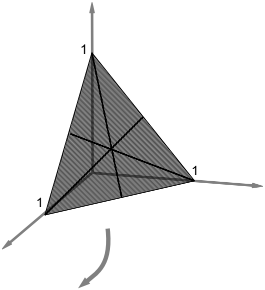
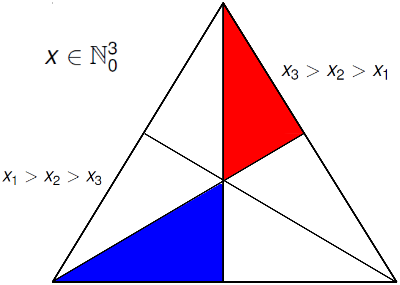
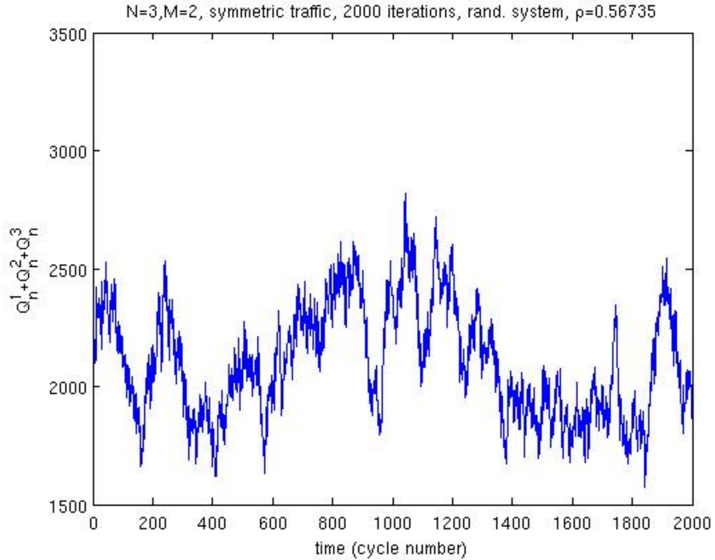


$x \in \mathbb{N}_0^3$



$\frac{1}{2} < \rho < \rho_\alpha = \frac{1}{\sqrt{3}}$



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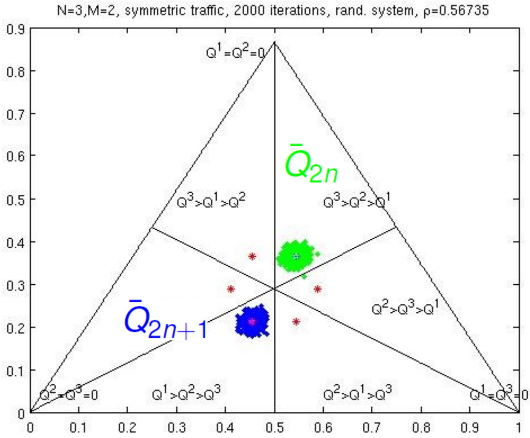

