A SUPERMARTINGALE ARGUMENT FOR CHARACTERIZING THE FUNCTIONAL HILL PROCESS WEAK LAW FOR SMALL PARAMETERS

Adja Mbarka FALL   Gane Samb LO
adjambarkafall@gmail.com   ganesamblo@ganesamblo.net
LERSTAD, Gaston Berger University, Saint-Louis, SENEGAL   MAPMO, Orleans University, Orleans, FRANCE
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Problem

The extreme value theory has its foundations in finding the asymptotic law of the maximum observation $X_{n,n} = \max(X_1, ..., X_n)$. It is said that the underlying *df* F of the observations is attracted to some *df* H if for some sequences $(a_n > 0)_{n\geq 1}$ and $(b_n)_{n\geq 1}$, we have for any continuity point $x \in \mathbb{R}$ of H,

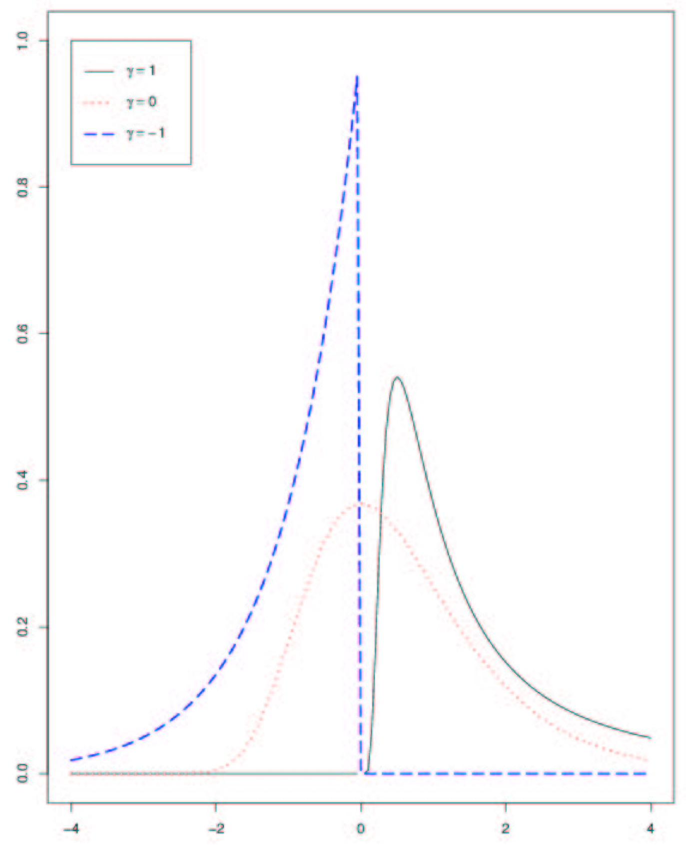
$$\lim_{n\to\infty} \mathbb{P}\left(\frac{X_{n,n} - b_n}{a_n} \leq x\right) = \lim_{n\to\infty} F^n(a_n x + b_n) = H(x).$$

Non-degenerated H can be parametrized as for $\gamma \in \mathbb{R}$

$$G_\gamma(x) = \begin{cases} \exp(-(1+\gamma x)^{-1/\gamma}), 1+\gamma x > 0, & for \quad \gamma > 0 \\ exp(-exp(-x)), & for \quad \gamma = 0 \end{cases}$$

G is named as the Generalized Extreme Value (GEV) distribution. It is said that F is in the domain of attraction of G_γ, hereby denoted as : $F \in D(G_\gamma)$. The parameter γ is called the extreme value index (EVI).

Although the parameter γ in the GEV is continuous, the three cases $\gamma < 0$, $\gamma = 0$ and $\gamma > 0$, behave radically differently. These cases are respectively named the Weibull, Gumbel and Frechet cases. In all these cases, the Hill statistic is used to estimate γ. The following figure gives a representation of these three cases



Our problem consists to find the asymptotic law of the functional Hill stochastic process when F is in the Weibull domain. This stochastic process generalised the Hill's statistic of the EVI and is based on extreme values of independent and identically distributed rv's $X_1, \ldots, X_n$. The process is defined as follows :

$$T_n(f) = \sum_{j=1}^{k(n)} f(j) \left(\log(X_{n-j+1,n}) - \log(X_{n-j,n})\right).$$

