Traveling Waves in Stochastic Neural Field Equations

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A Spiking Neuron

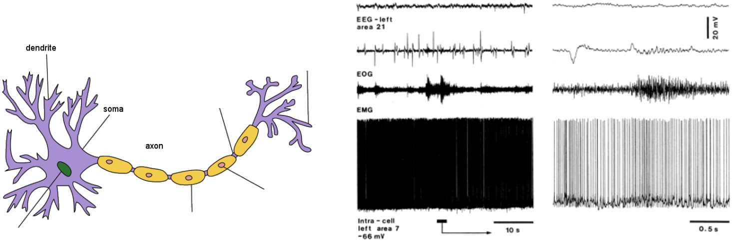


Figure: (a) Neuron [3](b) Membrane potential activity recording in an awake cat [4]

- neuron receives input from other neurons on the dendrites charging the membrane
- when the membrane potential crosses a certain threshold → action potential
- transmitted to other neurons via synapses

[4] from Quasar Jarosz at en.wikipedia

[4] from Steriade, Timofeev, Grenier, 2001

Traveling Waves in Brain Slices

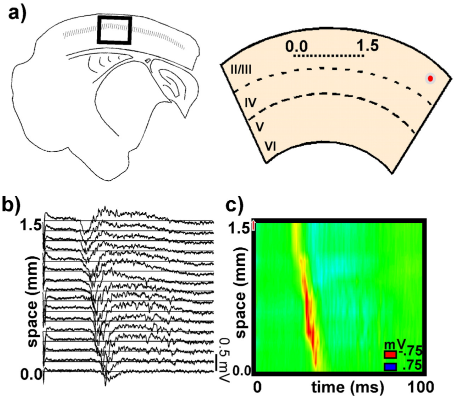


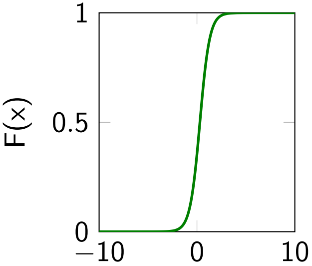
Figure: Propagating activity in a brain slice [5]

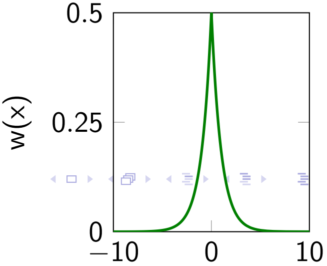
[5] From D.Pinto, S. Patrick, W. Huang, B. Connors: Initiation, Propagation, and Termination of Epileptiform Activity in Rodent Neocortex In Vitro Involve Distinct Mechanisms; The Journal of Neuroscience, 2005

The Neural Field Equation

$$\frac{\partial}{\partial t}u(x,t) = -u(x,t) + \int_{-\infty}^{\infty} w(x-y)F(u(y,t))dy$$

- $u(x,t)$: average membrane potential in the population of neurons at position $x \in \mathbb{R}$ at time t
- $F : \mathbb{R} \to [0,1]$: gain function; proportion of firing neurons in the population
- $w : \mathbb{R} \to [0,\infty)$: strength of synaptic connections between neurons at x and neurons at y





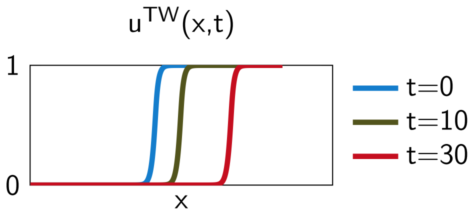
Traveling Waves

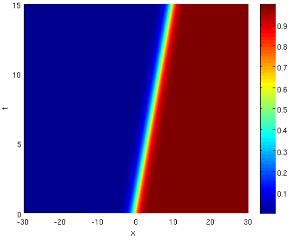
Theorem (Ermentrout, McLeod '93)

