Decomposition along the diameter of the Brownian Continuum Random Tree

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Motivation

limit theorems for height and diameter

- Let $n \geq 1$. There are n^{n-1} rooted trees on vertices $1, 2, \cdots, n$ (Cayley, 1889).
  Let T_n be a uniformly picked tree.
- Endow each edge with length 1.
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where $\mathbf{P}\left(\mathrm{M}^* > x\right) = 2\sum_{n=1}^{\infty}\left(n^2x^2 - 1\right)e^{-n^2x^2/2}, \ x > 0.$

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  where $\mathbf{P}\left(\Delta > x\right) = \sum_{n=1}^{\infty}(n^2 - 1)(\frac{n^4x^4}{24} - n^2x^2 + 2)e^{-n^2x^2/8}$.

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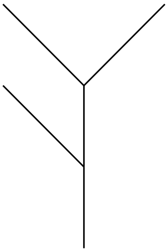

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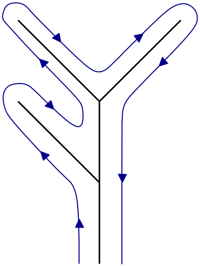

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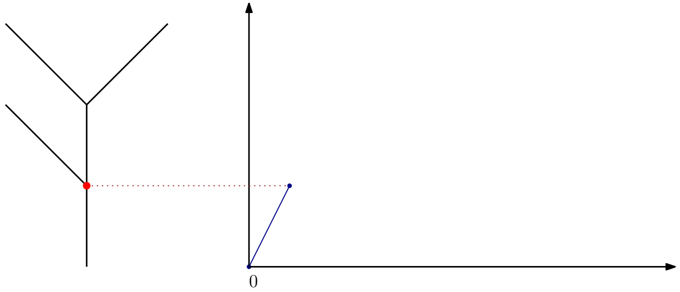

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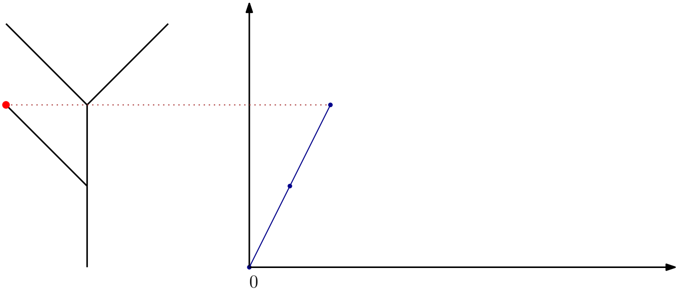

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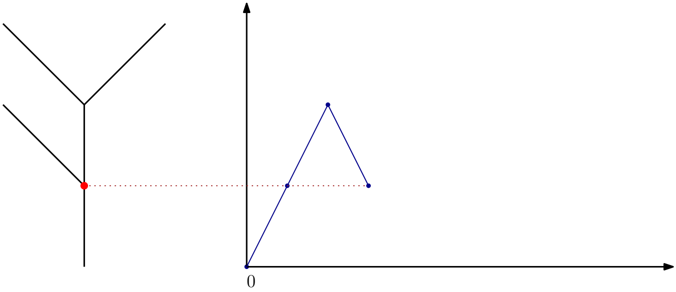

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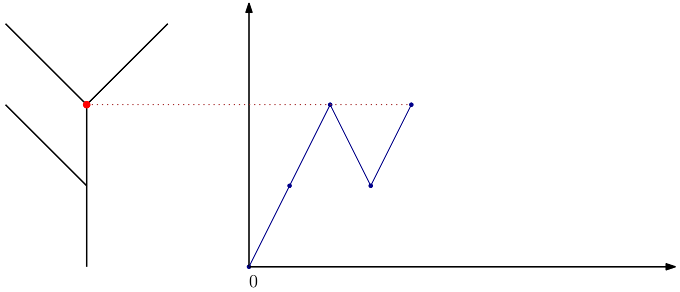