$k(n)$ is a sequence of integers satisfying $1 \leq k(n) < n$. $f(j)$ is a real and increasing function of $j \in \mathbb{N}$ such that $f(0) = 0$.
The functional Hill process was introduced by Deme et al. (2012) [2]. From this processus is derived the Diop and Lo (2006) [1] generalization of Hill's statistic. Its corresponds to the continuous statistic $T_n(f)$ for $f(j) = j^\tau$, $\tau > 0$. Diop and Lo proved its asymptotic normality for any γ, but only for $\tau > 1/2$. The Hungarian Gaussian Approximation used could not allow to find the asymptotic law for $\tau \leq 1/2$.
The functional form $T_n(f)$ has been extensively studied for Frechet and Gumbel cases by Deme *et al.* (2012). Some conditions under wich $T_n(f)$ has a Gaussian limiting process were established. When particularized for $f(j) = j^\tau$, Deme *et al.* get asymptotic normality for $\tau \geq 1/2$ and not for $0 < \tau < 1/2$. Deme et al. results are based on sums of independent *rv's*, and then the Kolmogorov Theorem for centred *rv's* applies
In our work we show that the law of the functional Hill process is derived when F is in the Weibull domain from the following process :

$$W_k(f) = \sum_{j=1}^{k-1} \Delta_f(j) \left(\exp(-\gamma \sum_{h=j}^{k-1} \frac{E_h}{h}) - \mathbb{E}\exp(-\gamma \sum_{h=j}^{k-1} \frac{E_h}{h}) \right).$$

where $E_1, E_2, ...$ are independent exponential standard rv's, $\gamma > 0$ and $\Delta_f(j) = f(j) - f(j-1)$.
Our best achievement is the characterization of $W_k(f)$ and its use to find the asymptotic behavior of $T_n(f)$.

Contribution

We describe the asymptotic behavior of the functional Hill process when F is in the weibull domain and particulary when $f(j) = j^\tau$ for small parameters $0 < \tau < 1/2$. By this way we complete the open problem consisting to characterize the asymptotic behavior of the functional Hill process $T_n(f)$ for each domain of attraction.

Methods

We use martingale results to characterize the asymptotic law of the process $W_k(f)$. We next apply the findings to determine the asymptotic behavior of the functional Hill process for small parameters within the Extreme Value Theory (EVT) field.
Consider the filtration $\mathcal{F}_k = \sigma(E_1, ..., E_k), k \geq 1$ and remark the sequence $(W_k)_{k\geq 1}$ is adapted with respect to $(\mathcal{F}_k)_{k\geq 1}$. We have the following intermediate results.

Theorem 1 *The sequence $W_k(f)$ is a supermatingale with respect to $\mathcal{F}_k$. Furthermore, it converges almost-surely (a.s) to random variable $W_\infty(f)$ with finite expectation whenever*

$$\limsup_{k\to+\infty} k^{-\gamma} \sum_{j=L}^{k-1} \overline{f}(j) j^{\gamma - 1/2} < +\infty \tag{K1}$$

holds.

Corollary 1 *For $f(j) = f_\tau(j) = j^\tau, 0 < \tau < 1/2, W_k(f_\tau)$ converges almost surely to a finite random variable $W_\infty(\tau)$.*

Results

We resume the extreme value problem. We will suppose without any loss of generality that the observations X_i are greater than one so that $T_n(f)$. Now in the sequel we simplify the notation of $k(n)$ to k. We combine Renyi's and classical representation of $Y_j = \log X_j$. Denote by $G(y) = F(e^y)$ the *df* of $\log X_i$. Remind that $G \in D(G_{-\gamma})$ if and only if $F \in D(G_{-\gamma})$. We then start with the simplest case of functions $F \in D(G_{-\gamma})$, that is

$$y_0 - G^{-1}(1-u) = u^\gamma, 0 \leq u \leq 1, \tag{1}$$

where y_0 is the upper endpoint of G. We use here the index $-\gamma < 0$ instead of $\gamma < 0$. We are going to characterize the asymptotic law $T_n(f)$ under the condition $(K1)$.

Theorem 2 *Let $X_1, X_2, ...$ be a sequence of iid rv′s with common df G. Let $f(j)$ be an increasing function of the integer $j \geq 1$ such that $(K1)$ holds and let for any $\leq 1 \leq k \leq n$,*

$$A_{k,n}(f) = f(k-1) - \sum_{j=1}^{k-1} (f(j) - f(j-1)) \exp\left(- \sum_{h=j}^{k-1} \log(1+\gamma/h) \right).$$

