Two-sided Parisian stopping time and the pricing of Parisian options

Angelos Dassios, Jia Wei Lim

Dept of Statistics, London School of Economics
Email: *j.w.lim1@lse.ac.uk*

Abstract

In this paper, we obtain a recursive formula for the density of the two-sided Parisian stopping time. This formula does not require any numerical inversion of Laplace transforms, and is similar to the formula obtained for the one-sided Parisian stopping time derived in Dassios and Lim. However, when we study the tails of the two distributions, we find that the two-sided stopping time has an exponential tail, while the one-sided stopping time has a heavier tail. We derive an asymptotic result for the tail of the two-sided stopping time distribution and propose an alternative method of approximating the price of the two-sided Parisian option.

1 Introduction

A Parisian option is a path dependent option whose payoff depends on the path trajectory of the underlying asset. For example, the owner of a Parisian min-in option receives the payoff only if there is an excursion above or below the level L which is of length greater than the window length D. The key to pricing this is to find the distribution of the two-sided Parisian stopping time for a Brownian motion, which is the first time the Brownian motion makes an excursion above or below a barrier b that is of length longer than D. We assume a Black Scholes framework, as below.

Let S be the underlying asset following a geometric Brownian motion, and $\mathcal{Q}$ denote the risk neutral probability measure. The dynamics of S under $\mathcal{Q}$ is

$$dS_t = S_t(rdt + \sigma dW_t), \qquad S_0 = x,$$

where W_t is a standard Brownian motion under $\mathcal{Q}$, and r and σ are positive constants. For simplicity, we have assumed zero dividends. Let K denote the strike price of the option and we introduce the notations $m = \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right)$, $b = \frac{1}{\sigma}\ln\left(\frac{L}{x}\right)$, and $k = \frac{1}{\sigma}\ln\left(\frac{K}{x}\right)$, so that the asset price $S_t = xe^{\sigma(mt+W_t)}$. We use the notation

$$\begin{aligned} g^S_{L,t} &= \sup\{s \le t | S_s = L\} \\ d^S_{L,t} &= \inf\{s \ge t | S_s = L\}, \end{aligned}$$

with the usual convention that $\sup\emptyset = 0$ and $\inf\emptyset = \infty$. We are interested here in $t - g^S_{L,t}$, which is the age of the excursion at time t. We further denote by $g^W_{L,t}$ and $d^W_{L,t}$ the excursion lengths when the underlying process is the Brownian motion W.

Without loss of generality, we let the window length $D = 1$, we now define

$$\begin{aligned} \tau^+_b &= \inf\{t \ge 0 | \mathbf{1}_{W_t > b}(t - g^W_{b,t}) \ge 1\}, \\ \tau^-_b &= \inf\{t \ge 0 | \mathbf{1}_{W_t < b}(t - g^W_{b,t}) \ge 1\}, \\ \tau_b &= \tau^+_b \wedge \tau^-_b. \end{aligned}$$

Note that we have taken the window length of both sides to be the same (ie. 1 in our case).

2 Density of the two-sided Parisian stopping time

Let the barrier $b > 0$. We are only interested in the case when $\{T_b < 1\}$, where T_b is the first hitting time of level b, since if $T_b \ge 1$, $\tau_b = 1$. We have the following recursive solution for the density of τ_b on the set $\{T_b < 1\}$.

Theorem 2.1 *For $b > 0$, we denote by $f_b(t, T_b < 1)$ the probability density function of the two-sided stopping time τ_b on the set $\{T_b < 1\}$. We have*

$$f_b(t, T_b < 1) = \sum_{k=0}^{n-1} (-1)^k L_k(t-1),$$

for $n < t \le n+1$, $n = 1, 2, ...$, for $t > 0$, where $L_k(t)$ is defined recursively as follows:

$$\begin{aligned} L_0(t) &= \mathbf{1}_{\{0<t\le 1\}} \frac{1}{\pi\sqrt{t}} e^{-\frac{b^2}{2t}} \\ &\quad + \mathbf{1}_{\{t>1\}} \frac{2}{\pi\sqrt{t}} e^{-\frac{b^2}{2t}} \mathcal{N}\left(-b\sqrt{\frac{t-1}{t}}\right), \\ L_{k+1}(t) &= \int_1^{t-k} L_k(t-s) \frac{\sqrt{s-1}}{\pi s} ds, \end{aligned}$$

for $t > k+1$.

And for $b < 0$, we have

$$f_b(t, T_b < 1) = f_{-b}(t, T_{-b} < 1),$$

due to the symmetry of the standard Brownian motion. We prove that the two-sided stopping time τ_0 has an exponential tail, unlike the distribution of the one-sided stopping time τ^-_0. This is as expected because the one-sided case involves the hitting time of a Brownian motion, which is a heavy tailed distribution with infinite expectation, while the two-sided one involves the hitting time of a Brownian motion reflected in zero, which has an exponential tail.

