Large deviation principle for one-dimensional SDE's with discontinuous coefficients.

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Outline

Large deviation principle

Let $(\mathbb{X}, \rho)$ be a complete separable metric space with $\mathbb{X}$-valued random variables X^n, $n \geq 1$.

We say that family $\{X^n\}$ satisfies large deviation principle (LDP) with rate function $I : \mathbb{X} \to [0, \infty]$ if, for each opened set A,

$$\liminf_{n\to\infty} \frac{1}{n} \log P\{X^n \in A\} \geq - \inf_{x\in A} I(x), \tag{1}$$

and, for each closed set B,

$$\limsup_{n\to\infty} \frac{1}{n} \log P\{X^n \in B\} \leq - \inf_{x\in B} I(x). \tag{2}$$

If (1) holds and (2) holds for each compact set B only then family $\{X^n\}$ satisfies weak LDP. If each level set $\{x : I(x) \leq a\}$, $a \geq 0$, is compact rate functional I is called "good".

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The model

Consider one-dimensional SDE's

$$dX_t^n = a(X_t^n)dt + \frac{1}{\sqrt{n}}\sigma(X_t^n)dW_t \qquad (3)$$

with initial conditions $X_0^n = x_0$.

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Case $a \equiv 0$

Let $\mathbb{X} = \mathcal{C}([0,\infty))$, $\rho(f,g) = \sum_{k=1}^{\infty} \frac{1}{2^k}\big(\sup_{t\in[0,k]} |f(t) - g(t)| \wedge 1\big)$ and let $a \equiv 0$.

Theorem 1

Consider such measurable, bounded and separated from zero σ that the set Δ_σ of discontinuity points of σ has zero Lebesgue measure.
Then the family of solutions to (3) satisfies LDP with a good rate function I, which equals

$$I(x) = \frac{1}{2}\int_0^\infty \frac{(\dot{x}(t))^2}{\sigma^2(x(t))}dt$$

if $x \in \mathcal{C}([0,\infty))$ is an absolutely continuous function with $x(0) = x_0$, $\dot{x} \in L_2([0,\infty))$, and $I(x) = \infty$ otherwise.
This result was obtained using semicontraction principles (Kulik'05).
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Consider now case $a \not\equiv 0$. $\mathbb{X} = \mathcal{C}[0,T]$. The following result was obtained using Girsanov theorem on change of measure, Varadhan lemma and Bryc formula (Feng, Kurtz'06) and Theorem 1.

Theorem 2

Consider $\mathbb{X} = \mathcal{C}([0,T])$. Suppose that assumptions of Theorem 1 hold true and, additionally, $\frac{a}{\sigma^2}$ has bounded derivative.

Then the family of solutions to (3) satisfies LDP with a good rate function I, which equals

$$I(x) = \liminf_{y \to x} \frac{1}{2} \int_0^T \frac{(a(y(t)) - \dot{y}(t))^2}{\sigma^2(y(t))} dt$$

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Difference from Freidlin-Wentzell result

Remark.

Rate function in Theorem 2 differs from standart Freidlin-Wentzell result. The reason for this is that we require rate function to be lower semicontinuous and functional $Q(y) = \frac{1}{2}\int_0^T \frac{(a(y(t))-\dot{y}(t))^2}{\sigma^2(y(t))}dt$, in general, is not lower semicontinuous.

Example. Let $T = 1$, $x_0 = 0$ and

$$\sigma(y) = \begin{cases} c_1, & y < 0 \\ c_2, & y \geq 0 \end{cases}, \; 0 < c_1 < c_2, \quad a(y) = \sigma^2(y).$$

Then conditions of Theorem 2 are satisfied. Consider a sequence

$$y_n(t) = \begin{cases} -\frac{t}{n}, & t \in [0, \frac{1}{2}] \\ -\frac{1}{2n}, & t \in [\frac{1}{2}, 1] \end{cases},$$

we have $y_n \to y_0 \equiv 0$ as $n \to \infty$. For this sequence function Q is equal to
$Q(y_n) = \frac{1}{2}\left(c_1^2 + \frac{1}{n} + \frac{1}{2n^2}\frac{1}{c_1^2}\right) \to \frac{c_1^2}{2}$ as $n \to \infty$,
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References

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