Option Volatility and Pricing

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Abstract

This poster discusses three types of volatility : the historical, implied and stochastic volatility, and the concept of local volatility. Finally the pricing formula of Vanilla options with stochastic volatility model such as the model of Heston is presented.

Introduction

The study of the volatility of certain financial assets became very important subject in finance. This importance comes from the possibility to measure the uncertainty of evolution of the yield on an asset (share or index) and the fact that the fluctuations of prices can not be neglected. Nowadays, any investor is conscious of these fluctuations which introduce an element of risk into its portfolio. That is why the investors wish to choose the degree "of exposure" at the risk compatible with their level of tolerance to this risk. Thus the study of the volatility plays an essential role in evaluation and hedging of risk of an investment.

The definition of the volatility

The volatility is a measure for variation of price of a financial instrument over time.

The types of volatility :

– Historical volatility
– Implied volatility
– Stochastic volatility

Part 1. Historical volatility

The historical volatility reflects the past price movements of the underlying asset, it is calculated as a standard deviation of a stock's returns over a fixed number of days.

The estimate of historical volatility starting from data :

$n+1$: the number of observations;
S_t : the price at the time t;
u_t : the return at time t;
τ : duration of the time intervals in year.
Then

$$u_t = \ln\left(\frac{S_t}{S_{t-1}}\right)$$

for $t = 1, 2, ..., n$

The estimate s of the standard deviation of u_t is given by this formula :

$$s = \sqrt{\frac{1}{n-1}\sum_{t=1}^{n}(u_t - \overline{u})^2}$$

where $\overline{u}$ is the average of u_t

The Black and Scholes model assumes the following dynamic of the stock price :

$$dS_t = \mu S_t dt + \sigma S_t dz_t$$

where z_t is a Wiener process

Since $\frac{dS_t}{S_t}$ behaves like a normal distribution $\mathcal{N}(\mu dt, \sigma\sqrt{dt})$, the standard deviation of the return is equal to $\sigma\sqrt{dt}$

Therefore s is an estimator of $\sigma\sqrt{\tau}$. Thus we estimate σ by $\widehat{\sigma}$ where :

$$\widehat{\sigma} = \frac{s}{\sqrt{\tau}}$$

Part 2 : Implied volatility

The implied volatility is often interpreted as estimation of the future volatility. It means that this volatility is a volatility anticipated by the market maker. In other words it is the value of σ that equalizes the price calculated by Black & Scholes model with the observed prices on the market

$$C_t^{observed}(S_t, T, K) = C_t^{Black-Scholes}(S_t, T, K, \sigma^{impl})$$

Estimation of implied volatility

We can find the value of an European call, by using the Risk-Neutral valuation method i.e by considering that price of a call corresponds to its expected value of future return discounted :

$$C_0 = e^{-rT}\widehat{E}\left[\max(S_T - K; 0)\right] = e^{-rT}\widehat{E}\left[(S_T - K)^+\right]$$

where $\widehat{E}$ is the expectation operator under the probability risk–neutral.

The value of call and put option a time 0 is given by :

$$\begin{aligned} C_0 &= S_0\Phi(d_1) - Ke^{rT}\Phi(d_2) \\ P_0 &= Ke^{rT}\Phi(-d_2) - S_0\Phi(-d_1) \end{aligned}$$