Assume that the equation has two stable fixed points separated by an unstable fixed point. Then there exists a unique (up to translations) monotone increasing wave profile $\hat{u} : \mathbb{R} \to \mathbb{R}$ *and a unique wave speed* $c \in \mathbb{R}$ *such that*

$$u^{TW}(x,t) := \hat{u}(x - ct)$$

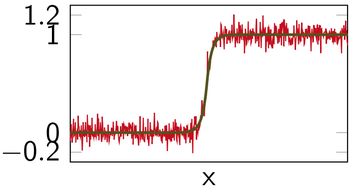
is a traveling wave solution connecting the two stable states

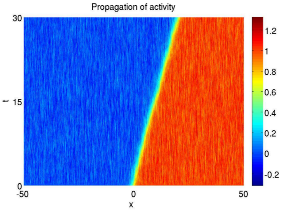




Goals

- determine the influence of noise on the dynamics
- stochastic stability
- stochastic differential equation for the wave speed
- attempts in this direction: P. Bressloff, M. Webber 2012 [6](not rigorous)





[6] P. C. Bressloff, M. A. Webber: Front Propagation in Stochastic Neural Fields; SIAM J. Applied Dynamical Systems, 2012

The Stochastic Neural Field Equation

- want to realise the neural field equation as a stochastic evolution equation

$$du(x,t) = (-u(x,t) + w * F(u_t)(x))\,dt + \sigma\,dW(x,t)$$

- one possibility: work in weighted $\mathcal{L}^2$ space (Faugeras, Inglis 2013)[7]
- consider the difference $v(x,t) = u(x,t) - u^{TW}(x,t)$ leading to the equation

$$dv_t = \left(-v_t + w * (F(v_t + u_t^{TW}) - F(u_t^{TW}))\right) dt + \sigma\,dW_t$$

- (W_t) Q-Wiener process on $\mathcal{L}^2$ with $tr(Q) < \infty$
- for any initial condition $v^0 \in \mathcal{L}^2$, $\exists!$ strong solution admitting a continuous modification

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[7] O. Faugeras, J. Inglis: Stochastic Neural Field Equations: A Rigorous Footing; arXiv:1311.5446, 2013

Stability

- is the wave solution stable under small perturbations?
- in the deterministic case: if $\|v^0\|$ is small, does $\|v_t\| \to 0$?
- in the stochastic case: $u_t =$ traveling wave $+$ fluctuations around wave profile

Linearisation

- Ansatz: linearise the equation around the traveling wave
$$\begin{aligned} dv_t &= \left(-v_t + w * \left(F(v_t + u_t^{TW}) - F(u_t^{TW})\right)\right) dt + \sigma dW_t \\ &= \left(-v_t + w * \left(F'(u_t^{TW}) v_t\right)\right) dt + R(t, v_t) dt + \sigma dW_t \end{aligned}$$
where
$$\|R(t, v_t)\| = O(\|v_t\|^2)$$
- then, in the deterministic case,
$$\frac{1}{2} \frac{d}{dt} \|v_t\|^2 \approx \langle L v_t, v_t \rangle$$
where
$$Lu = -u + w * (F'(u^{TW}) u)$$
- cannot expect L to be strictly dissipative (i.e. $\langle Lu, u \rangle \leq -\kappa \|u\|^2$)

Adaptation of the Wave Speed

- need to adapt to the right phase of the wave
- idea: minimise the $\mathcal{L}^2-$distance, i.e. find y^* such that

$$\|u_t - u_t^{TW}(\cdot - y^*)\| = \min_y \|u_t - u_t^{TW}(\cdot - y)\|$$

- note that

$$\tfrac{1}{2}\frac{d}{ds}\|u_t - u_t^{TW}(\cdot - y(s))\|^2$$
$$= y'(s)\langle u_t - u_t^{TW}(\cdot - y(s)), \partial_x u_t^{TW}(\cdot - y(s))\rangle \leq 0$$

for $y'(s) = -m\langle u_t - u_t^{TW}(\cdot - y(s)), \partial_x u_t^{TW}(\cdot - y(s))\rangle$ for any $m > 0$

- define the phase adaptation as the unique solution to the ode

$$C'(t) = c - m\langle u_t - \hat{u}(\cdot - C(t)), \hat{u}_x(\cdot - C(t))\rangle$$

(P-a.s. in the stochastic case)

