Energy Fluctuations in the Disordered Harmonic Chain

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Young Women In Probability
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Diffusive behavior and heat transport

Ludwig Boltzmann
[1844-1906]

Joseph Fourier
[1768-1830]

What is heat equation?

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Initially: $T_0(x) =$ temperature

What is heat equation?

$t > 0$: heat transfer

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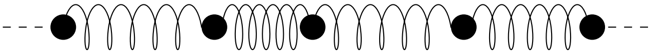
FOURIER LAW (1822):

$$\boxed{\mathcal{J}(x,t) = -\mathbf{D}(T)\frac{\partial T}{\partial x}(x,t)}$$

D = diffusion coefficient

Chain of N harmonic coupled oscillators on the torus

$$\mathbb{T}_N = \{1, \ldots, N\} \qquad\qquad 0 \equiv N$$



p_x : momentum of atom x,
r_x : distance between x et $x+1$,
m_x : mass of atom x.

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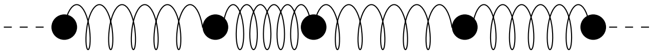

$$\begin{aligned} p_x &: \text{ momentum of atom } x, \\ r_x &: \text{ distance between } x \text{ et } x+1, \\ m_x &: \text{ mass of atom } x. \end{aligned}$$

Total energy = "temperature"

$$\mathcal{H} := \sum_x \left\{ \frac{p_x^2}{2m_x} + \frac{r_x^2}{2} \right\} = \sum_x e_x.$$

Hamiltonian dynamics

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- **Conserved quantities**

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- **Number of particules?** ... $N \propto 10^{23}$ (Avogadro) !!

Boltzmann idea!

Atomic description $\Rightarrow$ *Macroscopic* description

We define $\mu_t^N(\mathrm{d}\mathbf{r},\mathrm{d}\mathbf{p})$ as the probability law at time t on the space

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Equilibrium states?

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Equilibrium states? *Invariant* measures:

$\triangleright$ If there is only one conserved quantity (the *energy*) the invariant measures are the **Gibbs measures** defined as

$$\mu_T^N(\mathrm{d}\mathbf{r}, \mathrm{d}\mathbf{p}) := \frac{1}{Z(T)} \prod_{x=1}^{N} \exp\left(-\frac{e_x}{T}\right) \mathrm{d}r_x \mathrm{d}p_x.$$

$\triangleright$ One parameter $T > 0$, called *temperature*, and $e_x := \dfrac{p_x^2}{2m_x} + \dfrac{r_x^2}{2}$

Stochastic perturbation of the dynamics

Stefano Olla

S.R.S Varadhan

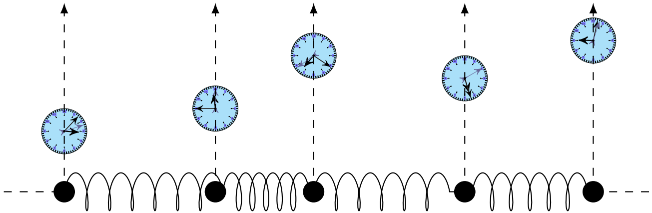
H.T. Yau

We add a **stochastic noise** $\Rightarrow$ provides ergodicity

II. Let's make noise!

We add a **stochastic noise** ⇒ provides ergodicity

Each atom x waits independently a random Poissonian time and then flips p_x into $-p_x$.



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Both still conserve $\sum \left\{ \frac{p_x^2}{2m_x} + \frac{r_x^2}{2} \right\}$

Final dynamics

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- Still degenerate... **No spectral gap!**

What's important?

1. $\left\{r_x(t), p_x(t)\ ;\ x \in \mathbb{T}_N\right\}_{t \geqslant 0}$ is a Markov process on $(\mathbb{R} \times \mathbb{R})^N$.

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2. The density f_t^N of the law μ_t^N satisfies the Fokker-Planck equation

$$\boxed{\frac{\partial f_t^N}{\partial t} = \mathcal{L}_N^{\mathbf{m}} f_t^N}$$

   where $\mathcal{L}_N^{\mathbf{m}} = -\mathcal{A}_N^{\mathbf{m}} + \gamma \mathcal{S}_N$ and $\gamma > 0$ is the *noise intensity*.

