Projective systems of functionals and applications to high order heat-type equations

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Outline

Generalizations of Feynman-Kac formula

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\Delta u(t,x) - V(x)u(t,x)$$

$$u(t,x) = \int_{C_t} e^{-\int_0^t V(\omega(s)+x)ds} u_0(\omega(t)+x)dW(\omega), \qquad \text{for a.e. } x \in \mathbb{R}^d$$

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$$\begin{cases} dX(t) = b(t,X(t))dt + \sigma(t,X(t))dB(t), \quad 0 \leq s < t \leq T \\ X(s) = x, \end{cases}$$

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) + \frac{1}{2}Tr[\sigma(t,x)\sigma^*(t,x)D_x^2 u(t,x)] + \langle b(t,x), D_x u(t,x)\rangle = 0 \\ u(T,x) = \phi(x) \end{cases}$$

$$u(s,x) = \mathbb{E}[\phi(X(T,s,x))]$$

Generalizations of Feynman-Kac formula

$\Delta \mapsto \Delta^N$ Heat equation $\mapsto$ "higher order heat equation"

$$\frac{\partial}{\partial t} u(t) = (-1)^{N+1} \Delta^N u(t) \tag{1}$$

$$\frac{\partial}{\partial t} u(t) = (-1)^{N+1} \Delta^N u(t) - V(x) u(t,x) \tag{2}$$

$$u(t,x) = \text{"}\mathbb{E}^x[u(0,X(t)) e^{-\int_0^t V(X(s))ds}]\text{"}$$

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Problem: the solutions of (1) do not satisfy a maximum principle.

Formal derivation of Feynman-Kac formula: Trotter product formula

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$$e^{(\frac{\Delta}{2}-V)t}u_0(x) = \lim_{n\to\infty}\int_{\mathbb{R}^n} u_0(x+x_0)e^{-\sum_{j=1}^n V(x_j)\frac{t}{n}}\prod_{j=1}^n G(t/n, x_j - x_{j-1})dx_j$$

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$$G(t,x-y) = \frac{e^{-\frac{(x-y)^2}{2t}}}{\sqrt{2\pi t}}$$

$$e^{(\frac{\Delta}{2}-V)t}u_0(x) = \mathbb{E}[u_0(x+W(t))e^{-\int_0^t V(x+W(s))ds}]$$

Signed measures on $\mathbb{R}^{[0,t]}$

Let's try to define a signed measure μ on $\Omega = \mathbb{R}^{[0,t]}$ defined on "cylindrical sets" $I_k \subset \Omega = \{x : [0, \infty) \to \mathbb{R}\}$,
$I_k := \{\omega \in \Omega : \omega(t_j) \in [a_j, b_j], j = 1, \dots k\}$, $0 < t_1 < t_2 < \dots t_k$,

$$\mu_{t_1,t_2,\dots,t_k}(I_k) = \int_{a_1}^{b_1} \dots \int_{a_k}^{b_k} \prod_{j=0}^{k-1} G(t_{j+1} - t_j, x_{j+1} - x_j) dx_1 \dots dx_k,$$

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"higher order heat equation" case $\dot{u}(t) = (-1)^{N+1} \Delta^n u(t)$

$$G(t, x - y) = \frac{1}{2\pi} \int e^{i(x-y)\xi} e^{-\xi^{2n} t} d\xi$$

in not positive!

Non existence of the limiting measure μ

- V. Yu. Krylov (1960): It does not exist a σ-additive resp. signed **bounded variation** measure μ on $\mathbb{R}^{[0,t]}$ whose cylindrical approximations are given by

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- E.Thomas, PTRF (2001): proof of the analogue of Prokhorov's criterion on the existence of the projective limit of a compatible system of **signed** or **complex** measures.

An alternative integration theory

- Realization of the "integral"

$$f \in C_0(X) \;\mapsto\; \int_X f(x)d\mu(x) := I_\mu(f),$$

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- Consequence: in the application to the generalized Feynman-Kac formulae for the high order heat type equations

$$u(t,x) = \mathbb{E}[u_0(x + \omega(t) e^{-\int V(x+\omega(s))ds}]$$

we expect some restrictions on the class of initial data u_0 and potentials V

Possible approaches and results

- V. Yu. Krylov (1960), K. Hochberg (1978): solution of $\frac{\partial}{\partial t} u(t) = (-1)^{N+1} \Delta^N u(t) - V(x) u(t,x)$ in terms of on integral on $\mathbb{R}^{[0,t]}$ w.r.t a signed measure with ∞ total variation

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- R. Léandre (2010): Hida-Connes calculus approach

A new probabilistic construction

S. Bonaccorsi and S. Mazzucchi, *High Order Heat-type Equations and Random Walks on the Complex Plane.* arXiv:1402.6140 [math.PR] (2014).

