Filter Based Volatility in the HMM

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Overview

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Hidden Markov Model in Continuous Time

- Observation process $Y = (Y_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{A}, \mathbb{P})$ with

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- Drift μ_t is governed by signal X_t, a Markov jump process with state space $\{e_1, \ldots, e_d\}$, so $\mu_t = \langle \mu, X_t \rangle$, and rate matrix Q.

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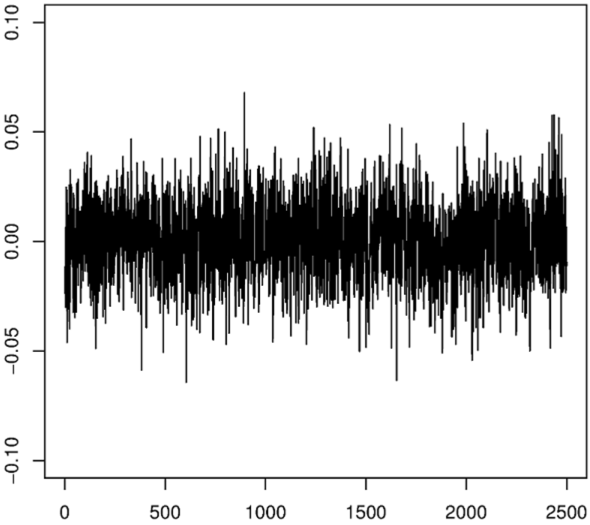
- W Brownian motion, independent of signal X.
- We are interested in the filter $\mathbb{E}\left[X_t \mid \mathcal{Y}_t\right]$, where $\mathcal{Y}_t = \sigma(Y_s | s \leq t)$ is the observable information.


0.10
0.05
0.00
−0.05
−0.10
0
500
1000
1500
2000
2500

Change of Measure

- We introduce a change of measure using the density

$$\left.\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_t} = Z_t = \exp\left(\int_0^t \langle \mu, X_s \rangle \mathrm{d}W_s - \frac{1}{2}\int_0^t \langle \mu, X_s \rangle^2 \mathrm{d}s\right).$$

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- Define the unnormalized filter as

$$\rho_t(X) := \tilde{\mathbb{E}}\left[Z_t^{-1} X_t | \mathcal{Y}_t\right] \propto \mathbb{E}\left[X_t \mid \mathcal{Y}_t\right].$$

Filtering Equations

- For the unnormalized filter we have that

$$\mathrm{d}\rho_t(X) = Q^T \rho_t(X)\mathrm{d}t + B\rho_t(X)\mathrm{d}Y_t,$$

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- In general, filtering equations are not finite dimensional.

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- The optimal strategy depends on the filter.

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- Filters can be derived for any $\sigma > 0$.
- But: σ has to be constant.
- So 'Stylized Facts' as e.g jumps in the volatility or volatility clustering cannot be modeled using a HMM!

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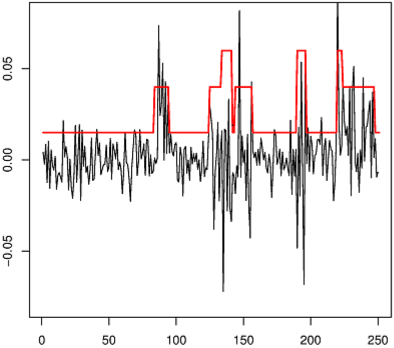