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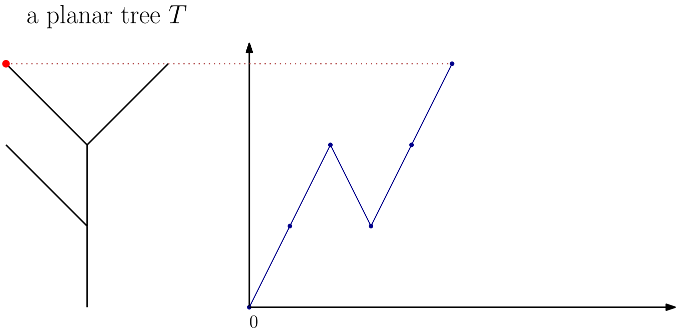

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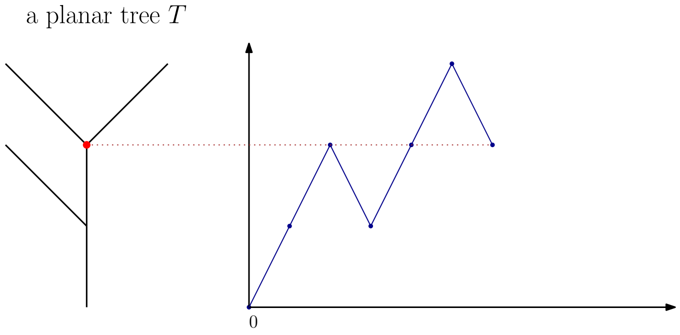

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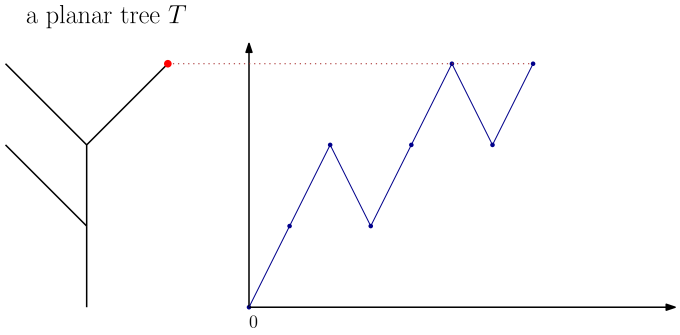

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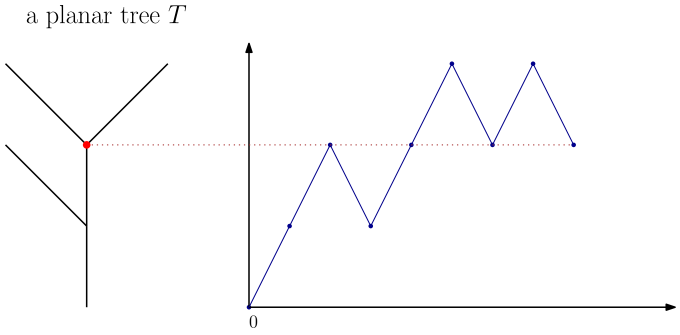

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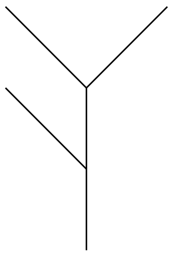
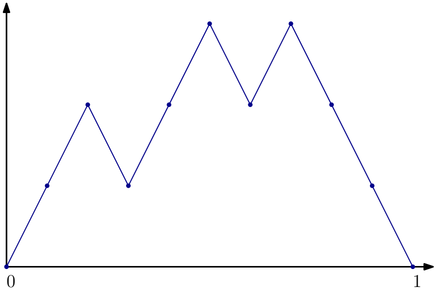


its contour function $\mathcal{C}(T)$



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$$\frac{1}{\sqrt{n}}\mathcal{C}(T_n) \Longrightarrow 2e \text{ in } \mathbf{C}([0,1]),$$

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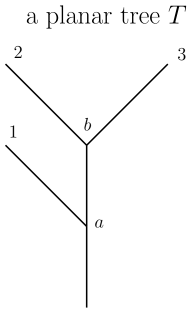

