Derivation of the Fick's law for the Lorentz model in a low density regime

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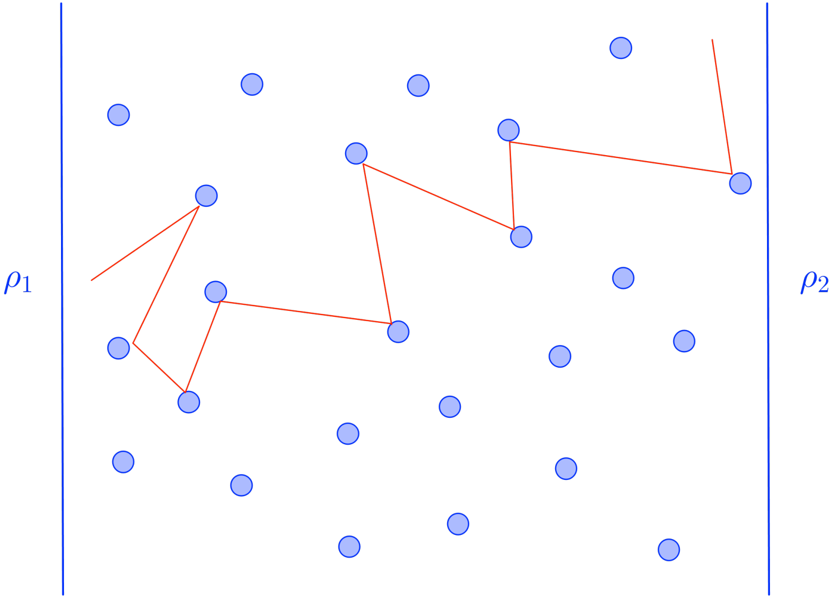
May 26, 2014

Joint work with G. Basile, F. Pezzotti, M. Pulvirenti


ρ_1
ρ_2

Nonequilibrium (mass reservoirs at density ρ_1, ρ_2)

ρ : macroscopic mass density

J : macroscopic mass current

$$J = -D\nabla\rho \qquad \text{(Fick's law)}$$

$D > 0$ diffusion coefficient and

$$\varrho(x) = \frac{\rho_1(L - x_1) + \rho_2 x_1}{L} \qquad \text{(Linear profile)}$$

- Bunimovich and Sinai, 1981 (Diffusion for the periodic Lorentz gas)

Microscopic description

$$\begin{cases} \dot{x} = v \\ \dot{v} = \mathrm{F}_{obs} \end{cases}$$

Newtonian dynamics
(Lorentz gas)

$$\Longrightarrow$$

Macroscopic description

$$\partial_t \varrho = D\Delta\varrho, \quad \varrho = \int f\, dv$$

Hydrodynamic equation
(diffusion equation)

Mesoscopic description

$$\underbrace{(\partial_t + v \cdot \nabla_x) f}_{transport} = \underbrace{\mathcal{L}(f)}_{collisions}$$

Linear kinetic equation

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Background

Test particle moving in a random configuration of obstacles in $\mathbb{R}^d$.
μ: intensity of the obstacles; $\varepsilon > 0$ ratio between micro and macro scale.

Low density limit: $t \to \varepsilon t$, $x \to \varepsilon x$, $\mu_\varepsilon = \varepsilon^{-(d-1)}\mu$.

(rarefied gas, mean free path $O(1)$)

$$\underbrace{(\partial_t + v \cdot \nabla_x) f(x,v,t)}_{transport} = \underbrace{\mathcal{L} f(x,v,t)}_{collisions} \qquad \textbf{(Linear Boltzmann)}$$

$$\mathcal{L} f(v) = \mu \int_{-1}^{1} d\rho \{ f(v') - f(v) \}$$

- Gallavotti, 1969 (Poisson distribution of hard sphere scatterers)
- Spohn, 1978 (more general distribution of scatterers)
- Desvillettes, Pulvirenti, 1999 (long range interaction)

Background

Scaling: $t \to \varepsilon t$, $x \to \varepsilon x$,
$\mu_\varepsilon = \eta_\varepsilon \varepsilon^{-(d-1)} \mu$, η_ε slowly diverging.