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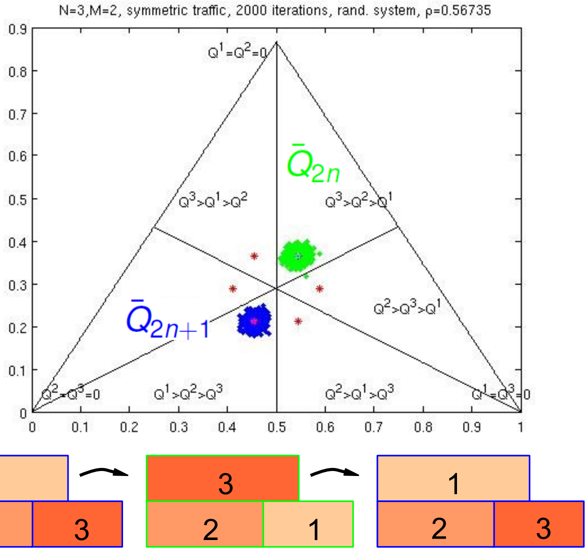
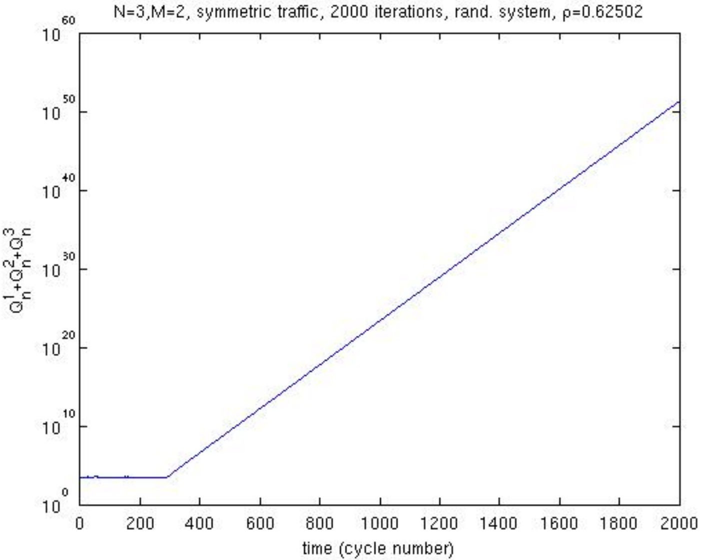


$\frac{1}{\sqrt{3}} = \rho_\alpha < \rho < \rho_\beta$



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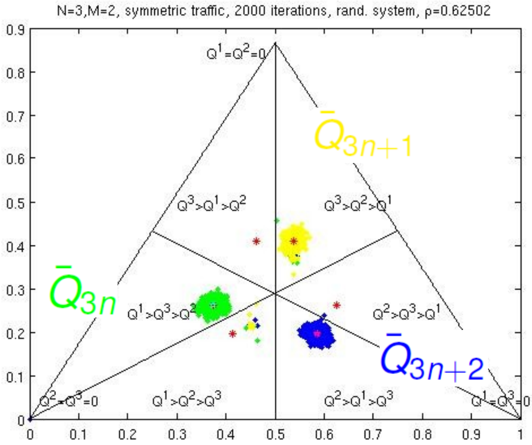

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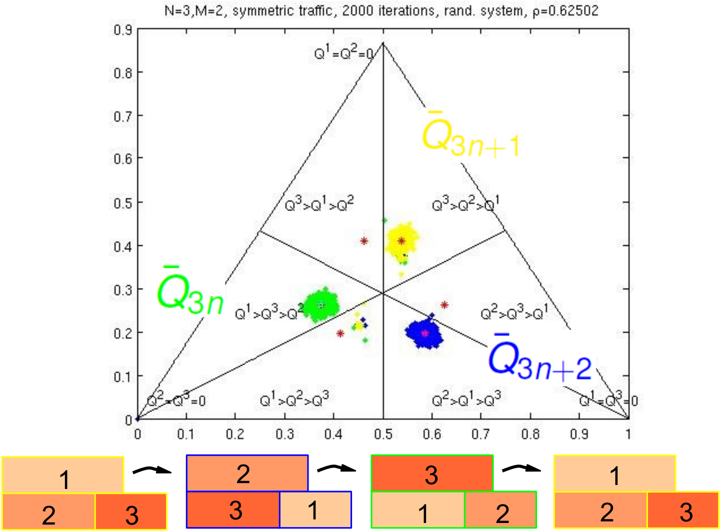
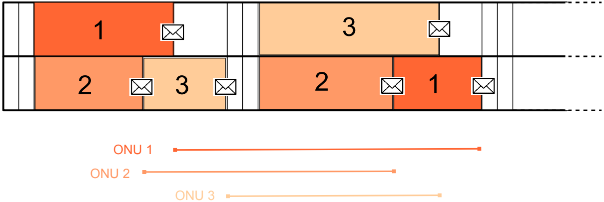


To take home:

$$\bar{Q}_{n+1} \sim \text{Poi}\left(\rho\left(A_{n+1}\bar{Q}_{n-1} + B_{n+1}\bar{Q}_n + 2\bar{\tau}\right)\right)$$

Critical value $\rho_c = \frac{1}{\sqrt{3}}$ is larger than expected, since **distinct waiting times** can fall into **stabilizing pattern** for $N = 3$ and $M = 2$!