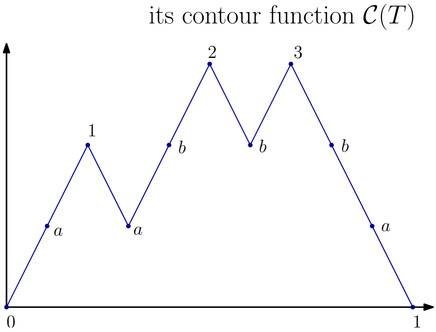

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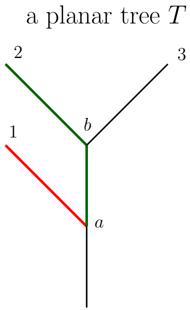

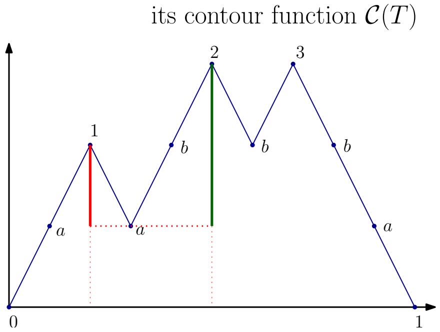

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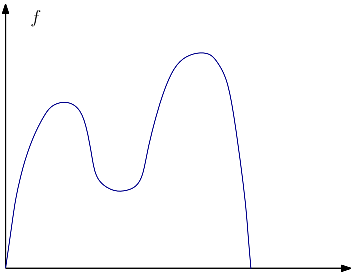

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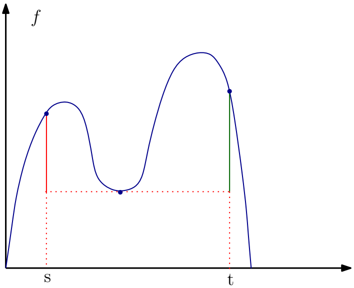

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more generally, $f \in \mathbf{C}_c$, $f \geq 0$, $d_f(s,t) = f(s) + f(t) - 2I_f(s,t)$
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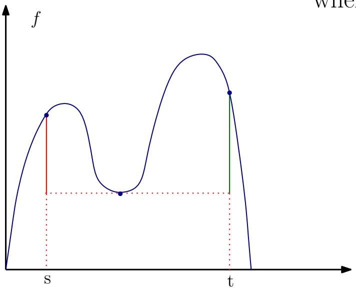

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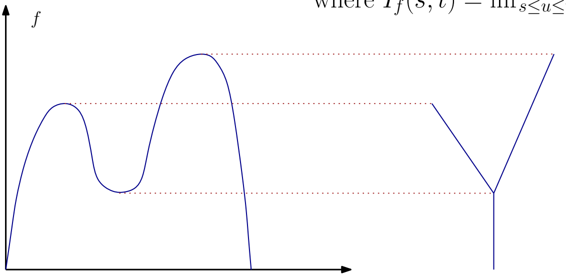

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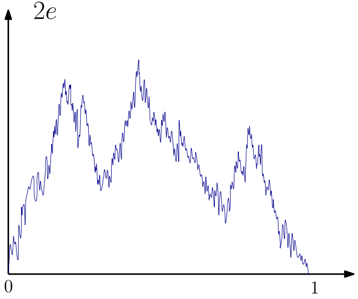

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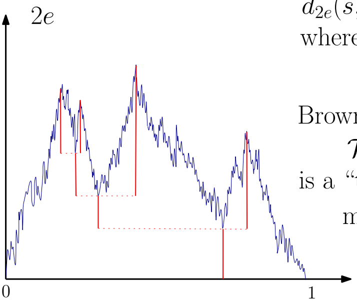

$d_{2e}(s,t) = 2(e_s + e_t - 2I_e(s,t))$
where $I_e(s,t) = \inf_{s\le u\le t} e_u$

Brownian CRT

$$\mathcal{T}^{br} := ([0,1]/\sim_d, d_{2e})$$

is a "tree-like" (random) compact metric space.

