Exit problem from a bounded domain

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Introduction

Behavior of diffusions with a small parameter noise

Let X^ϵ be a random perturbation of the deterministic system:

$$\frac{dX(t)}{dt} = b(X(t)), \quad X(0) = x. \tag{1}$$

We consider the following perturbation of the above deterministic system.

$$dX_t^\epsilon = b_\epsilon(X_t^\epsilon)\,dt + \sqrt{\epsilon}\sigma_\epsilon(X_t^\epsilon)\,dW_t \tag{2}$$

if $\epsilon \to 0$ then the above SDE transformed to a deterministic function. Let $\mathbb{Q}_\epsilon$ be the measure induced by $X_\epsilon(\cdot)$ on the space of $\mathbb{R}^d$-valued continuous functions on some arbitrary but finite interval. Then $\{X_t^\epsilon\}$ the unique strong solution of 2 satisfies LDP on $C[0,1]$ with good rate function:

$$I_x(f) = \frac{1}{2}\int_0^1 \left(\dot f(t) - b(f(t))\right)' a^{-1} f(t)\left(\dot f(t) - b(f(t))\right)$$

where $f \in \mathcal{H}$ and $\mathcal{H}$ is a Cameron-Martin space, otherwise $I_x(f) = \infty$.

A solution through Freidlin-Wentzell theory

Let us consider the problem of exit from a domain. We consider the system

$$dX_t^\epsilon = b(X_t^\epsilon)\,dt + \sqrt{\epsilon}\sigma(X_t^\epsilon)\,dW_t, \qquad X_t^\epsilon \in \mathbb{R}^d \quad X_0^\epsilon = x(3)$$

in the open, bounded $G \subseteq \mathbb{R}^d$ and let ∂G be its boundary, which we assume to be smooth for the sake of simplicity, $b(\cdot), \sigma(\cdot)$ are uniformly Lipschitz continuous functions of d-dimensions and W is d-dimensional BM. If we define the stopping time

$$\tau^\epsilon = \inf\{t : X_t^\epsilon \notin G\}$$

then events like this $\{\tau^\epsilon < T\}$ are rare events, indeed

$$\mathbb{P}[\tau^\epsilon < T] \to 0 \quad \text{as } \epsilon \to 0 \text{ and } T < \infty.$$

Motivated by Freidlin-Wentzell theory, we define the cost function

$$V(y,z,t) \triangleq \inf_{f \in C([0,1]): f_t = z} I_{y,t}(f) \tag{4}$$

$$= \inf_{g \in L_2([0,t]): f_t = z, f_s = y + \int_0^s b(f_u)du + \int_0^s \sigma(f_u)\dot g du} \frac{1}{2}\int_0^t |\dot g_s|^2 ds.$$

Basic Assumptions

A-1 The unique equilibrium point in G of the d-dimensional ordinary differential equation

$$\dot f_t = b(f_t) \tag{5}$$

is at $0 \in G$, and

$$f_0 \in G \Rightarrow \forall t > 0, f_t \in G \text{ and } \lim_{t\to\infty} f_t = 0$$

A-2 All the trajectories of the deterministic system 5 starting at $f_0 \in \partial G$ converge to 0 as $t \to \infty$.

A-3 $\bar V \triangleq \inf_{z \in \partial G} V(0,z) < \infty$, where $V(0,z) = \inf_{t \geq 0} V(0,z,t)$.