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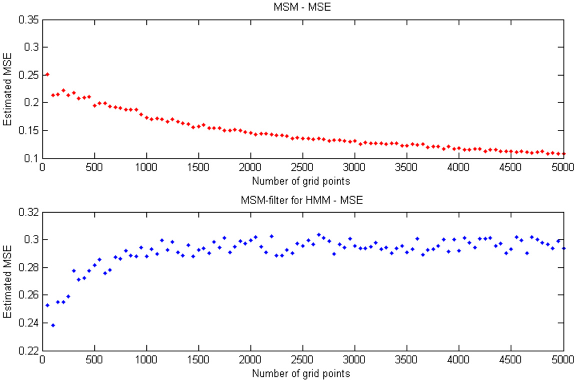
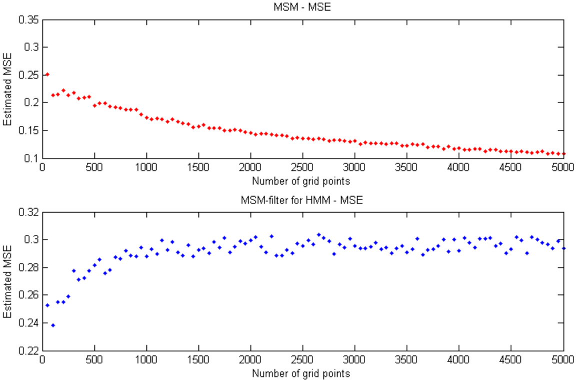
- For finer discretizations, the MSE in the MSM behaves differently than in the HMM.


MSM - MSE
Estimated MSE
0.35
0.3
0.25
0.2
0.15
0.1
0
500
1000
1500
2000
2500
3000
3500
4000
4500
5000
Number of grid points
MSM-filter for HMM - MSE
Estimated MSE
0.32
0.3
0.28
0.26
0.24
0.22
Number of grid points



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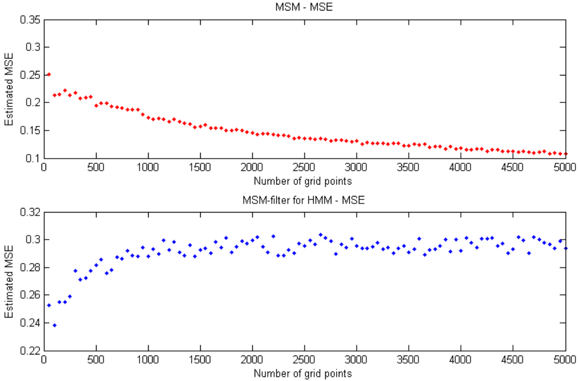

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- Model volatility as depending on an observable process.

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where $\mathrm{d}\tilde{W}_t = \mathrm{d}W_t + \sigma_t^{-1} \langle \mu, X_t \rangle \mathrm{d}t$ $\tilde{\mathbb{P}}$-Brownian motion.

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- Consider a model for $\sigma_t = \sigma(\xi_t)$ with

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$$\mathbb{E}[(Y_t^{MSM} - Y_t^{FV})^2] \stackrel{!}{=} \min_{\tilde{\xi}} \mathbb{E}\left[\left(\int_0^t \langle \sigma, X_t \rangle - \sigma(\tilde{\xi}_t) \mathrm{d}W_s \right)^2 \right].$$

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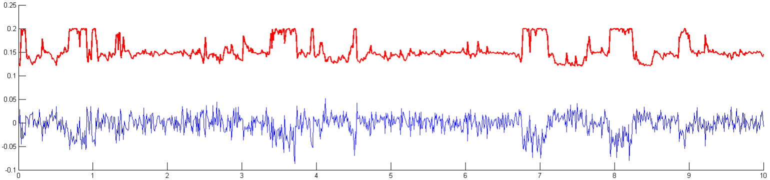
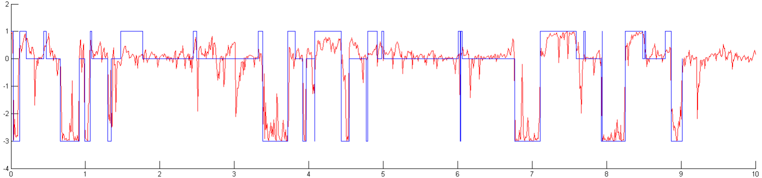
- In particular, it is linear in the filter.

Linear Model





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- Portfolio optimization can be done for several utility functions (Sass/Haussmann,2004).
- Robust discretizations are available.

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Literature

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