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limit theorems

- Aldous' Theorem:
$$\frac{1}{\sqrt{n}} T_n \Longrightarrow \mathcal{T}^{br},$$
in the sense that $\frac{1}{\sqrt{n}}\mathcal{C}(T_n) \Longrightarrow \mathcal{C}(\mathcal{T}^{br})$ in $\mathbf{C}([0,1])$.

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distribution of $\boldsymbol{\Gamma}(\mathcal{T}^{br})$

- We must have
$$\boldsymbol{\Gamma}(\mathcal{T}^{br}) \stackrel{d}{=} \mathrm{M}^*,$$
where M^* is the limit distribution of $\frac{1}{\sqrt{n}}\boldsymbol{\Gamma}(T_n)$, given by
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- On the other hand,
$$\boldsymbol{\Gamma}(\mathcal{T}^{br}) = 2\sup_{0\leq s\leq 1} e_s,$$
whose distribution has been deduced independently by Kennedy, Chung, Durrett et al. in the 70s.

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- Szekeres:

$$\frac{1}{\sqrt{n}}\mathbf{D}(T_n) \Longrightarrow \Delta.$$

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a question of Aldous

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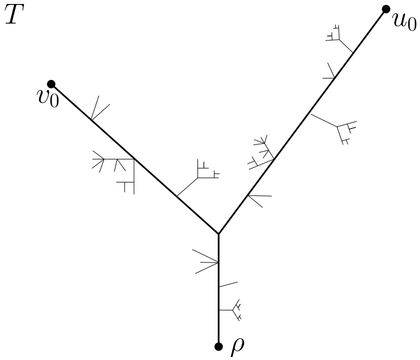
- Aldous (CRT II, '91): Can we deduce the distribution of

$$\mathbf{D}(\mathcal{T}^{br}) = 2\sup_{0\le s\le t\le 1}\Big(e_s + e_t - 2I_e(s,t)\Big)$$

from the Brownian excursion itself?

Distribution of the diameter of $\mathcal{T}^{br}$

first observation



$$d(u_0, v_0) = \mathbf{D}(T)$$
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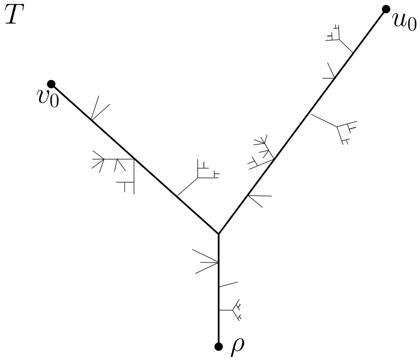

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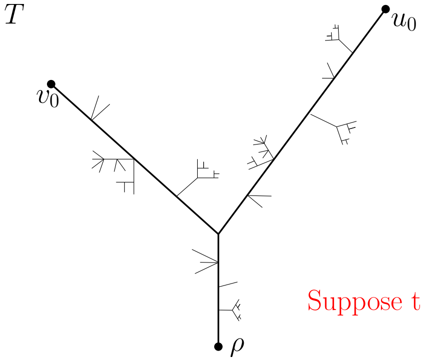

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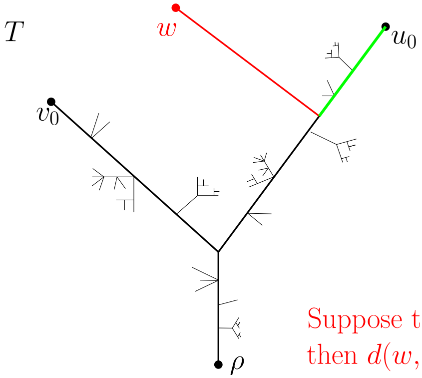