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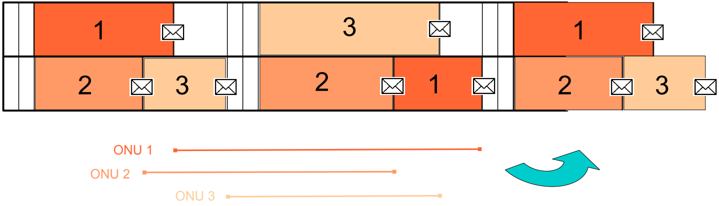

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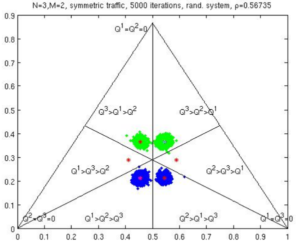
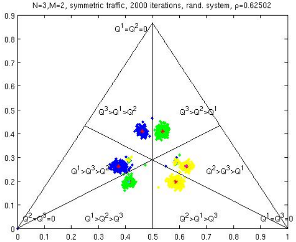
Problem: Changing A_n, B_n!





Outlook

To do:

- $\rho_c = \frac{1}{\sqrt{3}}$
- characterisation and analysis of $\frac{1}{\sqrt{3}} \leq \rho \leq \rho_\beta$
- limiting properties of the invariant distribution for $\rho \to \frac{1}{\sqrt{3}}$
- rate, delay...

THANK YOU!

Transience

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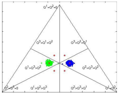
Now consider “$X_n := \text{sort}(\tilde{Q}_n)$” on a pattern  :

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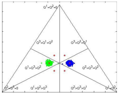
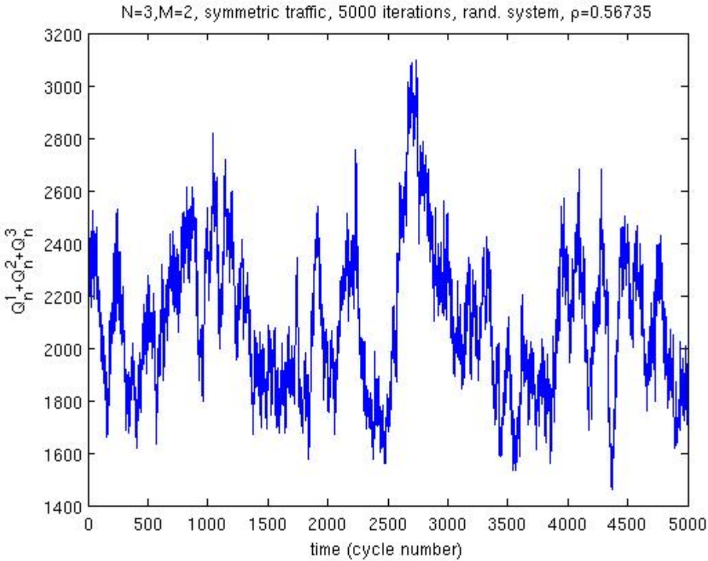
Now consider “$X_n := \text{sort}(\tilde{Q}_n)$” on a pattern :