Theorem 2.2 *We denote $\bar{F}_0(t)$ as the tail of the distribution of the two-sided Parisian stopping time τ_0 with barrier 0. It has an exponential tail. As $t \to \infty$, we have*

$$\bar{F}_0(t) \sim 2e^{-\beta^*} e^{-\beta^*(t-1)},$$

for some constant $\beta^* > 0$ such that $-\beta^*$ is the unique solution of the equation

$$\int_0^1 \frac{e^{-\beta s}}{\sqrt{s}} ds + \frac{e^{-\beta}}{\beta} = 0.$$

We can compute β^* numerically to be 0.854.

Hence, we have an approximation for the density. As t gets larger,

$$f_b(t, T_b < 1) \sim 2\beta^* e^{-\beta^*(t+1)} \int_0^1 e^{\beta^* s - \frac{b^2}{2s}} \frac{b}{\sqrt{2\pi s^3}} dt.$$

3 Numerical Results

We can then use this result to price two-sided Parisian options. The formula can be found in the paper, and we show some numerical results below. The table below presents the survival function for both τ_0 and τ^-_0, computed using a time step of $h = 0.001$ with R.

Table 1: One and two-sided survival functions for $0 < t \le 10$

t	$\bar{F}_0(t)$	$\bar{F}^-_0(t)$	t	$\bar{F}_0(t)$	$\bar{F}^-_0(t)$
1.5	0.555931	0.775033	6.0	0.015114	0.403422
2.0	0.369469	0.681770	6.5	0.010779	0.387956
2.5	0.242144	0.614236	7.0	0.007910	0.374142
3.0	0.159600	0.563552	7.5	0.006003	0.361704
3.5	0.105503	0.523602	8.0	0.004726	0.350429
4.0	0.070093	0.491082	8.5	0.003866	0.340146
4.5	0.046893	0.463944	9.0	0.003278	0.330718
5.0	0.031679	0.440854	9.5	0.002872	0.322033
5.5	0.021687	0.420896	10.0	0.002586	0.313997

We can see that the two-sided survival function goes to 0 much faster than the one-sided case.

The following graph compares the density functions of the one and two-sided case. The red line represents $f_0(t)$ while the black line $f^-_0(t)$, plotted against time. This graph suggests that $f^-_0(t)$ has a heavier tail.

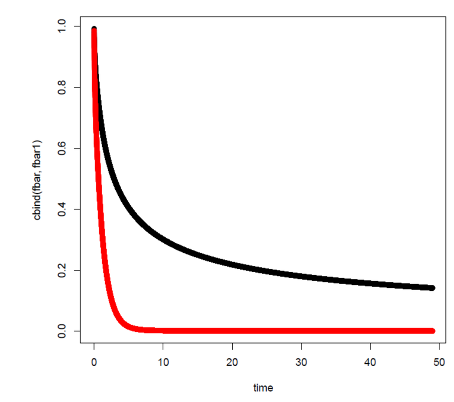

Figure 1: $f_0(t)$ and $f^-_0(t)$ for $0 < t \le 50$

The following graph depicts the tails $\bar{F}_0(t)$ (black) and the approximation $C_{\beta^*}e^{-\beta^* t}$ (red).

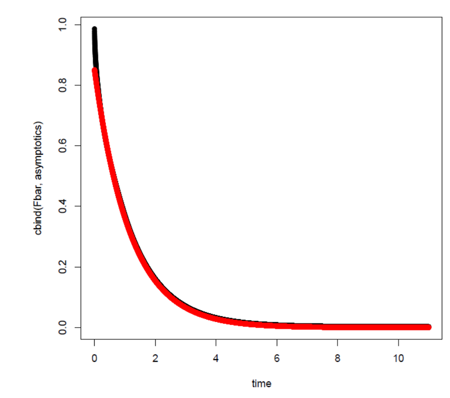

Figure 2: $\bar{F}_0(t)$ and $C_{\beta^*}e^{-\beta^* t}$ for $0 < t \le 20$

It suggests that the asymptotic provides a good approximation for the survival function.

We also provide some numerical results for the prices of the two-sided Parisian options. The following table gives the prices for different values of initial asset price S_0 and window length D, for parameters $K = 95$, $L = 90$, $T = 1$ year, $r = 0.05$ and $\sigma = 0.2$. These values are obtained using the recursive formula for $t \le 4$, and for $t > 4$, the asymptotics is used.

Table 2: Price of Parisian min-in call

S_0	1 week	2 weeks	1 month	2 months
80	2.817708	2.809610	2.660829	2.123282
82	3.471103	3.430688	3.145066	2.482966
84	4.203278	4.101558	3.737759	3.096815
86	5.050461	4.978642	4.724678	4.261088
88	6.535228	6.639547	6.589191	6.342500
90	6.897115	6.895460	6.891562	6.872088.