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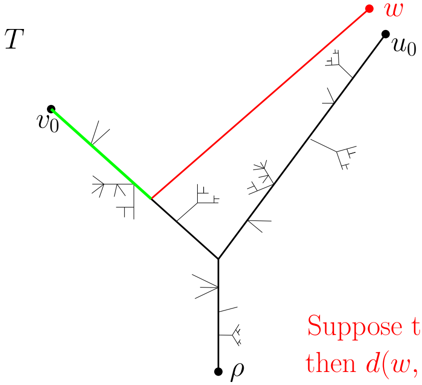

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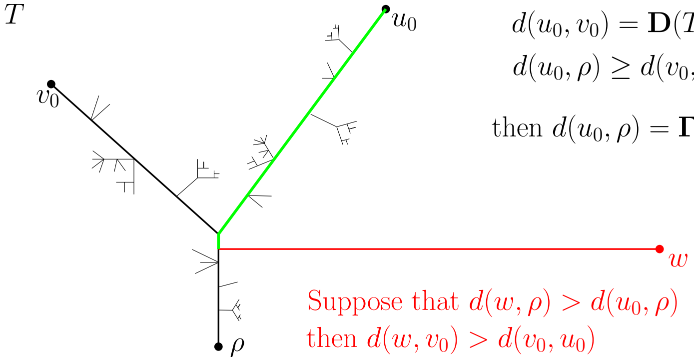
Distribution of the diameter of $\mathcal{T}^{br}$

proof on a picture: third case



$$d(u_0, v_0) = \mathbf{D}(T)$$

$$d(u_0, \rho) \geq d(v_0, \rho)$$

then $d(u_0, \rho) = \mathbf{\Gamma}(T)$

Suppose that $d(w, \rho) > d(u_0, \rho)$

then $d(w, v_0) > d(v_0, u_0)$

Distribution of the diameter of $\mathcal{T}^{br}$

first observation

- Let T be a tree rooted at ρ. If $u_0, v_0 \in T$ such that

$$d(u_0, v_0) = \mathbf{D}(T), \quad d(u_0, \rho) \geq d(v_0, \rho),$$

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- By the properties of the Brownian excursion, there exists almost surely a unique point s_0 such that

  $$\mathbf{\Gamma}(\mathcal{T}^{br}) = 2 \sup_{0 \leq s \leq 1} e_s = 2e_{s_0}.$$

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- By the properties of the Brownian excursion, there exists almost surely a unique point s_0 such that

$$\mathbf{\Gamma}(\mathcal{T}^{br}) = 2 \sup_{0 \leq s \leq 1} e_s = 2e_{s_0}.$$

- Therefore, almost surely,

$$\mathbf{D}(\mathcal{T}^{br}) = \sup_{0 \leq s \leq t \leq 1} d_{2e}(s, t) = \sup_{0 \leq t \leq 1} d_{2e}(s_0, t).$$

Distribution of the diameter of $\mathcal{T}^{br}$

Williams' decomposition

- Let n_+ be the Ito's excursion measure on $\mathbf{C}(\mathbf{R}_+, \mathbf{R}_+)$. Let $(\omega_s, 0 \leq s \leq \zeta)$ be the coordinate process.
- Under n_+, there exists a unique point s_0 such that

$$\omega_{s_0} = \sup_{0 \leq s \leq \zeta} \omega_s.$$

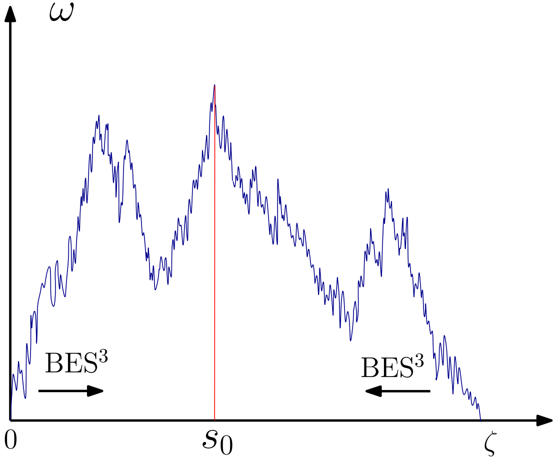
- Under n_+ and given $\omega_{s_0} = c$,

$$(\omega_s, 0 \leq s \leq s_0) \quad \text{and} \quad (\omega_{\zeta - s}, 0 \leq s \leq \zeta - s_0)$$

  are distributed as two independent $\mathrm{BES}^3(0)$ processes which run until hitting c.