$$X_{n+1} = C_\rho X_n + \xi_{n+1}(X_n, .)$$

$\frac{1}{2} < \rho < \rho_\alpha = \frac{1}{\sqrt{3}}$



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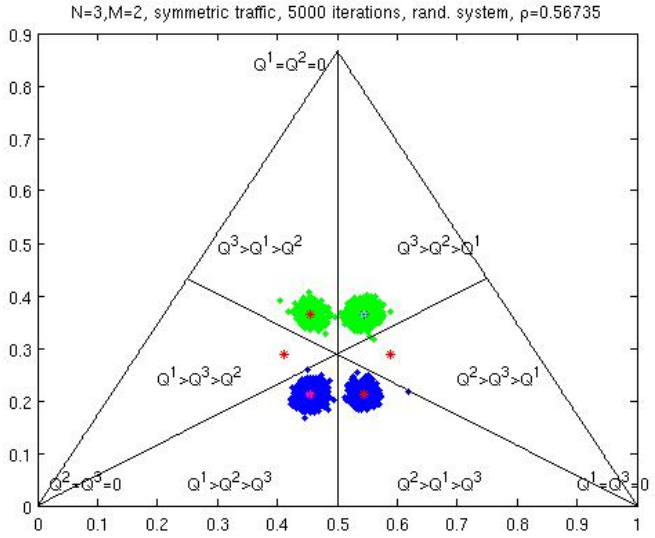
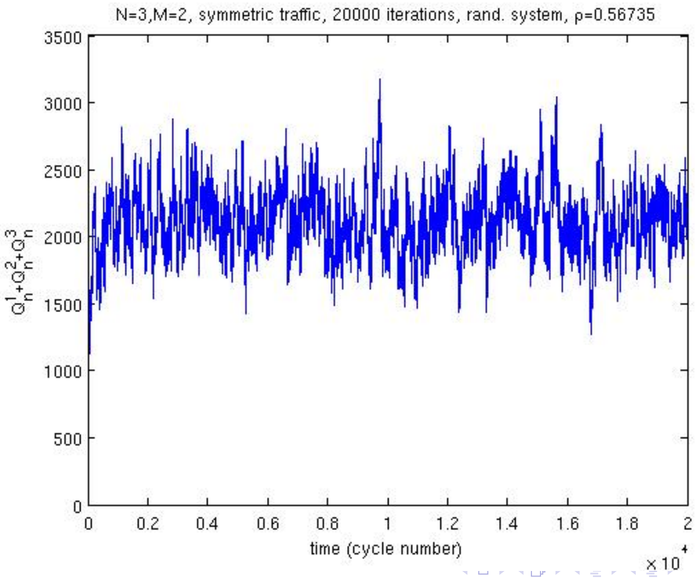


long simulation



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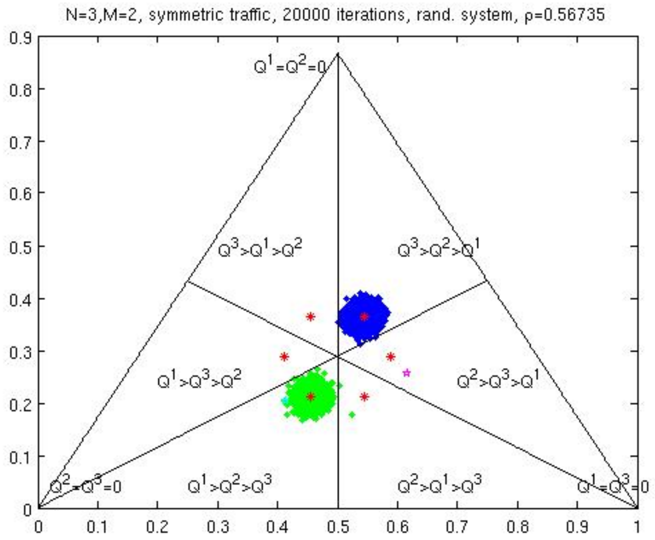
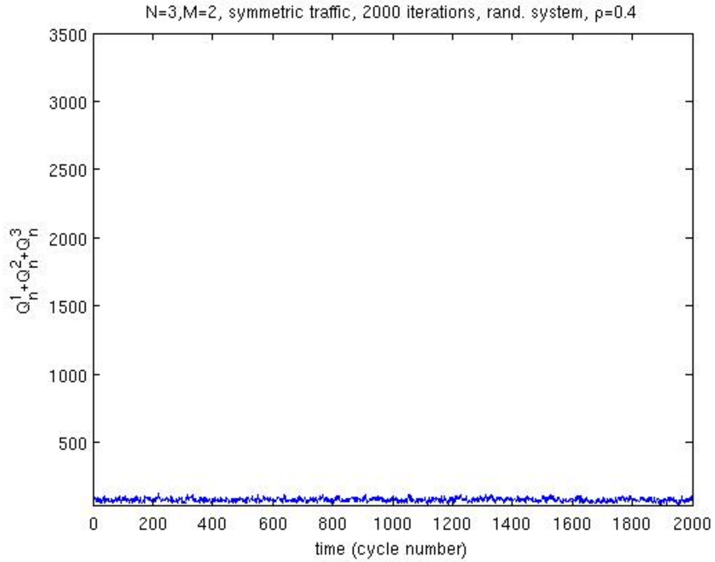