Asymptotic equation

$$(\partial_t + v \cdot \nabla_x) f(x, v, t) \sim \eta_\varepsilon \mathcal{L} f(x, v, t)$$

$\varepsilon \to 0$?

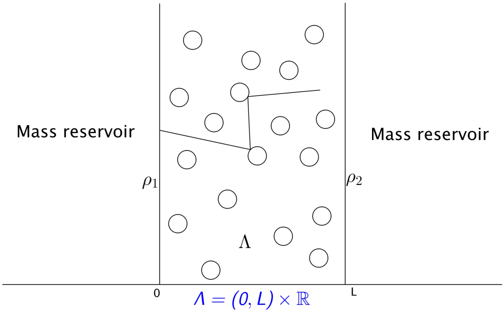
Idea: look at a longer time scale in which the equilibrium starts to evolve $t \to \eta_\varepsilon t$
$\Longrightarrow$ diffusion for the position variable

- Erdos, Salmhofer, Yau, 2008 (linear Quantum Boltzmann)
- Bodineau, Gallagher, and Saint-Raymond, 2013 (linear Boltzmann)
- Basile, N., Pulvirenti, 2013 (linear Landau)

The model

Poisson distribution of fixed hard disks $\mathbf{c}_N = (c_1, \dots, c_N)$ in $\Lambda = (0, L) \times \mathbb{R}$.
$\mu > 0$ intensity.

Mass reservoirs: free point particles at equilibrium at densities ρ_1, ρ_2.



$$\mathbb{P}(dc_N) = e^{-\mu|A|} \frac{\mu^N}{N!} dc_1 \dots dc_N, \qquad A \subset \Lambda.$$

Scaling: $\mu \to \mu_\varepsilon = \varepsilon^{-1} \eta_\varepsilon \mu$ $\qquad \left[Ass : \varepsilon^{\frac{1}{2}} \eta_\varepsilon^6 \underset{\varepsilon \to 0}{\longrightarrow} 0 \right]$

The model

Initial probability distribution $f_0 = f_0(x, v)$. Boundary value

$$f_B(x, v) := \begin{cases} \rho_1 M(v) & \text{if} \quad x \in \{0\} \times \mathbb{R}, \quad v_1 > 0 \\ \rho_2 M(v) & \text{if} \quad x \in \{L\} \times \mathbb{R}, \quad v_1 < 0 \end{cases}$$

We look at the one-particle correlation function

$$f_\varepsilon(x, v, t) = \mathbb{E}_\varepsilon[f_B(T_{\mathbf{c}_N}^{-(t-\tau)}(x, v))\chi(\tau > 0)] + \mathbb{E}_\varepsilon[f_0(T_{\mathbf{c}_N}^{-t}(x, v))\chi(\tau = 0)]$$

$\mathbb{E}_\varepsilon[\cdot]$: expectation w.r.t. the measure $\mathbb{P}_\varepsilon$.

$T_{\mathbf{c}_N}^{-t}(x, v)$: (backward) Hamiltonian flow.

$t - \tau$: first (backward) hitting time with $\partial\Lambda$. $\quad (\tau = \tau(x, v, t, \mathbf{c}_N))$

($\tau = 0$ if $T_{\mathbf{c}_N}^{-s}(x, v)$, $s \in [0, t]$, never hits the boundary)

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The Model

$f_\varepsilon^S(x, v)$ stationary solution of the microscopic dynamics

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Stationary mass flux: $$J_\varepsilon^S(x) = \eta_\varepsilon \int_{S_1} v\, f_\varepsilon^S(x,v)\, dv$$

Stationary mass density: $$\varrho_\varepsilon^S(x) = \int_{S_1} f_\varepsilon^S(x,v)\, dv$$

J_ε^S : total amount of mass through a unit area in a unit time.

