Two-sided Parisian stopping time and the pricing of Parisian options

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Abstract
In this paper, we obtain a recursive formula for the density of the two-sided Parisian stopping time. This formula does not require any numerical inversion of Laplace transforms, and is similar to the formula obtained for the one-sided Parisian stopping time derived in Dassios and Lim. However, when we study the tails of the two distributions, we find that the two-sided stopping time has an exponential tail, while the one-sided stopping time has a heavier tail. We derive an asymptotic result for the tails of the two-sided stopping time distribution, and propose an alternative method of approximating the price of the two-sided Parisian option.

1 Introduction
A Parisian option is a path dependent option whose payoff depends on the path trajectory of the underlying asset. For example, the owner of a Parisian min-in option receives the payoff only if there is an excursion above or below an upper barrier $b$, which is of length greater than the window length $D$. The key to pricing this is to find the distribution of the two-sided Parisian stopping time for a Brownian motion, which is the first time the Brownian motion makes an excursion above or below a barrier $b$ that is of length longer than $D$. We assume a Black-Scholes framework, as below.

Let $S$ be the underlying asset following a geometric Brownian motion, and $Q$ denote the risk neutral probability measure. The dynamics of $S$ under $Q$ is

$$dS_t = S_t(rdt + σdW_t), \quad S_0 = x,$$

where $W_t$ is a standard Brownian motion under $Q$, and $r$ and $σ$ are positive constants. For simplicity, we have assumed zero dividends. Let $K$ denote the strike price of the option and we introduce the notations $m = \frac{1}{2}(r - σ^2)$, $b = \frac{1}{2}\ln\left(\frac{S_0}{K}\right)$, and $k = \frac{1}{2}\ln\left(\frac{S_0}{K}\right)$, so that the asset price $S_t = xe^{m(t-t_0)}$. We use the notation $g_{2k}(t) = \sup\{s \leq t \mid S_s = L\}$.

With the usual convention that sup$\emptyset = 0$ and inf$\emptyset = \infty$. We are interested here in $t - g_{2k}(t)$, which is the age of the excursion at time $t$. We further denote by $g_{2k}^1(t)$ and $g_{2k}^2(t)$ the excursion lengths when the underlying process is the Brownian motion $W$.

Without loss of generality, we let the window length $D = 1$, we now define

$$τ^+_b = \inf\{t \geq 0 \mid W_t \geq s_b^W \} \geq 1,$$
$$τ^-_b = \inf\{t \geq 0 \mid W_t \leq s_b^W \} \geq 1,$$
$$τ_b = τ^+_b \wedge τ^-_b.$$

Note that we have taken the window length of both sides to be the same (ie. 1 in our case).

2 Density of the two-sided Parisian stopping time
Let the barrier $b > 0$. We are only interested in the case when $\{T_b < 1\}$, where $T_b$ is the first hitting time of level $b$, since if $T_b \geq 1$, $τ_b = 1$. We have the following recursive solution for the density of $τ_b$ on the set $\{T_b < 1\}$.

**Theorem 2.1** For $b > 0$, we denote by $f_0(t, T_b < 1)$ the probability density function of the two-sided stopping time $τ_b$ on the set $\{T_b < 1\}$. We have

$$f_0(t, T_b < 1) = \sum_{k=0}^{n-1} (-1)^k L_k(t, 1),$$

for $n < t \leq n+1$, $n = 1, 2, \ldots$, for $t > 0$, where $L_k(t)$ is defined recursively as follows:

$$L_0(t) = 1\{0 < t < 1\} \cdot \frac{1}{\sqrt{\pi t}} \exp(-t) + 1\{t > 1\} \cdot \frac{2}{\sqrt{\pi t}} \exp(-t) - t\varphi\left(\frac{1}{t}\right),$$

$$L_{k+1}(t) = \int_{t-k}^{t} L_k(s) - s \frac{\pi}{\sqrt{\pi t}} ds,$$

for $t > k + 1$.

And for $b < 0$, we have

$$f_b(t, T_b < 1) = f_{-b}(t, T_{-b} < 1),$$

due to the symmetry of the standard Brownian motion. We prove that the two-sided stopping time $τ_b$ has an exponential tail, unlike the distribution of the one-sided stopping time $τ^-_b$. This is as expected because the one-sided case involves the hitting time of a Brownian motion, which is a heavy tailed distribution with infinite variance, while the two-sided one involves the hitting time of a Brownian motion reflected in zero, which has an exponential tail.

**Theorem 2.2** We denote $F_0(t)$ as the tail of the distribution of the two-sided Parisian stopping time $τ_b$ with barrier 0. It has an exponential tail. As $t \to \infty$, we have

$$F_0(t) \sim 2e^{-\beta \sqrt{t}} - e^{-\beta t},$$

for some constant $\beta > 0$ such that $-\beta^* > 0$ is the unique solution of the equation

$$\int_{0}^{1} e^{-\beta s} \sqrt{s} ds + e^{-\beta} = 0.$$

We can compute $\beta^*$ numerically to be 0.854.

Hence, we have an approximation for the density. As $t$ gets larger,

$$f_0(t, T_b < 1) \sim 2\beta^* e^{-\beta^*(t+1)} \int_{0}^{1} e^{\beta s} \sqrt{s} ds \frac{b}{\sqrt{2\pi s^3}}.$$