same scale stable cases



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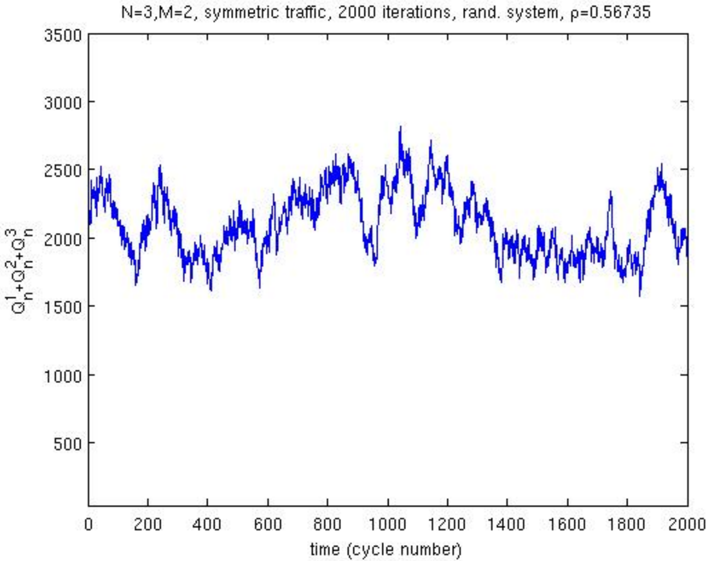
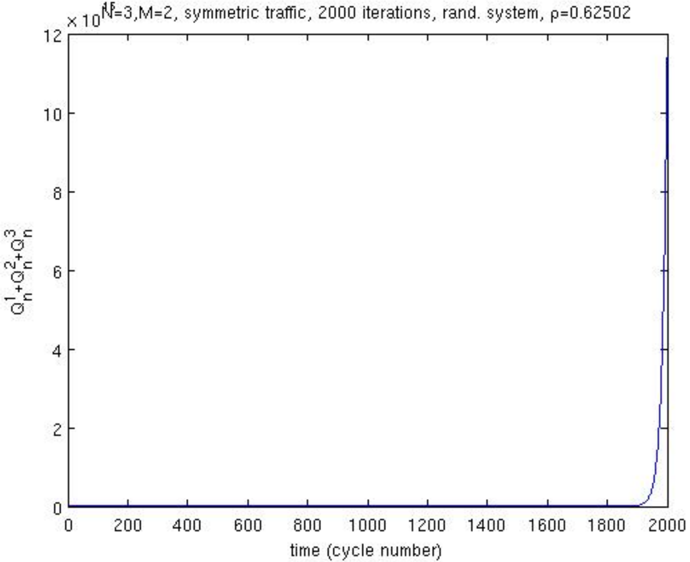