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The 1- dimensional Brownian motion can be realized as a weak scaling limit of a random walk on the real line:

$$B(t) = \lim_{n\to\infty} \frac{1}{n^{1/2}} \sum_{j=1}^{\lfloor nt \rfloor} \xi_j = \lim_{n\to\infty} S_n(t)$$

ξ_j i.i.d. random variables such that

$$P(\xi_j = 1) = P(\xi_j = -1) = \frac{1}{2}$$

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the Feynman-Kac formula can be written as:

$$u(t,x) = \lim_{n\to\infty} \mathbb{E}[e^{-i\int_0^t V(S_n(s)+x)ds} u_0(S_n(t)+x)]$$

A new probabilistic construction

Let α be a complex number and $N \in \mathbb{N}$ a given integer.

$$\frac{\partial}{\partial t} u(t,x) = \frac{\alpha}{N!} \frac{\partial^N}{\partial x^N} u(t,x), \quad x \in \mathbb{R}$$

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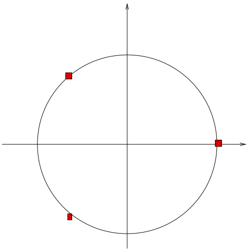
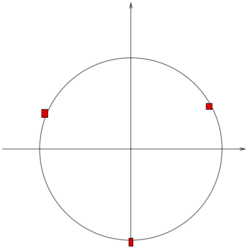
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $R(N) = \{e^{2i\pi k/N},\ k = 0, 1, \ldots, N-1\}$ be the roots of the unity. Then we consider the random variable ξ that has uniform distribution on the set $\alpha^{1/N} R(N)$:

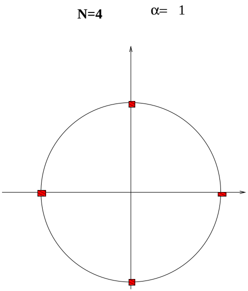
$$\mathbb{E}[f(\xi)] = \frac{1}{N} \sum_{k=0}^{N-1} f(\alpha^{1/N} e^{2i\pi k/N}). \tag{3}$$

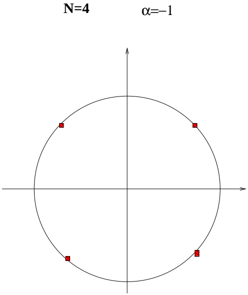
N=3 $\alpha=1$



N=3 $\alpha= i$




N=4
α= 1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Some interestin properties

The random variable ξ has finite moments of every order

$$\mathbb{E}[\xi^m] = \begin{cases} \alpha^{m/N}, & m = nN,\ n \in \mathbb{N}, \\ 0, & \text{otherwise} \end{cases}$$

Examples:

- $N = 2$: $\mathbb{E}[\xi^m] = 0$ if m odd, $\mathbb{E}[\xi^m] = \alpha^{m/2}$ if m even

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- $N = 6$: $\mathbb{E}[\xi^m] = 0$ for $m = 1, 2, 3, 4, 5$, $\mathbb{E}[\xi^6] = \alpha$

A non standard central limit theorem

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Then its distribution converges weakly to a stable distribution of order N in the sense that

$$\lim_{n\to\infty} \mathbb{E}[\exp(i\lambda \tilde{S}_n)] = \exp\left(\frac{i^N \alpha}{N!}\lambda^N\right).$$

Remarks

- We explicitly note that, for $N > 2$, the scaling exponent $1/N$ is weaker than that of the classical CLT and is related to the order of spatial derivative in the PDE.

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- We explicitly note that, for $N > 2$, the scaling exponent $1/N$ is weaker than that of the classical CLT and is related to the order of spatial derivative in the PDE.
- We also note that, for $N > 2$, the function $\exp(cx^N)$ is not a well defined characteristic function.
- In case $N = 2$, the limit of the random walk $\tilde{S}_n$ is a Wiener process; to be precise, since in the definition of $\tilde{S}_n$ no time is involved, it converges to the Wiener process at time $t = 1$. It is possible to extend this result to general times; however, it is not possible to talk about the limit process in case $N > 2$. We aim to construct a family of random walks $W_n(t)$ that generalizes, in a suitable sense, $\tilde{S}_n$ to a continuous time process

A family of complex jump processes

Let $\{\xi_j\}$ be a sequence of independent copies of the random variable ξ defined in (3). Then for any $n \in \mathbb{N}$ we set

$$W_n(t) = \frac{1}{n^{1/N}} \sum_{j=1}^{\lfloor nt \rfloor} \xi_j$$

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The sequence of processes $\{W_n\}$ should converge in a very weak sense to a N-stable process (which, we note again, does not exist for $N > 2$.

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Theorem

For any $t \in (-\infty, +\infty)$ and $\lambda \in \mathbb{C}$,

$$\lim_{n \to \infty} \mathbb{E}[\exp(i\lambda W_n(t))] = \exp\left(i^N \frac{\lambda^N}{N!} \alpha t \right).$$

Solution of higher order PDEs

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) &= \frac{\alpha}{N!}\frac{\partial^N}{\partial x^N}u(t,x), \\ u(t_0,x) &= f(x), \qquad\qquad x \in \mathbb{R}. \end{aligned} \tag{4}$$

We are going to show that for a suitable class of initial datum f, the limit

$$u(t,x) = \lim_{n\to\infty} \mathbb{E}[f(x + W_n(t-t_0))] \tag{5}$$

is well defined for any $x \in \mathbb{R}$ and t in a suitable neighbor of t_0, and it provides a representation for the solution of (4).