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3. The only "relevant" invariant measures are the Gibbs measures

$$\boxed{\mu_T^N(\mathrm{d}\mathbf{r}, \mathrm{d}\mathbf{p}) := \frac{1}{Z(T)} \prod_{x=1}^{N} \exp\left(-\frac{e_x}{T}\right) \mathrm{d}r_x \mathrm{d}p_x.}$$

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Consequence of CLT

If $\{r_x, p_x\}$ are distributed according to μ_T^N then

$$\mathcal{Y}^N(\cdot) := \frac{1}{\sqrt{N}} \sum_{x=1}^{N} \delta_{x/N}(\cdot) \left\{ e_x - T \right\}$$

converges in law towards a Gaussian field.

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Precisely: If F, G are smooth functions,

$$\left\langle \mathcal{Y}^N(F)\,\mathcal{Y}^N(G)\right\rangle_T \xrightarrow[N\to\infty]{} 2T^2\int_0^1 F(u)G(u)\mathrm{d}u$$

And when time evolves?

Energy fluctuation field

$$\mathcal{Y}_t^N(\cdot) := \frac{1}{\sqrt{N}} \sum_{x=1}^{N} \delta_{x/N}(\cdot) \left\{ e_x\left(tN^{\mathbf{a}}\right) - T \right\}$$

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Diffusion? $\mathbf{a} = \mathbf{2}$ and $\mathcal{Y}_t^N$ converges in law towards an infinite dimensional Ornstein-Uhlenbeck process $\mathcal{Y}_t$ solution to the SPDE

$$\boxed{\partial_t \mathcal{Y} = \mathbf{D}\ \partial_x^2 \mathcal{Y}\ \mathrm{d}t + \left(4\mathbf{D}T^2\right)^{1/2}\ \partial_x \mathcal{B}(x,t)}$$

where $\mathcal{B}$ is a standard space-time white noise.

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$\mathbf{D}$ = diffusion coefficient

Two results

Let F and G be smooth functions,

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[S. 2013]

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1. **Constant masses + flip of intensity γ**

$$\boxed{\mathbf{D}(\gamma) = \frac{1}{2\gamma}}$$

[S. 2013]

2. **Random masses + flip(γ) + strong exchange(λ)**

$$\boxed{0 < \mathbf{D}(\gamma,\lambda) < +\infty}$$

[S. 2014]

What about $\mathrm{D}(\gamma, \lambda)$?

1. **Does not depend** on the realization of masses

   ... only on moments $\mathbb{E}[m_0]$, $\mathbb{E}[m_0^2]$, ...

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3. **Has to satisfy**

$$\mathbb{E}^{\mathbf{m}} \left\langle \left(\int_0^t \sum_{x \in \mathbb{T}_N} \left[j_{x,x+1} - \left\{ \underbrace{\mathbf{D}(\gamma,\lambda)(e_{x+1} - e_x) + \mathcal{L}^{\mathbf{m}}(\tau_x f)}_{\mathit{fluctuation-dissipation}} \right\} \right] \mathrm{d}s \right)^2 \right\rangle_T$$

$\mathcal{L}^{\mathbf{m}} =$ generator of the dynamics

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In other words:

We construct a norm $\|\cdot\|_T$ and a Hilbert space $\mathcal{H}$ which contains $j_{0,1}$ such that

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- ▷ **only with the strong exchange**

Going further!

1. **Green-Kubo formula** for $\mathbf{D}(\gamma, \lambda)$ **[S. 2013]**

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   - ▷ the two definitions are equivalent $\checkmark$

Going further!

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Going further!

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Going further!

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   - Proving the *Fourier law*

$$\frac{\partial e(x,t)}{\partial t} = \frac{\partial}{\partial x}\left[\mathbf{D}(e)\frac{\partial e}{\partial x}\right]$$

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   - Harmonic chain with *constant* masses + flip ✓ **[S. 2013]**
     Harmonic chain with *random* masses **Open!**

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3. Cases of **anomalous diffusion** **[S. Bernardin Gonçalves Jara '14]**

Thank you for your attention!