A Brief Introduction to McKean-Vlasov Processes

and non-linear diffusions in general

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Young Women in Probability
Bonn, May 2014

What is a McKean Vlasov process?

Roughly speaking, McKean-Vlasov processes or McKean-Vlasov diffusions are stochastic process which can be described by SDEs of the form

$$\begin{cases} dX_t = \int \alpha(X_t, u)\mu_t(du)dB_t + \int \beta(X_t, u)\mu_t(du)dt, & X_0 \text{ given} \\ \mu_t = \mathcal{L}(X_t), \end{cases} \tag{1}$$

where B is a standard d-dimensional Brownian motion and $\mathcal{L}(X_t)$ denotes the marginal distribution of the process X at the time t.

In general one can think of processes which satisfy SDEs of the following form

$$\begin{cases} dX_t = a(X_t, \mu_t)dB_t + b(X_t, \mu_t)dt, & X_0 \text{ given} \\ \mu_t = \mathcal{L}(X_t). \end{cases} \tag{2}$$

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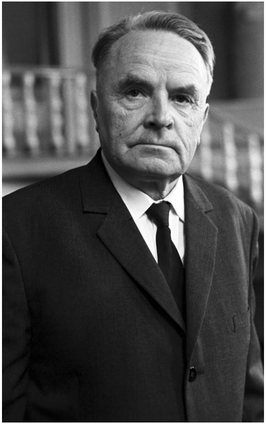
A little bit of history

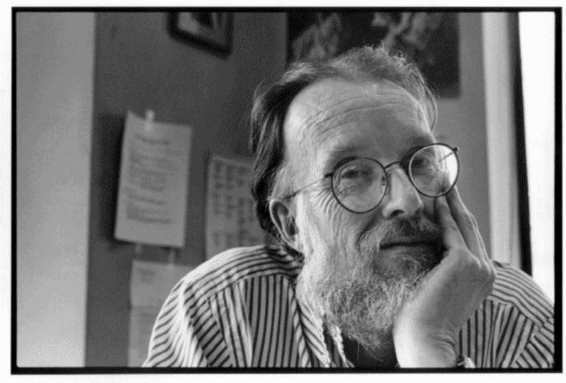
(a) Mark Kac



(b) Anatoly Vlasov

The story of these processes started with a stochastic toy model for the Vlasov equation of plasma proposed by Mark Kac in his paper *"Foundations of kinetic theory (1956)"*.

A little bit of history



In 1966 Henry P. McKean published his seminal paper *"A class of Markov processes associated with non-linear parabolic equations"*.

Why these processes are interesting?

- Theoretical interest
  - Existence, Uniqueness and Properties
  - Mean Fields
  - Stochastic Control
- Connection with non-linear parabolic PDEs
  - Vlasov equation of plasma
  - Granular media equation
- Applications in several areas
  - Physics
  - Finance
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Starting Simple

- For the rest of this talk we are going to assume that the diffusion coefficient is constant.
- Consider non-linear SDE

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where $\beta : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is bounded and Lipschitz continuous.

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Generator

- Family of generators of the form

$$L_t = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_j \partial x_i} + \sum_{i=1}^{d} \int \beta_i(x,y)\mu_t(dy)\frac{\partial}{\partial x_i},$$

  for all $t \geq 0$.
- Martingale Formulation
- PDE of the form

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= L_t u(t,x), \qquad t > 0, \\ u(0,x) &= u_0. \end{aligned}$$

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Particle System

- A natural way of associating a particle system is to consider one with mean field interaction.
- For each $N \in \mathbb{N}$ consider the particle system

$$\begin{cases} dX_t^{i,N} = \sqrt{2}dB_t^i + \int \beta(X_t^{i,N}, y)\Pi_t^N(dy)dt & i = 1, \ldots, N \\ X_0^{i,N} = X_0, & i = 1, \ldots, N \end{cases}$$

where

$$\Pi_t^N = \frac{1}{N}\sum_{j=1}^{N} \delta_{X_t^{j,N}}(dx)$$

- Clearly this can be written as follows

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Properties

- Existence and Uniqueness
- Moment Control
- Behaviour at Infinity
  - Existence of a stationary distribution
  - Uniqueness of the stationary distribution
  - Speed of convergence towards the invariant distribution
- All these properties depend on the assumptions on the coefficient β!
  - Bounded and Lipschitz continuous
  - Bounded and Locally Lipschitz
  - What about unbounded coefficients?
  - Linear growth, Polynomial growth?
  - We require additional assumptions!

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An interesting example

- Consider equations of the form

$$\begin{cases} dX_t = \sqrt{2}dB_t - [\nabla V(X_t) + \nabla W * \mu_t(X_t)]dt, & X_0 \text{ given} \\ \mu_t = \mathcal{L}(X_t), \end{cases} \tag{5}$$

  where $*$ denotes the convolution operator.
- Provided some regularities on V and W the existence of theses processes can be proved.
- Moreover, it is not difficult to prove that the laws $\mu_t, t \geq 0$ are absolutely continuous and their densities $u_t, \geq 0$ satisfy the so-called granular media equation

$$\frac{\partial u}{\partial t} = \nabla \cdot [\nabla u + u\nabla V + u(\nabla W * u)].$$

Some work done on this case

- 1998 - Benachour et al. studied equation (5) with $V = 0$ in the one-dimensional case.
- 2001 - Malrieu studied equation (5) by using a particle system and propagation of chaos approach.
- 2008 - Herrmann et al. generalised Benachour et al. results to the multidimensional case.
- 2008 - Cattiaux et al. generalised Malrieu's work.
- 2014 - Pierre del Moral and Tugaut proved uniform propagation of chaos for processes of the form (5) with $V = 0$.

My Work

Theorem

Let $\beta : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ be a function satisfying assumptions (I)- (III) and ξ a probability measure which belongs to $\mathcal{P}_q$ with $q = \max\{m + m_1 + 1, m_2 + 1\}$. Then there exists a unique strong solution to the non-linear stochastic differential equation

$$\begin{cases} dX_t = \sqrt{2}dB_t + \int \beta(X_t, u)\mu_t(du)dt, & \mathcal{L}(X_0) = \xi \text{ given} \\ \mu_t = \mathcal{L}(X_t). \end{cases}$$

Moreover, we have

$$\sup_{0 \le t \le T} \mathbb{E}[|X_t|^q] < \infty,$$

for all $T > 0$.

Final Comments

- Our approach consist in the application of a fixed-point argument in an appropriate space of curves of probability measures.
- It was inspired by the work of V. Kolokoltsov [2].
- Assumptions (I) and (II) are more or less standard and easy to prove.
- Assumption (III) might be difficult to check though.
- It is possible to extended this approach to more general non-linear diffusions.
- Work in progress
  - Behaviour at infinity
  - Other kinds of non-linear diffusions

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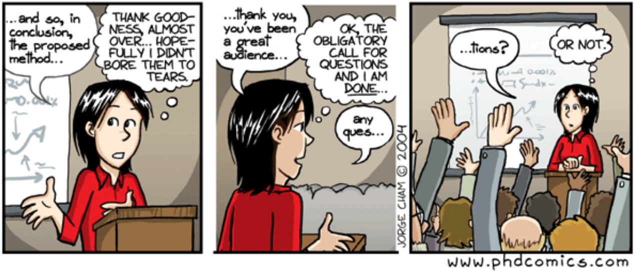
Thank you very much for listening!



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