another $\rho_\alpha < \rho < \rho_\beta$



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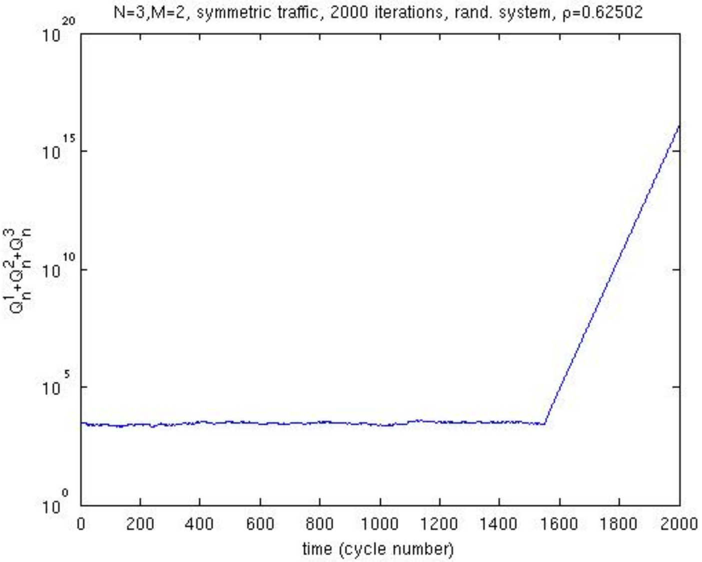

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Stability of networks

M W B

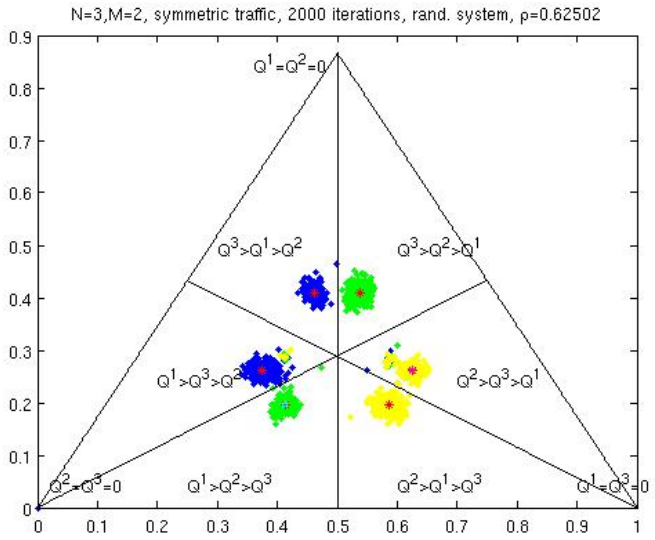
Motivation and Setting

Immediate reporting

The deterministic system

Simulations

Summary and Outlook



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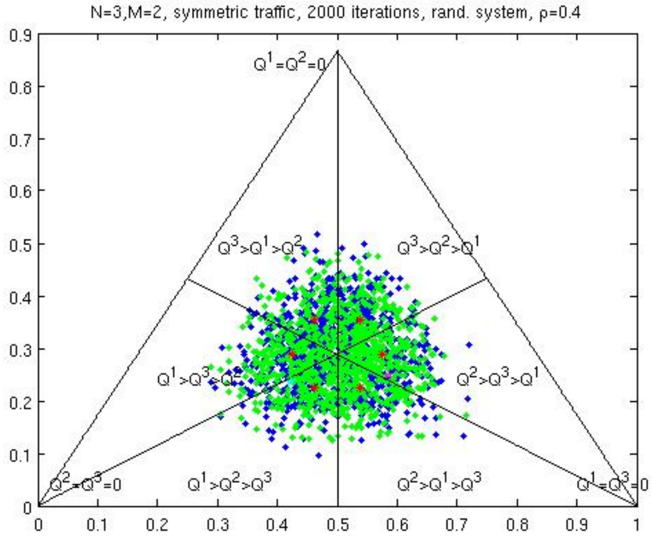
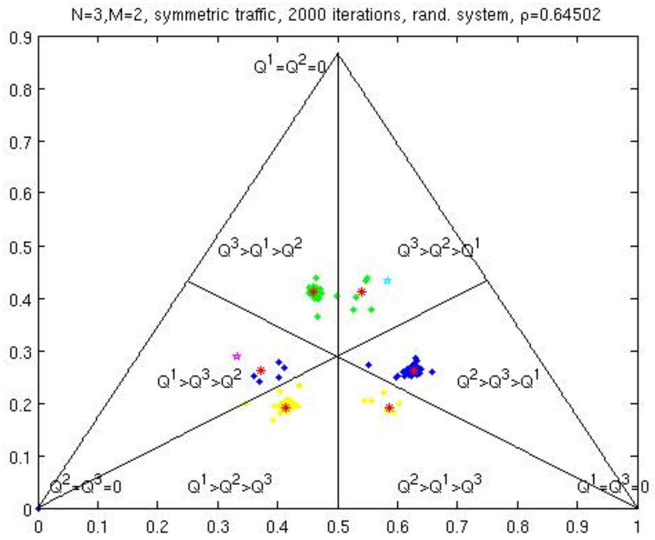