$\eta_\varepsilon \quad \rightsquigarrow$ (time scaling necessary for diffusion!)

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Main results

Theorem 1 [Basile, N., Pezzotti, Pulvirenti]

For ε sufficiently small there exists a unique L^∞ stationary solution f_ε^S for the microscopic dynamics. Moreover

$$f_\varepsilon^S \to \varrho^S \quad \text{as} \quad \varepsilon \to 0$$

where ϱ^S is the stationary solution of the heat equation

$$\varrho^S(x) = \frac{\rho_1(L - x_1) + \rho_2 x_1}{L}.$$

The convergence is in $L^2((0,L) \times S_1)$.

- Boundary conditions depend on the space variable only through x_1
  $\Rightarrow$ f_ε^S, ϱ^S inherits the same feature
  $\Rightarrow$ convergence in $L^2((0,L) \times S_1)$ instead of $L^2(\Lambda \times S_1)$.
- For the convergence of the stationary solutions it is enough $\varepsilon^{\frac{1}{2}} \eta_\varepsilon^5 \to 0$.

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Theorem 2 [**Fick's law**] [Basile, N., Pezzotti, Pulvirenti]

- $$J^S_\varepsilon + D\nabla_x \varrho^S_\varepsilon \longrightarrow 0 \quad \text{as} \quad \varepsilon \longrightarrow 0 \qquad \text{in} \quad \mathcal{D}'(0,L)$$

  where $$D = \frac{1}{4\pi}\int_{S_1} dv\, v \cdot \big(-\mathcal{L}\big)^{-1} v \qquad \text{(Green-Kubo).}$$

- $$J^S = \lim_{\varepsilon\longrightarrow 0} J^S_\varepsilon(x) \qquad \text{in} \quad L^2(0,L)$$

  and $J^S = -D\,\nabla\varrho^S = -D\,\frac{\rho_2-\rho_1}{L}$ (ϱ^S linear profile).

- J^S does not depend on the space variable.
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The Boltzmann equation as a bridge

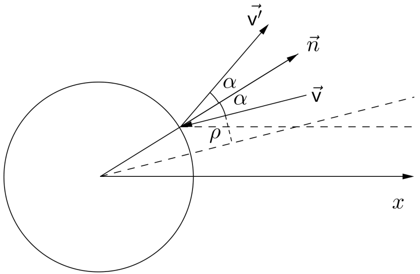
Introduce the stationary linear Boltzmann equation

$$\begin{cases} (v \cdot \nabla_x) h_\varepsilon^S(x,v) = \eta_\varepsilon \, \mathfrak{L} h_\varepsilon^S(x,v), \\ h_\varepsilon^S(x,v) = \rho_1, \qquad x \in \{0\} \times \mathbb{R}, \quad v_1 > 0, \\ h_\varepsilon^S(x,v) = \rho_2, \qquad x \in \{L\} \times \mathbb{R}, \quad v_1 < 0, \end{cases}$$

$$\mathfrak{L} f(v) = \mu \int_{-1}^{1} d\rho \big[f(v') - f(v) \big]$$

$$f \in L^1(S_1)$$

$$v' = v - 2(n \cdot v)n, \quad n = n(\rho)$$



$\{v(t)\}_{t \geq 0}$ Markov jump process
$x(t) = \int_0^t v(s) ds$

Strategy

1. There exists a unique stationary solution $h_\varepsilon^S \in L^\infty((0,L)\times S_1)$ of the linear Boltzmann equation. As $\varepsilon \to 0$

$$h_\varepsilon^S \to \varrho^S \quad \text{in} \quad L^2((0,L)\times S_1) \qquad (\textbf{Markov part})$$

2. There exists a unique stationary solution $f_\varepsilon^S \in L^\infty((0,L)\times S_1)$ for the microscopic dynamics such that

$$\|h_\varepsilon^S - f_\varepsilon^S\|_\infty \leq C\varepsilon^{\frac{1}{2}}\eta_\varepsilon^5 \qquad (\textbf{Markovian approximation})$$