A-4 There exists $M < \infty$ such that for all $\rho > 0$ small enough and all x, y with $|x - y| \leq \rho$ for some $z \in \partial G \cup \{0\}$ there is a function $g \in \mathcal{L}_2$ such that $\|\dot g\|_{\mathcal{L}_2} < M$ where

$$f_t = x + \int_0^t b(f_s)\,ds + \int_0^t \sigma(f_s)\,\dot g ds.$$

- Lower bound. For any $\rho > 0$ small enough, there is $T < \infty$ such that

$$\liminf_{\epsilon\to 0} \epsilon \log \inf_{x \in B_\rho} \mathbb{P}_x[\tau^\epsilon < T] > -\bar V.$$

- Let $\sigma_\rho \triangleq \inf\{t : X_t^\epsilon \in B_\rho \cup \partial G\}$. Then

$$\lim_{t\to\infty} \limsup_{\epsilon\to 0} \log \sup_{x \in G} \mathbb{P}_x(\sigma_\rho > t) = -\infty.$$

- Upper bound. For any closed set $N \subset \partial G$

$$\lim_{\rho\to 0} \limsup_{\epsilon\to 0} \epsilon \log \sup_{y \in S_{2\rho}} \mathbb{P}_y[X_{\sigma_\rho}^\epsilon \in N] \leq -\inf_{z \in N} V(0,z)$$

where $\sigma_\rho \triangleq \inf\{t : X_t^\epsilon \in B_\rho \cup \partial G\}$.

- For every $\rho > 0$ such that $B_\rho \subset G$ and all $x \in G$

$$\lim_{\epsilon\to 0} \mathbb{P}[X_{\sigma_\rho}^\epsilon \in B_\rho] = 1$$

Theorem

Assume (A-1)-(A-4) are satisfied. Then for all $x \in G$ and all $\delta > 0$,

$$\lim_{\epsilon\to 0} \mathbb{P}_x\left(e^{\frac{(\bar V+\delta)}{\epsilon}} > \tau^\epsilon > e^{\frac{(\bar V-\delta)}{\epsilon}}\right) = 1.$$

Moreover, for all $x \in G$

$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{E}_x(\tau^\epsilon) = \bar V.$$

Connection with viscosity solution

We formulate the above result in the language of PDEs, in particular as a result of viscosity solution of a parabolic problem. We set

$$\Phi^\epsilon(t,x) = \mathbb{P}[\tau^\epsilon < t_1] \tag{6}$$

where $t_1 < \infty$ with boundary data

$$\Phi^\epsilon(t,x) = 1 \quad (t,x) \in [0,t_1) \times \partial G$$

$$\Phi^\epsilon(t_1,x) = 0 \quad x \in \bar G$$

Then 6 satisfies a linear equation

$$-\frac{\partial \Phi^\epsilon(t,x)}{\partial t} - b(t,x)\,D_x\Phi^\epsilon(t,x) - \tag{7}$$

$$-\frac{\epsilon}{2}\sum_j\sum_i a_{ij}(t,x)\frac{\partial^2\Phi^\epsilon(t,x)}{\partial x_i \partial x_j} = 0.$$

We, now, make the logarithmic transformation:

$$V^\epsilon = -\epsilon \log \Phi^\epsilon(t,x) \tag{8}$$

then the dynammic programming PDE becomes

$$-\frac{\partial V^\epsilon(t,x)}{\partial t} - b(t,x)\,D_x V^\epsilon(t,x) -$$

$$-\frac{\epsilon}{2}\sum_i\sum_j a_{ij}(t,x)\frac{\partial^2 V^\epsilon(t,x)}{\partial x_i \partial x_j}$$

$$+\frac{1}{2}\sum_i\sum_j a_{ij}(t,x)\,D_x V^\epsilon\,(D_x V^\epsilon)' = 0 \tag{9}$$

and the boundary data become

$$V^\epsilon(t,x) = 0, \quad (t,x) \in (0,t_1) \times \partial G$$

$$\lim_{t\to t_1} V^\epsilon(t,x) = \infty, \qquad x \in G.$$

As $\epsilon \to 0$ we have a first order PDE

$$-\frac{\partial V^0(t,x)}{\partial t} - b(t,x)\,D_x V^0(t,x) +$$

$$\frac{1}{2}\sum_i\sum_j a_{ij}(t,x)\left(D_x V^0\right)'\left(D_x V^0\right) = 0 \tag{10}$$

and V^0 has a representation in terms of control theory. We consider the Hamiltonian function:

$$H(t,x,p) = -b(t,x)\,p + \frac{1}{2}p'\sigma\sigma'(t,x)\,p.$$

so that

$$-\frac{\partial V^0}{\partial t}(t,x) + H(t,x,p) = 0$$

Since the Hamiltonian is quadratic and particular convex in p, we can use the Legendre transform and may rewrite:

$$H(t,x,p) = \sup_{u \in \mathbb{R}^d}\{-up - L(t,x,u)\}$$

$$= -\inf_{u \in \mathbb{R}^d}\{up + L(t,x,u)\}$$

where

$$L(t,x,u) = \sup_{p \in \mathbb{R}^d}\{-up - H(t,x,u)\}$$

$$= \frac{1}{2}(u - b(t,x))\left(\sigma\sigma'(t,x)\right)^{-1}(u - b(t,x))'$$

$$-\frac{\partial V^0(t,x)}{\partial t} - \inf_{u \in \mathbb{R}^d}\{up + L(t,x,u)\} = 0 \tag{11}$$

where

$$V_0 = \inf \int_0^{t_1} \frac{1}{2}(\dot x(s) - b(x,s))\left(\sigma\sigma^T\right)^{-1}(t,x)\,(\dot x(s) - b(x(s)))'\,ds.$$

which together with the boundary data is associated to the value function for a calculus of variation problem and $(t,x,u) \in [0,t_1] \times G \times \mathbb{R}^d$. Then from control theory the solution to the Hamilton-Jacobi-Bellman equation is represented by a unique Lipschitz viscosity solution V^0. Therefore, the large deviation results stated as

$$\lim_{\epsilon\to 0} \epsilon \log \Phi^\epsilon(t,x) = -V^0(t,x)$$

where $V^0(t,x)$ is the rate function. We continue with two estimates of V^ϵ:

Lemma

Suppose that ∂G is smooth. Then there exists $K > 0$ satisfying

$$V^\epsilon(t,x) \leq \frac{K\text{dist}(x,\partial G)}{t_1 - t}, \tag{12}$$

Lemma

For any $M > 0$ and $d(x) = \text{dist}(x,\partial G)$ in $C^2(\bar G)$ with $d(x) = 0$ for all $x \in \partial G$, there exists $K_M > 0$ such that

$$V^\epsilon(t,x) \geq M d(x) - K_M(t_1 - t). \tag{13}$$

We use the Barles-Perthame procedure in order to define a viscosity supersolution and subsolution of 11. Define

$$V^*(t,x) = \limsup_{(s,y)\to(t,x)} V^\epsilon(s,y),$$

$$V_*(t,x) = \liminf_{(s,y)\to(t,x)} V^\epsilon(s,y).$$

These functions however are not necessarily continuous. Therefore we conclude that V^*, V_* are respectively subsolution and supersolution of 11 in $(0,T) \times \bar G$ for every $T < t_1$. Then using equation 7 and its boundary data yields that any viscosity subsolution is dominated by any viscosity supersolution, $V_* \geq V^*$. However, by consrtuction $V_* \leq V^*$. Although the terminal data of the problem is infinite the stability result still holds. Hence, 12 implies that V^*, V_* converges to ∞ as $t \to t_1$ uniformly on compact subsets of G. However this convergence is controlled by 13. These properties are used to show the convergence of $V^\epsilon \to V^0 = V^* = V_*$ which is the unique viscosity solution of HBJ equation and Lipschitz continuous on $[0,T] \times \bar G$.

Theorem

Assume that the properties of b, a, a^{-1} satisfied. Then V^ϵ converges to V^0 uniformly on compact subsets of $[0,t_1] \times \bar G$ as $\epsilon \to 0$.

Open Questions

It should be noted here that the choice of the model is highly arbitrary. In particular, it can developed for Poisson process and more generally for Lévy process. But, what happened when the model is driven by a rough path or the coefficients of the SDE are not Lipschitz continuous?

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