Is There an Optimal Trading Strategy for Behavioural Investors?

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Based on joint work with M. Rásonyi

Abstract

We study the optimal portfolio problem in a continuous-time market for a behavioural investor whose probability distortions and utilities are **power-like** functions. We provide sufficient and necessary conditions for **well-posedness**. Under such easily verifiable conditions, we also establish the **existence** of an optimal portfolio. We conclude with an example of a model to which our results apply.

1 Introduction

We consider a frictionless continuous-time economy with trading interval $[0,T]$, and characterised by a complete filtered space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. The market consists of $d+1$ traded assets: the money market account (assumed to be constant w.l.o.g.) and d risky assets. Suppose further that there is an equivalent martingale measure $\mathbb{Q}$ with Radon-Nikodym derivative $\rho = d\mathbb{Q}/d\mathbb{P}$.

Portfolio optimisation is a classic problem, and a predominantly used model for decision making under uncertainty has been Expected Utility Theory ([2]). However, its basic tenets have been contradicted by empirical evidence, leading to alternative approaches such as **Cumulative Prospect Theory** ([4]). Our *CPT-investor* is thus assumed to have:

- a **reference point**, represented by the integrable, $\mathcal{F}_T$-measurable r.v. B, with respect to which payoffs are evaluated at the maturity, and *gains* and *losses* are defined;
- a strictly increasing and continuous **utility function** $u : \mathbb{R} \to \mathbb{R}$ given by
$$u(x) = u_+(x^+)1_{[0,+\infty)}(x) - u_-(x^-)1_{(-\infty,0)}(x), \qquad x \in \mathbb{R},$$
with $u_\pm(0) = 0$, $u_+(x) \leq \kappa_+ x^\alpha$ and $u_-(x) \geq \kappa_- x^\beta$ for all $x \in [0,+\infty)$ – **Fig. 1**;
- a distorted perception of the actual probabilities, modelled with the strictly increasing and continuous **probability distortions** $w_+(x) \leq k_+ x^\gamma$ and $w_-(x) \geq k_- x^\delta$ for all $x \in [0,1]$, with $w_\pm(0) = 0$ and $w_\pm(1) = 1$ – **Fig. 2**;

where $\alpha, \beta, \gamma, \delta > 0$, $\kappa_+, \kappa_- > 0$, $k_+ \geq 1$ and $k_- \in (0,1]$.

Fig. 1 – A piecewise power-like utility function.

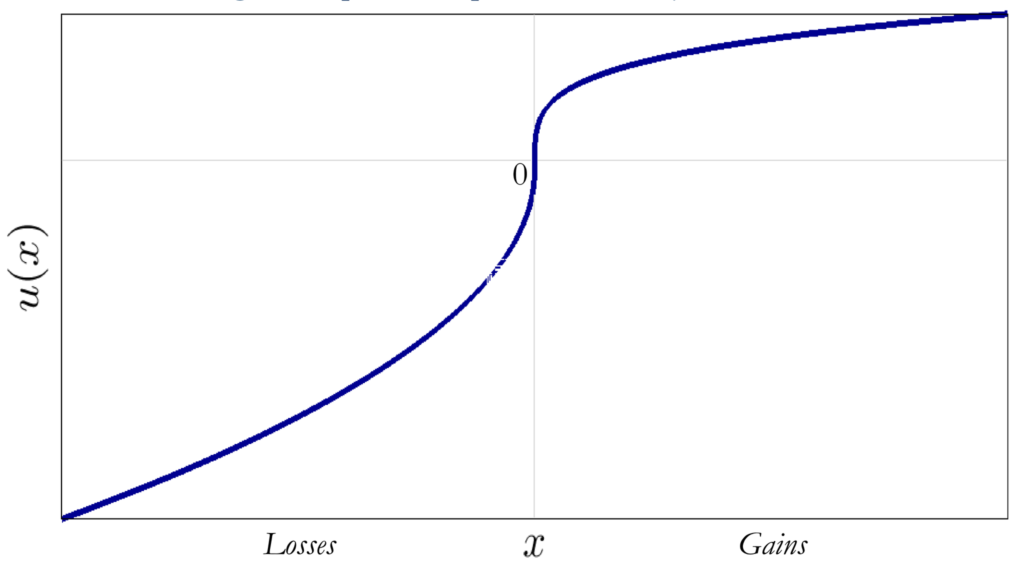
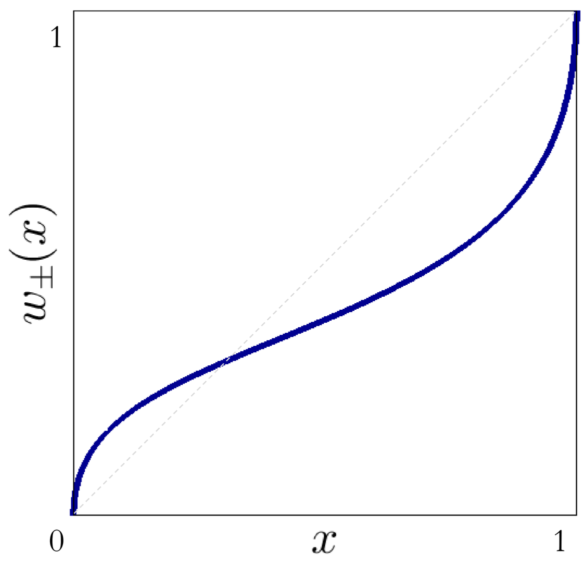


Fig. 2 – A power-like distortion function (the original CPT distortion proposed in [4]).