Then

$$W^*_{k-1,n}(f) = A_{k,n}(f) - T_n(f)/(y_0 - Y_{n-k+1,n})$$

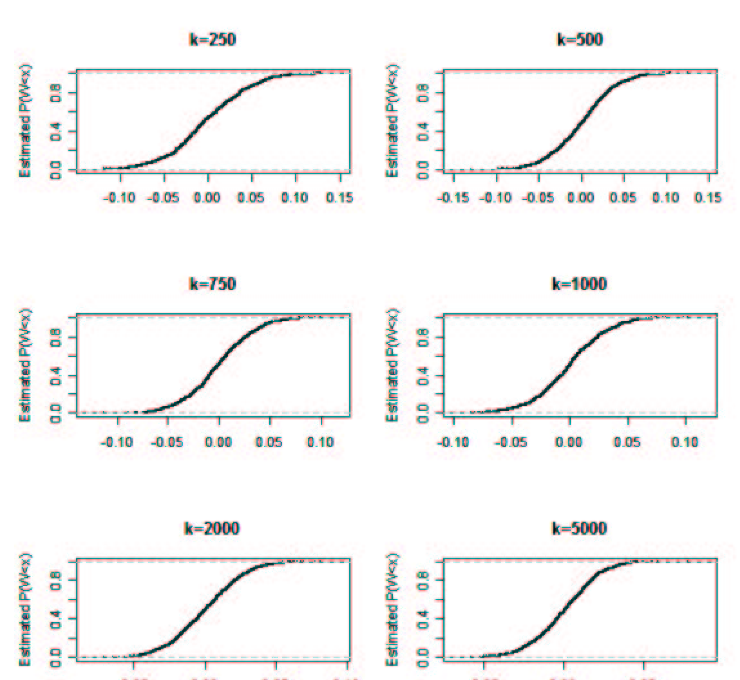
converges in distribution to the finite random variable $W_\infty(f)$ defined in Theorem 1. Further if $f(j) = f_\tau(j) = j^\tau$, $0 < \tau \leq 1/2$, then $W^*_{k-1,n}(f_\tau)$ converges in distribution to $W_\infty(\tau)$ defined in Corollary 1.

The law of $W_\infty(f)$ is unusual. We use computer-based methods for approximating this law. Simulation studies show that the empirical *df* based of $B0 = 1000$ replications are very stable from $k = 2000$. We proceed as follows. We fix $\tau, 0 < \tau < 1/2$, $\gamma > 0$ and $k \geq 2000$. At each step B from 1 to $B0 = 1000$, we generate standard exponential samples $E_1(B), ..., E_k(B)$. Then we compute W^*_k denoted by $W^*_k(B)$. We finally consider the empirical *df*, denoted by G_k, based on $W^*_k(1), ..., W^*_k(B0)$. Since G_k is stable in the sense that it does not significally change from $k = 2000$, we do approximate the *df* G_∞ of $W_\infty(\tau)$ by G_k for k large enough. We illustrate in the next figure the *df* G_k for $k = 250$, 500, 750, 1000, 2000, 500 for $\gamma = 1$ and $\tau = 1/4$. Here for instance, we infer that the support of G_∞ is $[-0.5, 0.5]$. On the whole, the figures clearly establish stability and support our proposal.

For users interested in using our method, we provide an executable file located at :

http ://www.ufrsat.org/lerstad/resources/lmhfw1.exe

for the computation of $P(W_\infty(\tau) \leq x) = G_\infty(x)$ and $P(|W_\infty(\tau))| \leq |x|) = G_\infty(|x|) - G_\infty(-|x|)$ for $x \in \mathbb{R}$.



For an application we illustrate how the law G_∞ of $W_\infty(1/4)$ may be used to do a statistical test for four models. We test the hypothesis that $F \in D(G_{-\gamma})$. We use here the following approximation :

$$T^*_n(f) = \frac{T_n(f)}{y_0 - \log X_{n-k+1,n}} \approx \frac{T_n(f)}{\log X_{n,n} - \log X_{n-k+1,n}}.$$

Here is the result :

Models	Quantile functions	$T^*_n(f_\tau)$	P-values
Weibull 1	$F^{-1}(1-u) = \exp(1-u^\gamma)$	3.16	67.4%
Weibull 2	$F^{-1}(1-u) = \exp(1-u^\gamma(1+u^9))$	0.0367	38.9%
	$F^{-1}(1-u) = \exp(1-u^\gamma(1+u^8))$	0.048	27.3%
	$F^{-1}(1-u) = \exp(1-u^\gamma(1+u^7))$	3.063	13.2%
	$F^{-1}(1-u) = \exp(1-u^\gamma(1+u^6))$	3.0725	10.4%
	$F^{-1}(1-u) = \exp(1-u^\gamma(1+u^5))$	3.097	2%
	$F^{-1}(1-u) = \exp(1-u^\gamma(1+u^4))$	3.17	0%
Stand. Exp.	$F^{-1}(1-u) = -\log u$	3.77	0%
Pareto	$F^{-1}(1-u) = u^{-1}$	19.755	0%

Discussion

We indeed remark that for the Weibull simple case, the law of the functional Hill process is found and particulary for $f(j) = j^\tau, 0 < \tau < 1/2$. For the general case, we have the following Karamata representation when F is in the Weibull case of parameter $\gamma > 0 : x_0(F) < \infty$ and

$$y_0 - G^{-1}(1-u) = c(1+p(u))u^\gamma \exp\left(\int_u^1 b(t) t^{-1} dt\right), \tag{2}$$

where $(p(u), b(u)) \to (0, 0)$ as $u \to 0$. So the results depends on the auxilliary functions p and b. If some further conditions on b and p are fullfilled, $T^*_n(f) = \frac{T_n(f)}{y_0 - Y_{n-k+1,n}}$ behaves as $W^*_{k,n}$ as in the present case and then the law should be the same.
Nevertheless, we can include in the statistical tests some models with specific forms of $b(\cdot)$ as shown in the precedent table.

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Références

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