(*The memory effects preventing the Markovianity are negligible!*)

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Markov part

h_ε solution of the linear Boltzmann equation
with boundary conditions $\rho_1,\ \rho_2$ and initial data f_0.

$$h_\varepsilon(x,v,t) = h_\varepsilon^{out}(x,v,t) + h_\varepsilon^{in}(x,v,t) = h_\varepsilon^{out}(x,v,t) + \underset{\text{Markov sgr.}}{S_\varepsilon^0(t)} f_0$$

h_ε^{out}: backward trajectories hitting the boundary.

h_ε^{in}: trajectories which never leave Λ.

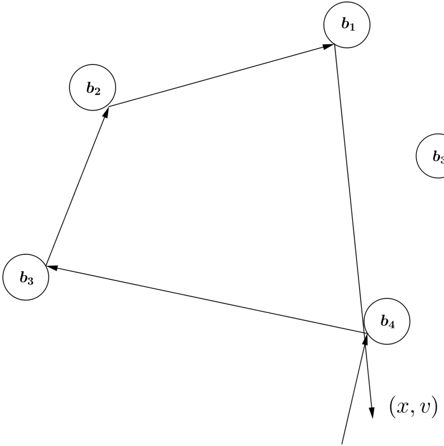
h_ε^S $$h_\varepsilon^S = h_\varepsilon^{out}(t_0) + S_\varepsilon^0(t_0) h_\varepsilon^S, \qquad t_0 > 0$$

$\Rightarrow$ $$h_\varepsilon^S = \sum_{n\geq 0} (S_\varepsilon^0(t_0))^n h_\varepsilon^{out}(t_0) \quad (\textit{Neumann series})$$

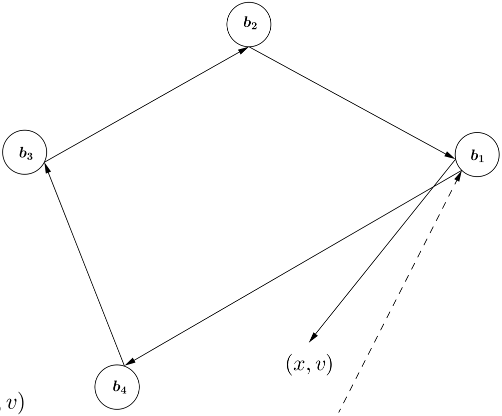
Convergence of the Neumann series $\Rightarrow$ existence and uniqueness of h_ε^S

$h_\varepsilon^S \to \varrho^S$ Hilbert expansion technique

Pathological configuration in the Markovian approximation



Backward Interference



Backward Recollision

Proof of Theorem 2

By standard computations (Hilbert expansion technique)

$$h_\varepsilon^S = \varrho^S + \frac{1}{\eta_\varepsilon} h^{(1)} + \frac{1}{\eta_\varepsilon} R_{\eta_\varepsilon}$$

But

$$h^{(1)}(v) = \mathcal{L}^{-1}(v \cdot \nabla_x \varrho^S) = \frac{\rho_2 - \rho_1}{L} \mathcal{L}^{-1}(v_1)$$

$$R_{\eta_\varepsilon} = O(\frac{1}{\sqrt{\eta_\varepsilon}}), \qquad \int_{S_1} v \varrho^S dv = 0$$

$$\Downarrow$$

$$\underbrace{\eta_\varepsilon \int_{S_1} v h_\varepsilon^S(x,v) dv}_{\substack{\sim J_\varepsilon^S(x) \quad \text{in } L^\infty \\ (Ass.)}} = \underbrace{-D \nabla_x \varrho^S}_{\substack{\sim D\nabla_x \varrho_\varepsilon^S \quad \text{in } \mathcal{D}' \\ (Theorem\ 1)}} + O(\frac{1}{\sqrt{\eta_\varepsilon}})$$

Thanks for your attention!