Distribution of the diameter of $\mathcal{T}^{br}$

Williams' decomposition



Distribution of the diameter of $\mathcal{T}^{br}$

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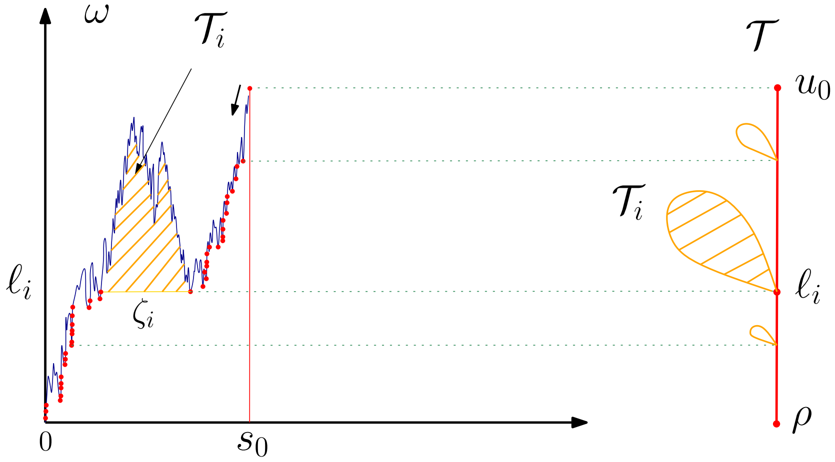


Distribution of the diameter of $\mathcal{T}^{br}$

Williams' decomposition: a representation by tree

Under $n_+(\cdot|\boldsymbol{\Gamma}(\mathcal{T}) = c)$,

$$\sum_{i\geq 1} \delta_{(\ell_i,\mathcal{T}_i)}$$

is a Poisson point measure of intensity

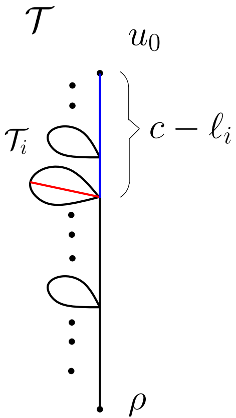
$$dt \cdot n_{c-t}$$

where $n_a = n_+(\cdot|\boldsymbol{\Gamma}(\mathcal{T}) < a)$ is the restriction of n_+ on $\{\boldsymbol{\Gamma}(\mathcal{T}) < a\}$.

Distribution of the diameter of $\mathcal{T}^{br}$

observation on the diameter

$$\mathbf{D}(\mathcal{T}) = \sup_{i\geq 1}\left(\mathbf{\Gamma}(\mathcal{T}_i) + c - \ell_i\right)$$



Distribution of the diameter of $\mathcal{T}^{br}$

calculation

Notice that $\zeta = \sum_{i\geq 1} \zeta_i$. Then, for $\lambda > 0$ and $y > c$,

$$n_+\Big(e^{-\lambda\zeta}\mathbf{1}_{\{\mathbf{D}(\mathcal{T})\leq y\}}\Big|\mathbf{\Gamma}(\mathcal{T}) = c\Big) = \mathbf{E}\Big[\prod_{i\geq 1} e^{-\lambda\zeta_i}\mathbf{1}_{\{\mathbf{\Gamma}(\mathcal{T}_i)+c-\ell_i\leq y\}}\Big]$$

$$= \exp\left(-\int_0^c dt\cdot n_{c-t}\Big(1 - e^{-\lambda\zeta}\mathbf{1}_{\{\mathbf{\Gamma}(\mathcal{T})+c-t\leq y\}}\Big)\right)$$

$$= \begin{cases} \frac{\sqrt{2}c^2\lambda\sinh^2((y-c)\sqrt{2\lambda})}{\sinh^4(y\sqrt{\lambda/2})}, & y < 2c \\ \frac{\sqrt{2}c^2\lambda}{\sinh^2(c\sqrt{2\lambda})}, & y \geq 2c. \end{cases}$$