Solution of higher order PDEs

Let $f : \mathbb{R} \to \mathbb{C}$ be a (complex-valued) function of the form

$$f(x) = \int_{\mathbb{R}} e^{ixy} \,\mathrm{d}\mu(y),$$

where μ is a measure of bounded variation on $\mathbb{R}$ satisfying following assumptions:

1. $\displaystyle\int_{\mathbb{R}} |e^{ixz}| \,\mathrm{d}|\mu|(x) < \infty$ for all $z \in \mathbb{C}$,

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1. $\displaystyle\int_{\mathbb{R}} |e^{ixz}| \, \mathrm{d}|\mu|(x) < \infty$ for all $z \in \mathbb{C}$,
2. there exists a time interval (T_1, T_2), with $T_1 < t_0 < T_2 \in \mathbb{R}$, such that

   $$\int_{\mathbb{R}} \left| \exp\left(i^N \alpha \frac{x^N}{N!} (t - t_0) \right) \right| \, \mathrm{d}|\mu|(x) < \infty$$

   for all $t \in (T_1, T_2)$

Solution of higher order PDEs

Then the function

$$u(t, x) = \lim_{n\to\infty} \mathbb{E}[f(x + W_n(t - t_0))]$$

is a representation of the solution of the parabolic problem (4) for any time $t \in (T_1, T_2)$ in the sense that

$$u(t, x) = \int_{\mathbb{R}} e^{ixy} \exp\left(\frac{i^N \alpha}{N!}(t - t_0)y^N\right) \mathrm{d}\mu(y) \qquad (6)$$

and the integral in (6) is absolutely convergent. Suppose further that

3. $\int_{\mathbb{R}} \left| x^N \exp\left(i^N \alpha \frac{x^N}{N!}(t - t_0)\right)\right| \mathrm{d}|\mu|(x) < \infty$ for all $t \in (T_1, T_2)$.

Then the function

$$u(t, x) = \lim_{n\to\infty} \mathbb{E}[f(x + W_n(t - t_0))]$$

is a is a classical solution for the problem (4) for any time $t \in (T_1, T_2)$

The boundary value problem

The technique can be applied to the study of the boundary value problem

$$\lim_{n\to\infty} \mathbb{E}[f(x + W_n(t - t_0))]$$

$$\frac{\partial}{\partial t} u(t,x) = \frac{\alpha}{N!} \frac{\partial^N}{\partial x^N} u(t,x)$$

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- on $\mathbb{R}^+$, with Dirichelet or Neumann boundary conditions, if N is even
- on $[0, L]$ with periodic boundary conditions, for any $N \in \mathbb{N}$

Further developments

Study of the "speed of convergence" and implementation of a Montecarlo type technique for the numerical computation of the solution.

$$u_n(t,x) := \mathbb{E}[f(x + W_n(t-t_0))]$$

$$|u(t,x) - u_n(t,x)| \leq (1+\epsilon)\frac{C(t)}{n} \qquad \forall x \in \mathbb{R},\ t \in (T_1, T_2)$$

where

$$C(t) = \frac{|\alpha|}{N!}\int |x|^N \left|\exp\left(i^N \alpha \frac{x^N}{N!}(t-t_0)\right)\right| \mathrm{d}|\mu|(x) +$$

$$+|\alpha|^2(t-t_0)\left(\frac{1}{2(N!)^2} - \frac{1}{(2N)!}\right)\int |x|^{2N} \left|\exp\left(i^N \alpha \frac{x^N}{N!}(t-t_0)\right)\right| \mathrm{d}|\mu|(x)$$

Further developments

Introduction of a potential V and construction of a generalized Feynman-Kac formula.

$$\begin{cases} \partial_t u(t,x) = \frac{\alpha}{N!}\,\partial_x^N u(t,x) + V(x)u(t,x) \\ u(0,x) = u_0(x), \qquad x \in \mathbb{R}, \end{cases} \tag{7}$$

$$u(t,x) = \lim_{n\to\infty} \mathbb{E}[u_0(x + W^n(t))e^{\int_0^t V(x+W^n(s))ds}]$$

Further developments

Implementation of a generalized stochastic calculus (Ito integral, Ito formula) for the process $W_n(t)$:

$$f(W^n_t) - f(W^n_0) \sim \int_0^t f'(W^n_s)dW^n_s + \frac{1}{2!}\int_0^t f''(W^n_s)(dW^n_s)^2 + \cdots + \frac{1}{N!}\int_0^t f^{(N)}(W^n_s)(dW^n_s)^N \quad n \to \infty$$