$\rho_\beta < \rho$



N=3,M=2, symmetric traffic, 2000 iterations, rand. system, ρ=0.64502
Q1=Q2=0
Q3>Q1>Q2
Q3>Q2>Q1
Q1>Q3>Q2
Q2>Q3>Q1
Q2=Q3=0
Q1>Q2>Q3
Q2>Q1>Q3
Q1=Q3=0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

the deterministic model

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Let λ_α, λ_β be the largest eigenvalue of C_α, resp. C_β.

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$$\lambda_\alpha \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\} 1 \Leftrightarrow \rho \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\} \frac{1}{\sqrt{3}} \qquad \lambda_\beta \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\} 1 \Leftrightarrow \rho \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\} \rho*$$

simulations: a detailed explanation

1. $Q_n^1 + Q_n^2 + Q_n^3$

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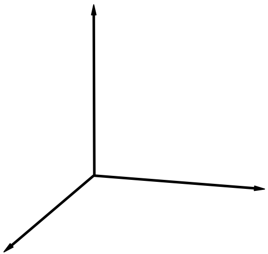

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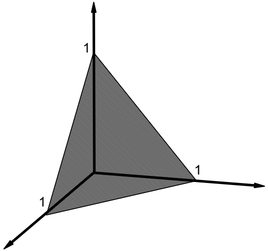

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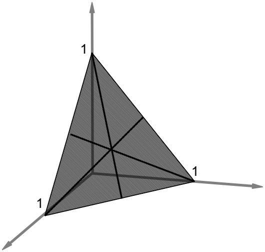

simulations: a detailed explanation

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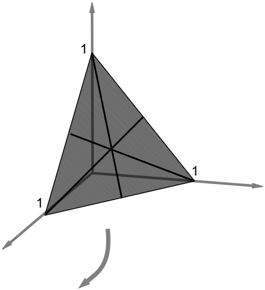
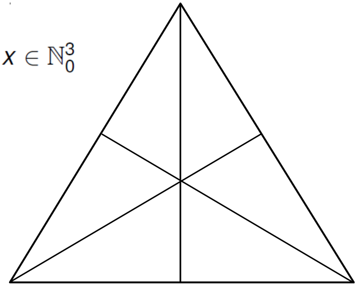


$x \in \mathbb{N}^3_0$



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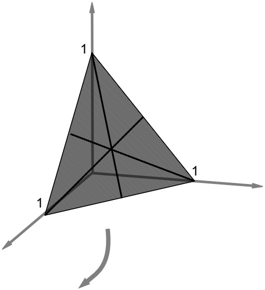

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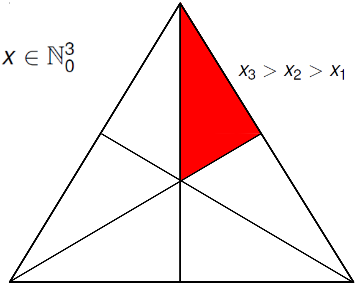

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