where

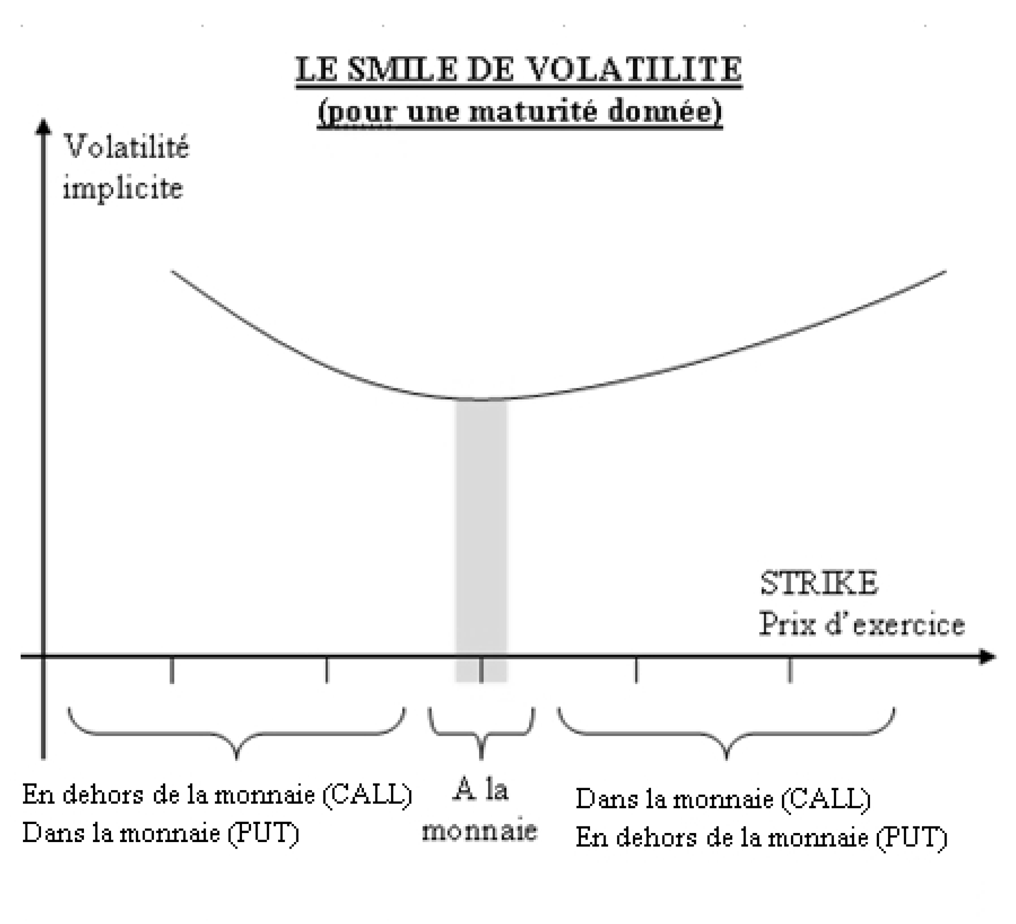
$$d_1 = \frac{\ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}},\ d_2 = d_1 - \sigma\sqrt{T}$$

and Φ is a the normal probability distribution function.

It is not possible to reverse the preceding equation and to express σ according to S_0, K, r, T and C_0. However, it is possible to determine the value of this implied volatility by using methods of interpolation like the method of Newton & Raphson.

Part 3 : Smile volatility

The implied volatility of an option evolves according to the strike and the maturity of the option, now when we draw the implied volatility according to strike for a given maturity, generally we do not obtain a horizontal line, which corresponds to the assumption of consistency of implied volatility.



Implied volatility smile

The Smile of volatility is a phenomenon observed on the markets of options vanillas which contradicts the assumption of Black and Scholes according to which the volatility of an option is constant and is not influenced by the value of other parameters. From a statistical point of view such a form of curve of volatility according to the strike price corresponds to a value of Kurtosis higher than 3, therefore risk-neutral dynamics of Black & Scholes and Merton are not compatible with the phenomenon of smile which exists on all markets options.