Finally, we impose the following assumptions throughout.

Assumption 1. The CDF of ρ under $\mathbb{P}$ $(F_\rho^{\mathbb{P}})$ is continuous, both ρ and $1/\rho$ are in $\mathcal{W} = \{Y \in \mathcal{F}_T \colon \mathbb{E}_{\mathbb{P}}[|Y|^p] < +\infty \ \forall p > 0\}$, $\operatorname{ess\,inf}_{\mathbb{P}} \rho = 0$ and $\operatorname{ess\,sup}_{\mathbb{P}} \rho = +\infty$.

Assumption 2. B and all integrable $\sigma(\rho)$-measurable r.v. are hedgeable.

2 The Optimal Portfolio Problem

Let Π^ϕ denote the wealth process associated with an admissible portfolio ϕ. The **behavioural portfolio choice problem** is formalised as follows.

$$\text{Maximise} \left\{ V\left(\Pi_T^\phi - B\right) = V_+\left(\left[\Pi_T^\phi - B\right]^+\right) - V_-\left(\left[\Pi_T^\phi - B\right]^-\right) \right\} \quad \textbf{(BPP)}$$

over the set of feasible strategies, where $V_\pm\left(\left[\Pi_T^\phi - B\right]^\pm\right)$ is the *Choquet integral* of $u_\pm \circ \left[\Pi_T^\phi - B\right]^\pm$ with respect to the *capacity* $w_\pm \circ \mathbb{P}$,

$$V_\pm\left(\left[\Pi_T^\phi - B\right]^\pm\right) = \int_0^{+\infty} w_\pm\left(\mathbb{P}\left\{u_\pm\left(\left[\Pi_T^\phi - B\right]^\pm\right) > y\right\}\right) dy.$$

3 Well-Posedness

A maximisation problem is termed **well-posed** if its supremum is finite. Problems that are not well-posed are said to be **ill-posed.**

For an ill-posed problem, maximisation does not make sense. Intuitively, the investor can obtain an arbitrarily high degree of satisfaction from the available trading strategies. Hence, first we must identify and exclude the ill-posed cases.

Theorem 1. (Necessary and sufficient conditions)

The problem **(BPP)** is well-posed if and only if $\alpha < \beta$ and $\dfrac{\alpha}{\gamma} < 1 < \dfrac{\beta}{\delta}$.

These conditions, besides being very easy to check, also admit an economic interpretation. For example, the first one implies that large losses loom larger than corresponding gains, indicating **loss aversion**.

4 Existence

It is clear that, even if the problem **(BPP)** is well-posed, it may still happen that an optimal solution does not exist. So we state our second main result.

Theorem 2.

Under $\alpha < \beta$ and $\dfrac{\alpha}{\gamma} < 1 < \dfrac{\beta}{\delta}$, there exists an optimal trading strategy.

Sketch of proof: *(Inspired by the works of Carassus and Rásonyi [1], and of Reichlin [3])*

Assume $B = 0$ *w.l.o.g. (by* **Assump. 2***), and let* $V^* = \sup_\phi V(\Pi_T^\phi) < +\infty$ *(by* **Theor. 1***).*

Then, take $\{\phi_n;\ n \in \mathbb{N}\}$ *such that* $\lim_{n\to+\infty} V(X_n) = V^*$, *where* $X_n = \Pi_T^{\phi_n}$.

The family of laws of X_n *is* tight, *hence it admits a* weak sublimit ν.

Denoting by q_ν *the quantile function of* ν, *and setting*

$$X_* = q_\nu\left(1 - F_\rho^{\mathbb{P}}(\rho)\right) \sim \nu,$$

its replicating strategy *is optimal.* □

5 Example

Let $k \geq d$ be an integer and let the i-th stock price satisfy

$$dS_t^i = \mu_t^i S_t^i\, dt + \textstyle\sum_{j=1}^k \sigma_t^{ij} S_t^i\, dW_t^j, \qquad i \in \{1,\ldots,d\}, \qquad \textbf{(1)}$$

where $W = \{W_t;\ 0 \leq t \leq T\}$ is a k-dimensional $\mathbb{P}$-Brownian motion. We are assuming $\mathbb{F}$ to be the natural filtration of W, and we consider the cases where either

(i) $k = d$, or

(ii) $k > d$ and all the coefficients μ^i and σ^i are **deterministic**.

Finally, we assume that, for Lebesgue a.e. $t \in [0,T]$, the $d \times k$ *volatility matrix* with entries σ_t^{ij} has rank d a.s..

Then from our theorems we deduce the following.

Corollary.

In the multidimensional diffusion model **(1)**, the behavioural problem is well-posed and there exists an optimal portfolio, exactly when $\alpha < \beta$ and $\alpha/\gamma < 1 < \beta/\delta$.

6 Future Work

Positive results can also be obtained for bounded u_+. Extending the above theorems to any unbounded u_+ or to incomplete diffusion models with stochastic coefficients is work in progress.

References

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