Here, we have used the fact that
$n_+(1 - e^{-\lambda\zeta}\mathbf{1}_{\{\mathbf{\Gamma}(\mathcal{T})<a\}}) = \sqrt{\lambda/2}\coth(a\sqrt{2\lambda})$
$(\Leftrightarrow n_+(e^{-\lambda\zeta}|\mathbf{\Gamma}(\mathcal{T}) = a) = \left(\frac{a\sqrt{2\lambda}}{\sinh(a\sqrt{2\lambda})}\right)^2)$.

Distribution of the diameter of $\mathcal{T}^{br}$

calculation

By integrating with respect to $n_+(\mathbf{\Gamma}(\mathcal{T}) > c) = 1/(2c)$, we find that for each $y > 0$,

$$n_+\left(e^{-\lambda\zeta}\mathbf{1}_{\{\mathbf{D}(\mathcal{T})>y\}}\right) = \sqrt{\lambda/2}\left(\coth(\sqrt{\lambda/2}y) - 1\right.$$
$$\left. - \frac{1}{\sinh^2(\sqrt{\lambda/2}y)}\left(\sqrt{\lambda/8}\coth(\sqrt{\lambda/2}y) - \frac{\lambda y}{4\sinh^2(\sqrt{\lambda/2}y)}\right)\right).$$

Distribution of the diameter of $\mathcal{T}^{br}$

spinal decomposition along the height

Since $(\zeta^{-1/2} \cdot \omega_{s\zeta})_{0 \le s \le 1} \overset{d}{=} (e_s)_{0 \le s \le 1}$,

$$\zeta^{-1/2} \cdot \mathbf{D}(\mathcal{T}) \overset{d}{=} \mathbf{D}(\mathcal{T}^{br})/2.$$

Therefore,

$$n_+\left(e^{-\lambda\zeta} \mathbf{1}_{\{\mathbf{D}(\mathcal{T})>y\}}\right)$$
$$= \int e^{-\lambda x} n_+(\zeta \in dx) \cdot \mathbf{P}\left(\sqrt{x}\,\mathbf{D}(\mathcal{T}^{br})/2 > y\right).$$

Distribution of the diameter of $\mathcal{T}^{br}$

conclusion

By the inverse Laplace transform and the fact that $n_+(\zeta \in dx) = (2\sqrt{2\pi x^3})^{-1} dx$, we find

$$\mathbf{P}\left(\mathbf{D}(\mathcal{T}^{br}) > y\right) = \sum_{n=1}^{\infty} (n^2 - 1)\left(\frac{n^4 y^4}{24} - n^2 y^2 + 2\right) e^{-n^2 y^2/8}, \quad y > 0.$$

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Recall Jacobi's identity on the theta function:

$$\text{if} \quad \theta(x) = \sum_{n=-\infty}^{\infty} e^{-n^2 \pi x}, \quad \text{then} \quad \sqrt{x}\theta(x) = \theta(x^{-1})$$

for each $x > 0$. It follows that

$$\mathbf{P}\left(\mathbf{D}(\mathcal{T}^{br}) > y\right) = 1 - 2^{37/2}\pi^{5/2} y^{-9} \sum_{n=1}^{\infty}\Big(\frac{1024}{3}\pi^4 n^4 - 24\pi^2 n^2 y^2 + \frac{2}{3}\pi^2 n^2 y^4 + \frac{1}{4} y^4\Big) e^{-64\pi^2 n^2/y^2}, \quad y > 0.$$

Other consequences

joint law of $\mathbf{\Gamma}(\mathcal{T}^{br})$ and $\mathbf{D}(\mathcal{T}^{br})$

- We have calculated $n_+\left(e^{-\lambda\zeta}\mathbf{1}_{\{\mathbf{D}(\mathcal{T})\leq y\}}\middle|\mathbf{\Gamma}(\mathcal{T})=c\right)$.