Part 4 : Local volatility

Dupire formula :

Fokker-Planck equation :

$$\begin{cases} \frac{\partial f}{\partial T} + r\frac{\partial}{\partial x}(xf) - \frac{1}{2}\frac{\partial^2}{\partial x^2}(x^2\sigma^2(x,T)f) = 0 \\ f(x,t) = \delta(S_t - x) \qquad \text{sur } [0,+\infty]\text{x}[t,+\infty] \end{cases}$$

where δ is the Dirac function

Theorem : for every (t,s) fixed the function :

$$C(T,K) = e^{-r(T-t)}E^Q[(S_T - K)^+ / S_t = s]$$

is a solution of Dupire equation :

$$\frac{\partial C}{\partial T} - r(C - K\frac{\partial C}{\partial K}) - \frac{1}{2}\sigma^2(T,K)K^2\frac{\partial^2 C}{\partial K^2} = 0$$

In particular :

$$\sigma(T,K) = \sqrt{2\frac{\frac{\partial C}{\partial T} + r(C - K\frac{\partial C}{\partial K})}{K^2\frac{\partial^2 C}{\partial K^2}}}$$

Part 4 : The link between local volatility and implied volatility

Let us consider a model with non-deterministic volatility. That is the risk-neutral dynamics of S is written as :

$$\frac{dS_t}{S_t} = \mu dt + \sigma_t dW_t$$

Instantaneous volatility σ is a process such as :

$$\int_0^T \sigma_s^2 ds < \infty, \forall\ T > 0$$

Thus we can define the local variance as conditional expectation of the future instantaneous variance

$$\sigma_L^2(T,K) = E^Q\left[\sigma_T^2 \middle| S_T = K\right]$$

This definition and that based on Dupire's formula are equivalent.

Part 5 : Heston model

Heston model (1993)

The Heston stochastic volatility model is based on the following stock price and variance dynamics

$$\begin{cases} dS(t) = \mu(t)S(t)dt + \sqrt{v(t)}S(t)dZ_1 \\ dv(t) = \kappa(\theta - v(t))dt + \sigma\sqrt{v(t)}dZ_2 \end{cases}$$

where $\langle dZ_1, dZ_2\rangle = \rho dt$, θ : the long-run average of $v(t)$,
κ : controls the speed by which $v(t)$ returns to its long-run mean and σ : the volatility of volatility.

The fundamental partial differential equation (PDE) verified by option price is :

$$\frac{\partial C}{\partial t} + \frac{1}{2}CS^2\frac{\partial^2 C}{\partial S^2} + \rho\sigma vS\frac{\partial^2 C}{\partial S\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 C}{\partial v^2} + rS\frac{\partial C}{\partial S} - rC$$

$$-\frac{\partial C}{\partial v}[\kappa(\theta - v) - \lambda v] = 0$$

We seek to solve the preceding PDE in the case of a European call option of strike K and maturity T, by analogy with the Black & Scholes formula, the solution of this option is of the form :

$$C(S,v,t) = SP_1 - Ke^{-r(T-t)}P_2$$

By injecting it in the PDF, and From the Fourier inversion theorem, we have :

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} Re\left[\frac{e^{-i\phi\ln(K)}f_j}{i\phi}\right]d\phi$$

For $j = 1, 2$, P_j are probabilities , and f_j are characteristics functions such as

$$f_j(x,v,T,\phi) = \exp\left(C(\tau,\phi) + D(\tau,\phi)v + i\phi x\right)$$

and $C(\tau,\phi) = r\phi i\tau + \frac{a}{\sigma^2}\left[(b_j - \rho\sigma\phi i + d)\tau - 2\ln\left(\frac{1-ge^{dr}}{1-g}\right)\right]$

$$D(\tau,\phi) = \frac{b_j - \rho\sigma\phi i + d}{\sigma^2}\left[\frac{1 - e^{dr}}{1 - ge^{dr}}\right]$$

$$g = \frac{b_j - \rho\sigma\phi i + d}{b_j - \rho\sigma\phi i - d} \qquad \text{and} \qquad d = \sqrt{\left(\rho\sigma\phi i - b_j\right)^2 - \sigma^2\left(2u_j\phi i - \phi^2\right)}$$

$$u_1 = \frac{1}{2}\ ,\ u_2 = -\frac{1}{2}\ ,\ a = \kappa\theta\ ,\ b_1 = \kappa + \lambda - \rho\sigma\ ,\ b_2 = \kappa + \lambda$$