The Case $c = 0$

- $L(\hat{u}_x) = 0$ and (under some assumptions)

$$\langle Lu, u\rangle \leq -\kappa \|u\|^2 + K\langle u, \hat{u}_x\rangle^2$$

(L., Stannat 2013)

- $\tilde{v}_t := u_t - \hat{u}(\cdot - C(t))$ satisfies

in the deterministic case:

$$\frac{1}{2}\frac{d}{dt}\|\tilde{v}_t\|^2 \lesssim -\kappa\|\tilde{v}_t\|^2 + (K - m)\langle \hat{u}_x(\cdot - C(t)), \tilde{v}_t\rangle^2 \leq -\kappa\|\tilde{v}_t\|^2$$

in the stochastic case:
$u_t \approx \hat{u}(\cdot - C(t)) +$ Ornstein-Uhlenbeck process

The Case $c = 0$

- noise influences the speed of the wave
- for large relaxation constants m,

$$C(t) \approx -\frac{1}{\|\hat{u}_x\|^2}\left(\langle \tilde{v}_0, \hat{u}_x\rangle + \sigma \int_0^t \langle \hat{u}_x(\cdot - C(s)), dW_s\rangle\right)$$

- $$\left[\sigma \int_0^\cdot \langle \hat{u}(x - C(t)), dW_s\rangle\right]_t \approx \sigma^2 \langle u^{TW}, Qu^{TW}\rangle t$$

- $\rightarrow \approx$ Brownian motion

The Case $c > 0$

- for sufficiently small c

$$\langle -v + w * (F'(\hat{u})v), v\rangle \leq -\kappa_c \|v\|^2 + K_c \langle v, \hat{u}_x\rangle^2$$

(L., 2013)

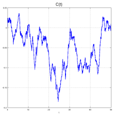
- $c = \frac{\int_0^1 u - F(u)du}{\int \hat{u}_x^2(x)F'(\hat{u}(x))dx}$ is generally not known
- control over c in terms of F: for $w(x) = \frac{1}{2\alpha}e^{-\frac{|x|}{\alpha}}$

$$\frac{\alpha}{\sqrt{2}} \frac{\int_0^1 u - F(u)du}{\sqrt{\int_0^a u - F(u)du}} \leq c \leq \frac{\alpha}{4} \frac{\int_0^1 u - F(u)du}{\int_a^1 F(u) - udu}$$

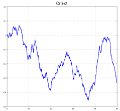
where $F(a) = a$

Numerical Simulations: $c = 0$

$c = 0$:



$c > 0$:



green: reference wave profile
red: wave profile moving at adapted speed
blue: solution to the stochastic neural field equation

The Noise

$$\frac{d}{dt}u^i(t) = \left(-u^i(t) + \sum_j w_{i,j}F(u^j(t))\right) + \text{"}\textit{noise}\text{"}$$

- source of the noise?
- source of intrinsic noise: finite size effects
- idea: set up a continuous time Markov chain describing the evolution of the activity a (= fraction of 'active' neurons) of M coupled populations of finite size N that yields the NFE for $u = F^{-1}(a)$ in the limit
  - Buice, Cowan '07; Bressloff '09; Buckwar, Riedler '13

The Noise

- Markov chain: M populations à N neurons
- activity in population i jumps from a^i to $a^i \pm \frac{1}{N}$ at rate $NF'(F^{-1}(a^i))\left(-F^{-1}(a^i) + \sum_j w_{i,j} a^j\right)_\pm$
- $da_t^i = F'(F^{-1}(a^i(t)))\left(-F^{-1}(a_t^i) + \sum_j w_{i,j} a^j(t)\right) + dM_t^i$
- $\sqrt{N}[M^i]_t \xrightarrow{N\to\infty} \int_0^t F'(F^{-1}(a_s^i)) \left|-F^{-1}(a_s^i) + \sum_j w_{i,j} a_s^j\right| ds$
- suggests the diffusion approximation

$$du_t^i = -u_t^i + \sum_j w_{i,j} F(u_t^j) + \frac{1}{\sqrt{N}} \sqrt{\frac{\left|-u_t^i + \sum_j w_{i,j} F(u_t^j)\right|}{F'(u_t^i)}} dB_t^i$$

- yields a stochastic NFE in the continuum limit