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joint law of $\mathbf{\Gamma}(\mathcal{T}^{br})$ and $\mathbf{D}(\mathcal{T}^{br})$

- We have calculated $n_+\left(e^{-\lambda\zeta}\mathbf{1}_{\{\mathbf{D}(\mathcal{T})\leq y\}}\middle|\mathbf{\Gamma}(\mathcal{T})=c\right)$.
- By integration, we find an expression for

$$n_+\left(e^{-\lambda\zeta}\mathbf{1}_{\{\mathbf{D}(\mathcal{T})>y,\mathbf{\Gamma}(\mathcal{T})>z\}}\right).$$

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- By the scaling property,

$$n_+\left(e^{-\lambda\zeta}\mathbf{1}_{\{\mathbf{D}(\mathcal{T})>y,\mathbf{\Gamma}(\mathcal{T})>z\}}\right)$$
$$=\int e^{-\lambda x}n_+(\zeta\in dx)\cdot\mathbf{P}\left(\sqrt{x}\,\mathbf{D}(\mathcal{T}^{br})/2>y,\sqrt{x}\,\mathbf{\Gamma}(\mathcal{T}^{br})/2>z\right).$$

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joint law of $\mathbf{\Gamma}(\mathcal{T}^{br})$ and $\mathbf{D}(\mathcal{T}^{br})$

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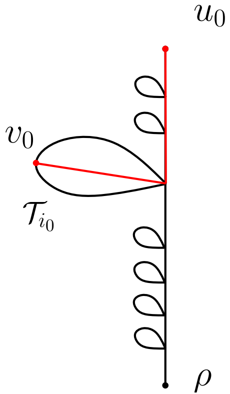
- We deduce for $m+n>1$,

$$2^{(m+n)/2}\cdot\mathbf{E}\left[\mathbf{D}(\mathcal{T}^{br})^m\cdot\mathbf{\Gamma}(\mathcal{T}^{br})^n\right]=\frac{2\sqrt{\pi}}{\Gamma(\frac{m+n-1}{2})}\int_0^\infty du\int_{u/2}^u dv\,u^m v^n$$
$$\cdot\frac{\sinh(2(u-v))-2\sinh^2(u-v)\coth(u/2)}{\sinh^4(u/2)}.$$

Other consequences

decomposition along the diameter of $\mathcal{T}^{br}$

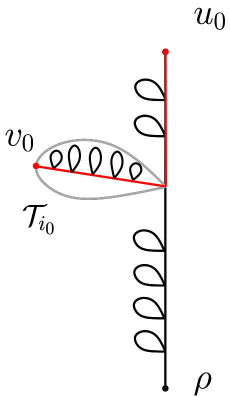
a spinal decomposition along the maximal height of $\mathcal{T}$ under n_+



Other consequences

decomposition along the diameter of $\mathcal{T}^{br}$

a spinal decomposition along the maximal height of $\mathcal{T}_{i_0}$ under $n_{c-\ell_i}$



Other consequences

decomposition along the diameter of $\mathcal{T}^{br}$

- We obtain a spinal decomposition along the diameter of $\mathcal{T}$ under n_+.

Other consequences

decomposition along the diameter of $\mathcal{T}^{br}$

- We obtain a spinal decomposition along the diameter of $\mathcal{T}$ under n_+.
- By the scaling property, we deduce a spinal decomposition along the diameter of $\mathcal{T}^{br}$, which can be written as a conditioned Poisson point measure.

Generalization

- Lévy trees are the scaling limits of Galton-Watson trees, generalizing the Brownian CRT:
  - A decomposition along the diameter of a Lévy tree under the excursion measure.

- An important subclass: stable tree:
  - Laplace transforms for the height and the diameter of a stable tree;
  - Asymptotics of the probabilities for the height (resp. diameter) to be large.

Thank you!